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Admissible initial operators for superpositions of right invertible operators

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Abstract. Suppose that D is a linear right invertible operator acting in a linear space X and that F is an initial operator for D corresponding to a right inverse R of D. Then the set of all right inverses of D is of the form: $\mathscr{R}_D = \{R+FA: A \in L_0(X)\}$, where $L_0(X)$ denotes the space of all linear operators determined on the whole space X and mapping X into itself. The set of all initial operators for D is of the form: $\mathscr{F}_D = \{F(I-AD: A \in L_0(X)\}$. Having already this characterization of the sets $\mathscr{R}_D$ and $\mathscr{F}_D$, we can describe all admissible initial operators for a superposition of a finite number of right invertible operators. A generalization of a theorem about the existence and uniqueness of a solution of an initial value problem with a right invertible operator is also given. Applications to multi-waves equations are indicated (even in the case of variable speed of waves).

In the present paper we determine the class of all initial operators for a superposition of given right invertible operators (cf. [2]). The problem in question was suggested to the author by Professor S. Kaliski.

This paper contains some generalizations of the results given in [2] and [5].

Let $\mathfrak{X}$ be an arbitrary ring with a unit e. The following characterization of right inverses is due to Arens [1]:

Suppose that $t \in \mathfrak{X}$ has a right inverse s. Then any right inverse of t is of the form:

$$\hat{s} = a+s(e-ta) \quad \text{for every } a \in \mathfrak{X}. \tag{1}$$

This characterization is also true in an arbitrary pararing with units (cf. [3]) if we consider all $a \in \mathfrak{X}$ such that the product ta is well-determined. In particular, we obtain the following characterization of right inverses of right invertible linear operators:

Let X be a linear space over a field $\mathscr{F}$ of scalars. Denote by $L(X)$ the set of all linear operators defined on linear subsets $\mathscr{D}_A$ of X, called the *domain* of X and mapping $\mathscr{D}_A$ into X (1).

(1) It is a pararing with units (moreover, it is a para-algebra).

Denote by $\boldsymbol{R}(X)$ the set of all right invertible operators belonging to $L(X)$. The set of all right inverses of an operator $D\in\boldsymbol{R}(X)$ will be denoted by $\mathscr{R}_D$. As in [2], we assume here that $\mathscr{D}_R = X$, $\mathscr{D}_D \subset RX$ for $R\in\mathscr{R}_D$. Write: $L_0(X) = \{A\in L(X)\colon \mathscr{D}_A = X\}$.

Let $D\in\boldsymbol{R}(X)$ and $R\in\mathscr{R}_D$. Then every right inverse of D is of the form

$$(2)\qquad \hat{R} = A+R(I-DA) \quad \text{for an } A\in L_0(X),$$

where I denotes the identity operator.

Let $\mathscr{F}_D$ denote the family of all initial operators for an operator $D\in\boldsymbol{R}(X)$. We recall (cf. [2]) that $F\in L(X)$ is an initial operator for $D\in\boldsymbol{R}(X)$ corresponding to an $R\in\mathscr{R}_D$ if it is a projection onto the kernel of D, i.e., if $F^2 = F$, $FX = Z_D$, where $Z_D = \{x\in\mathscr{D}_D\colon Dx = 0\}$ ([2]) and, moreover, $FR = 0$. By Theorem 2.2 of [2] an operator F belongs to $\mathscr{F}_D$ if and only if

$$(3)\qquad F = I-RD \quad \text{on } \mathscr{D}_D \quad \text{for an } R\in\mathscr{R}_D.$$

An initial operator $\hat{F}$ corresponding to a right inverse $\hat{R}$ defined by formula (1.2) is of the form

$$(4)\qquad \hat{F} = F(I-AD), \quad \text{where } A\in L_0(X),$$

and F is defined by formula (3).

Indeed, formula (3) implies that on the domain $\mathscr{D}_D$ we have

$$\begin{aligned}\hat{F} &= I-\hat{R}D = I-[A+R(I-DA)]D\\ &= I-AD-R(I-DA)D = I-AD-RD+(RD)(AD)\\ &= (I-RD)(I-AD) = F(I-AD).\end{aligned}$$

Proposition 1. *Let $D\in\boldsymbol{R}(X)$. Then $\hat{R}\in\mathscr{R}_D$ if and only if there is an $R\in\mathscr{R}_D$ and an $A\in L_0(X)$ such that*

$$(5)\qquad \hat{R} = R+FA$$

where F is an initial operator for D corresponding to R. Moreover, an initial operator $\hat{F}$ corresponding to $\hat{R}$ is

$$(6)\qquad \hat{F} = F(I-AD).$$

Proof. *Sufficiency.* Suppose that $R\in\mathscr{R}_D$ and $A\in L_0(X)$ are arbitrary. Write: $\hat{R} = R+FA$. Observe that

$$\hat{R} = R+FA = R+(I-RD)A = R+A-RDA = A+R(I-DA),$$

as in Formula (2). Since $DF = 0$, we have, by definition,

$$D\hat{R} = D(R+FA) = DR+DFA = I.$$

([2]) The kernel Z_D is said to be the *space of constants* for the operator D.

Hence $\hat{R}\in\mathscr{R}_D$. The initial operator $\hat{F}$ corresponding to $\hat{R}$ is, by definition,

$$\hat{F} = I-\hat{R}D = I-(R+FA)D = I-RD-FAD$$
$$= F-FAD = F(I-AD).$$

Necessity. Let $\hat{R}\in\mathscr{R}_D$ be given and let $R\in\mathscr{R}_D$ be arbitrary chosen. Write

$$F = I-\hat{R}D, \quad \hat{F} = I-\hat{R}D \quad \text{on } \mathscr{D}_D \quad \text{and} \quad A = R-\hat{R}\in L_0(X).$$

Since $DR = I$ and $FR = 0$, we conclude that

$$R+FA = R+F(R-\hat{R}) = R+FR-F\hat{R} = R-F\hat{R}$$
$$= R+(I-RD)\hat{R} = R+\hat{R}-R(D\hat{R}) = R+\hat{R}-R = \hat{R}.$$

Thus $\hat{R}\in\mathscr{R}_D$ is of the required form. Formula (4) implies that $\hat{F} = F(I-AD)$, which was to be proved.

An immediate consequence of Proposition 1 is

THEOREM 2. *Let $D\in\boldsymbol{R}(X)$. If $F\in L(X)$ is an arbitrary initial operator for D corresponding to an $R\in\mathscr{R}_D$, then*

$$(7) \qquad \mathscr{R}_D = \{R+FA\colon\ A\in L_0(X)\},$$

$$(8) \qquad \mathscr{F}_D = \{F(I-AD)\colon\ A\in L_0(X)\}.$$

A similar characterization can be given for left invertible operators (cf. [4]).

THEOREM 3. *Suppose that $D_1, \ldots, D_m\in\boldsymbol{R}(X)$ and that F_j is an initial operator for D_j corresponding to a right inverse $R_j\in\mathscr{R}_{D_j}$ $(j = 1, \ldots, m)$. Write*

$$(1.9) \qquad D = D_1\ldots D_m; \quad R = R_m\ldots R_1;$$

$$(1.10) \qquad F = F_m+R_mF_{m-1}D_m+\ldots+R_m\ldots R_2F_1D_2\ldots D_m.$$

Then $D\in\boldsymbol{R}(X)$, $R\in\mathscr{R}_D$, F is an initial operator for D corresponding to R and, moreover,

$$(1.11) \qquad \mathscr{R}_D = \{R+FA\colon\ A\in L_0(X)\},$$

$$(1.12) \qquad \mathscr{F}_D = \{F(I-AD)\colon\ A\in L_0(X)\}.$$

Proof. Since, by our assumption, $D_jR_j = I$ for $j = 1, \ldots, m$, we conclude that

$$DR = D_1\ldots D_{m-1}D_mR_mR_{m-1}\ldots R_1 = D_1\ldots D_{m-1}R_{m-1}\ldots R_1 = \ldots$$
$$\ldots = D_1R_1 = I.$$

Thus $D \in \boldsymbol{R}(X)$ and $R \in \mathcal{R}_D$. Formula (3) implies that on $\mathcal{D}_D$ we have

$$\begin{aligned} F &= F_m + R_m F_{m-1} D_m + \ldots + R_m \ldots R_2 F_1 D_2 \ldots D_m \\ &= I - R_m D_m + R_m (I - R_{m-1} D_{m-1}) D_m + \ldots + \\ &\qquad + R_m \ldots R_2 (I - R_1 D_1) D_2 \ldots D_m \\ &= I - R_m D_m + R_m D_m - R_m R_{m-1} D_{m-1} D_m + \ldots + \\ &\qquad + R_m \ldots R_2 D_2 \ldots D_m - R_m \ldots R_2 R_1 D_1 D_2 \ldots D_m \\ &= I - R_m \ldots R_1 D_1 \ldots D_m = I - RD. \end{aligned}$$

Hence F is an initial operator for D corresponding to R. This and Theorem 2 together imply that the sets $\mathcal{R}_D$ and $\mathcal{F}_D$ are of the form (11) and (12) respectively.

Theorems 2 and 3 together imply another characterization of initial operators for a superposition of right invertible operators.

THEOREM 4. *Suppose that all the assumptions of Theorem 3 are satisfied and that D is defined by Formula (9). Write:*

$$R = (R_m + F_m A_m) \ldots (R_1 + F_1 A_1), \tag{13}$$

$$\begin{aligned} F &= F_m (I - A_m D_m) + \ldots + \\ &\qquad + (R_m + F_m A_m) \ldots (R_2 + F_2 A_2) F_1 (I - A_1 D_1) D_2 \ldots D_m, \end{aligned} \tag{14}$$

where $A_1, \ldots, A_m \in L_0(X)$ are arbitrary.

Then $R \in \mathcal{R}_D$ and F is an initial operator for D corresponding to R. Moreover,

$$\mathcal{R}_D = \{(R_m + F_m A_m) \ldots (R_1 + F_1 A_1) \colon A_1, \ldots, A_m \in L_0(X)\}, \tag{15}$$

$$\begin{aligned} \mathcal{F}_D = \{ & F_m (I - A_m D_m) + \ldots + R_m + F_m A_m \ldots \\ & \ldots (R_2 + F_2 A_2 F_1)(I - A_1 D_1) D_2 \ldots D_m \colon A_1, \ldots, A_m \in L_0(X)\}. \end{aligned} \tag{16}$$

Indeed, Theorem 2 implies that every operator of the form $F_j(I - A_j D_j)$, where $A_j \in L_0(X)$ ($j = 1, \ldots, m$) is an initial operator for D_j corresponding to the right inverse $R_j + F_j A$ of D_j. This and Theorem 3 together imply that the operator F determined by formula (14) is an initial operator for $D = D_1 \ldots D_m$ corresponding to the right inverse R of D determined by formula (13).

Suppose we are given an operator $D \in \boldsymbol{R}(X)$ and an initial operator F for D corresponding to an $R \in \mathcal{R}_D$. Write

$$Q(D) = \sum_{k=0}^{N} Q_k D^k, \quad \text{where } Q_0, \ldots, Q_{N-1} \in L(X),\ Q_N = I. \tag{17}$$

We recall (cf. [2]) that and *initial value problem* for the operator $Q(D)$ is to find all solutions of the equation

$$Q(D)x = y, \quad y \in X, \tag{18}$$

satisfying the condition

$$FD^k x = y_k, \quad y_k \in Z_D \quad (k = 0, 1, \ldots, N-1). \tag{19}$$

This problem is said to be *well-posed* if it has a unique solution for every $y \in X$, $y_0, \ldots, y_{N-1} \in Z_D$. This means that a well-posed homogeneous initial value problem has only zero as a solution. By Corollary 3.1 of [2] an initial value problem (13)–(14) is well posed if -1 is not an eigenvalue of the operator

$$\hat{Q} = \sum_{k=0}^{N-1} Q_k R^{N-k}. \tag{20}$$

Indeed, in this case the operator $I + \hat{Q}$ is invertible and a unique solution of the problem under consideration is

$$x = R^N (I+\hat{Q})^{-1}\Big[y - \sum_{m=0}^{N-1} \Big(\sum_{k=0}^{m} Q_k R^{m-k}\Big) y_m\Big] + \sum_{k-0}^{N-1} R^k y_k. \tag{21}$$

In particular, if $Q_0 = Q_1 = \ldots = Q_{N-1} = 0$, then $\hat{Q} = 0$. We therefore conclude that in this case problem (1.18)–(1.19) is well-posed and its solution is

$$x = R^N y + \sum_{k=0}^{N-1} R^k y_k. \tag{22}$$

The Corollary 3.1 of [2], mentioned above, can easily be generalized to a larger class of operators. Namely, we have

THEOREM 5. *Suppose that the operator $D \in \boldsymbol{R}(X)$, that F is an initial operator for D corresponding to an $R \in \mathscr{R}_D$ and that the operator $Q(D)$ is defined by formula* (1.17). *If -1 is not an eigenvalue of the operator $\hat{Q}$ defined by formula* (20), *then the initial value problem*

$$Q(D) D^M x = y, \quad y \in X \quad (M \geqslant 0), \tag{23}$$

$$FD^k x = y_k, \quad y_k \in Z_D \quad (k = 0, 1, \ldots, N+M-1) \tag{24}$$

is well-posed and its unique solution is of the form

$$x = R^{N+M}(I+\hat{Q})^{-1}\Big[y - \sum_{m=0}^{N-1} \Big(\sum_{k=0}^{m} Q_k R^{m-k}\Big) y_{M+m}\Big] + \sum_{m=0}^{N+M-1} R^m y_m. \tag{25}$$

Proof. Put $u = D^M x$. Then we have

$$FD^k u = FD^{M+k} x = y_{k+M} \quad \text{for } k = 0, 1, \ldots, N-1.$$

Therefore we can rewrite problem (23)–(24) as follows:

$$Q(D)u = y, \tag{26}$$

$$FD^k u = y_{k+M} \quad (k = 0, 1, \ldots, N-1). \tag{27}$$

According to Corollary 3.1 of [2] and formula (21) this problem is well-posed and has a unique solution

$$u = R^N(I+\hat{Q})^{-1}\Big[y - \sum_{m=0}^{N-1}\Big(\sum_{k=0}^{m} Q_k R^{m-k}\Big) y_{M+m}\Big] + \sum_{k=0}^{N-1} R^k y_{k+M}. \tag{28}$$

Having already determined $u = D^M x$, we consider the following initial value problem:

$$D^M x = u, \quad FD^k x = y_k \quad (k = 0, 1, \ldots, M-1). \tag{29}$$

This problem is also well-posed and formula (22) implies that its unique solution is of the form:

$$x = R^M u + \sum_{m=0}^{M-1} R^m y_m.$$

This and formula (28) together imply that problem (23)–(24) is well-posed and has a unique solution

$$\begin{aligned} x &= R^M u + \sum_{m=0}^{M-1} R^m y_m \\ &= R^M\Big\{R^N(I+\hat{Q})^{-1}\Big[y - \sum_{m=0}^{N-1}\Big(\sum_{k=0}^{m} Q_k R^{m-k}\Big) y_{M+m}\Big] + \\ &\qquad + \sum_{k=0}^{N-1} R^k y_{k+M}\Big\} + \sum_{m=0}^{M-1} R^m y_m \\ &= R^{N+M}(I+\hat{Q})^{-1}\Big[y - \sum_{m=0}^{N-1}\Big(\sum_{k=0}^{m} Q_k R^{m-k}\Big) y_{M+m}\Big] + \\ &\qquad + \sum_{k=0}^{N-1} R^{k+M} y_{k+M} + \sum_{m=0}^{M-1} R^m y_m \\ &= R^{N+M}(I+\hat{Q})^{-1}\Big[y - \sum_{m=0}^{N-1}\Big(\sum_{k=0}^{m} Q_k R^{m-k}\Big) y_{M+m}\Big] + \\ &\qquad + \sum_{m=0}^{N+M-1} R^m y_m, \end{aligned}$$

which was to be proved.

A similar statement is also true for the operator $D^M Q(D)$.

In [5] we have shown some applications of Theorems 3 and 4 to one-dimensional multi-waves operators, i.e., to operators of the form

$$D = D_1 \dots D_m, \tag{30}$$

where

$$D_j = \frac{\partial^2}{\partial s^2} - \frac{1}{c_j^2}\frac{\partial^2}{\partial t^2} \quad (j = 1, \dots, m)$$

and c_j are given constant coefficients. Namely, if we choose operators $A_1, \dots, A_m$ in formulae (13) and (14) in an appropriate way, we obtain various probleme for the operator D, such the generalized Cauchy problem, the Darboux–Picard problem, etc.

Here we show only that this method can be used for equations with variable coefficients. It is enough to consider one operator of the form

$$\hat{D} = \frac{\partial^2}{\partial \xi^2} - b(\tau)\frac{\partial^2}{\partial \tau^2}, \tag{31}$$

where the function $1/b(\tau)$ is measurable and locally integrable, the function

$$a(\tau) = \int \frac{b(\theta)}{b\theta} \tag{32}$$

is differentiable at each point and, moreover, the function

$$a(\tau) = a(\tau)\tau \tag{33}$$

is one-to-one.

Write

$$s = \xi - a(\tau)\tau, \quad t = \xi + a(\tau)\tau. \tag{34}$$

It is easy to check that this change of variables permits us to rewrite the operator $\hat{D}$ as follows:

$$\begin{aligned} \hat{D} &= \frac{\partial^2}{\partial \xi^2} - b(\tau)\frac{\partial^2}{\partial \tau^2} \\ &= 4\frac{\partial^2}{\partial t \partial s} + \frac{1}{b\left[a^{-1}\left(\frac{t-s}{2}\right)\right]}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial s}\right) = 4(D+H), \end{aligned} \tag{35}$$

where

$$D = \frac{\partial^2}{\partial t \partial s}, \quad H = \frac{1}{4b\left[a^{-1}\left(\frac{t-s}{2}\right)\right]}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial s}\right). \tag{36}$$

The operator D is obviously right invertible and has a right inverse, for instance of the form

$$(Rx)(t, s) = \int_0^t \int_0^s x(\xi, \eta) d\eta d\xi. \tag{37}$$

It is not difficult to verify that under our assumptions the operator $I + RH$ is invertible (in the space of continuous functions). Thus the operator $\hat{D} = 4(D+H)$ has a right inverse

$$\hat{R} = \tfrac{1}{4}(I+RH)^{-1}R. \tag{38}$$

Indeed,

$$\hat{D}\hat{R} = 4(D+H)\tfrac{1}{4}(I+RH)^{-1}R = (D+DRH)(I+RH)^{-1}R$$
$$= D(I+RH)(I+RH)^{-1}R = DR = I.$$

Thus we can apply all the results of papers [2], [5] and the present paper to operators $\hat{D}$ of the form (31).

In a similar way we can consider the case where the coefficient of the operator $\hat{D}$ defined by formula (31) depends on the variable ξ:

$$b = b(\xi).$$

References

[1] R. Arens *Dense Limit Rings*, Michigan Math. J. 5 (1958), p. 159–182.

[2] D. Przeworska-Rolewicz, *Algebraic theory of right invertible operators*, Studia Math. 48 (1973), p. 129–144.

[3] — *Pseudocategories*; *Para-algebras and Perturbations of linear operators*, Preprint No. 60, Institute of Mathematics Polish Academy of Sciences, Warszawa 1973.

[4] — *Concerning left invertible operators*, Bull. Acad. Polon. Sci. 20 (1972), p. 837–839.

[5] — *A characterization of admissible initial value problems for one-dimensional multi-waves equations* (Polish), Biuletyn WAT 7 (263) (1974), p. 23–30.

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