ANNALES
POLONICI MATHEMATICI
XLVII (1987)

The controllability of a quasilinear functional differential system

by VITALI C. DANNON (Rock Hill) and ATHANASSIOS G. KARTSATOS (Tampa)

Abstract. Functional control systems of the type:

(S) $$x' = A(t, x, u, x_t, u_t)x + B(t, x, u, x_t, u_t)u + Q(t, x, u, x_t, u_t),$$
$$x_0 = \varphi, \quad u_0 = \psi$$

are studied. A, B, Q are $n\times n$, $n\times p$, $n\times 1$ matrices, respectively.

The controllability of such systems is shown under assumptions involving the controllability of the associated linear systems.

The method involves the Schauder–Tychonov theorem in connection with a result of Opial concerning the compactness of certain sets of L_1 matrices.

1. Introduction. Preliminaries. This paper is concerned with the controllability of quasilinear systems of the type:

(S) $$x' = A(t, x, u, x_t, u_t)x + B(t, x, u, x_t, u_t)u + Q(t, x, u, x_t, u_t),$$

where $x_t(s) = x(t+s)$, $s\in[-r_1, 0]$, $t\in[0, T]$, $u_t(s) = u(t+s)$, $s\in[-r_2, 0]$, $t\in[0, T]$. Here r_1, r_2, T are three fixed positive constants. We denote by $\boldsymbol{R}^1$, $\boldsymbol{R}_+$ the sets $(-\infty, \infty)$, $[0, +\infty)$, respectively. We use the symbol $C^j[a, b]$, $j = 1, 2, \ldots$, to denote the space of all continuous functions f: $[a, b]\to\boldsymbol{R}^j$ with the sup-norm $\|\cdot\|_\infty$. The matrices A, B, Q have dimensions $n\times n$, $n\times p$, $n\times 1$, respectively, where n, p are fixed positive integers. For every vector $x\in\boldsymbol{R}^m$ and every matrix $A = [a_{ij}]$ of dimension $n\times m$ we set

$$\|x\| = \sum_{i=1}^{m} |x_i|, \quad \|A\| = \sum_{i=1}^{n}\sum_{j=1}^{m} |a_{ij}|.$$

The symbol L_1^m stands for the space of all x: $D\to\boldsymbol{R}^m$ ($D = [0, T]$) such that $x(t)$ is measurable and $\|x(t)\|$ is Lebesgue integrable on D. The space L_1^m is associated with the norm:

$$|||x||| = \int_D \|x(t)\|\,dt.$$

Similarly,

$$L_1^{n\times m} = \{A(t) = [a_{ij}(t)],\ i = 1, \ldots, n,\ j = 1, \ldots, m;$$
$$a_{ij}(t) \text{ is defined and Lebesgue integrable on } D\}.$$

We associate $L_1^{n\times m}$ with the norm

$$\|A\|_m = \max_{t\in D} \{\sum_{i=1}^{n} \sum_{j=1}^{m} |\int_0^t a_{ij}(s)\,ds|\}.$$

In $L_1^{n\times m}$ we also consider the norm:

$$|||A||| = \int_D \|A(t)\|\,dt.$$

From Opial [12] we quote the following result:

LEMMA 1. *Let* $\alpha\colon D \to \boldsymbol{R}_+$ *be Lebesgue integrable. Then the set*

$$K = \{A \in L_1^{n\times m};\ \|A(t)\| \leqslant \alpha(t) \text{ a.e. in } D\}$$

is compact in the norm $\|\cdot\|_m$.

Actually Opial proved Lemma 1 in the case $m = n$, but his proof carries over in the present case with the obvious modifications.

Condition (I) below will be assumed to hold throughout the sequel.

(I) For each $(x_1, x_2, \varphi, \psi) \in \boldsymbol{R}^n \times \boldsymbol{R}^p \times C^n[-r_1, 0] \times C^p[-r_2, 0]$ the functions $A(t, x_1, x_2, \varphi, \psi)$, $B(t, x_1, x_2, \varphi, \psi)$, $Q(t, x_1, x_2, \varphi, \psi)$ are integrable in t. On the other hand, these functions are continuous in x_1, x_2, φ, ψ for almost all $t \in D$.

System (S) is said to be *controllable* if for every pair of points $\bar{x}_0, \bar{x}_T \in \boldsymbol{R}^n$ and every pair of functions $\varphi \in C^n[-r_1, 0]$, $\psi \in C^p[-r_2, 0]$ with $\varphi(0) = \bar{x}_0$, $\psi(0) = 0$, there exists a continuous control function $u \in C^p[-r_2, T]$ such that $u_0 = \psi$ and such that system (S) has a response $x \in C^n[-r_1, T]$ such that $x_0 = \varphi$ and $x(T) = \bar{x}_T$.

It is evident that under Condition (I), the initial value problem

$$(\mathrm{S})_{f,g} \quad x'(t) = A(t, f(t), g(t), f_t, g_t)x(t) + B(t, f(t), g(t), f_t, g_t)u(t) + Q(t, f(t), g(t), f_t, g_t),$$

$$x(0) = \bar{x}_0$$

has a solution $x(t)$ (in the Caratheodory sense) on $[0, T]$ for every $(f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]$, every $u \in C^p[0, T]$, and every $\bar{x}_0 \in \boldsymbol{R}^n$. We set $\|(f, g)\| = \|f\|_\infty + \|g\|_\infty$.

Our purpose in this paper is to prove the controllability of (S) via a fixed point theorem applied on an operator associated with the linear control system $(\mathrm{S})_{f,g}$.

Although the interval $[0, T]$ here is fixed, our controllability results can be easily modified to include arbitrary intervals $[t_0, t_1] \subset R^1$. An interesting aspect of the study of (S) is that the control $u(t)$ can be chosen to equal an arbitrary continuous function ψ on $[-r_2, T]$ with $\psi(0) = 0$.

The reader is referred to the book of Conti [4] and paper [3] for an excellent account of results in linear control theory. For a nonfunctional result about (S) we cite the paper of Kartsatos [5]. For other related results on the subject, the paper of Anichini [1] and the references therein are suggested. Various quasilinear results concerning ordinary differential systems are contained in Opial [12], Kartsatos [5]–[8], Kartsatos and Parrott [9]–[10] and Becker [2]. Interesting quasilinear problems were also considered by Sager in his dissertation [13]. Sager has an excellent bibliography of relevant results in finite as well as infinite dimensional spaces.

2. Continuity with respect to A, B, Q. We denote by $X(t; A)$ the fundamental matrix of solutions of the system

$$x' + A(t)x = 0, \qquad t \in [0, T], \tag{1}$$

where $A \in L_1^{n \times n}$ is a given matrix. We further assume that $X(0; A) = I$ (= the $n \times n$ identity matrix). We also set, for $t \in [0, T]$,

$$U(t; A, B) = \int_{T-t}^{T} X(T; A) X^{-1}(s; A) B(s)\, ds,$$

$$S(A, B) = \int_0^T U(s; A, B) U^*(s; A, B)\, ds,$$

$$V(t; A, B) = \int_{T-t}^{T} U^*(s; A, B)\, ds,$$

$$W(t; A, B) = \int_0^t X(t; A) X^{-1}(s; A) B(s) V(s; A, B)\, ds,$$

where $B \in L_1^{n \times p}$ is another given matrix.

Now, let $\bar{x}_0$, $\bar{x}_T \in R^n$ be fixed and consider the operator F: $(A, B, Q) \to (x, u)$, where

$$x(t) = X(t; A)\bar{x}_0 + W(t; A, B) y(A, B) + \int_0^t X(t; A) X^{-1}(s; A) Q(s)\, ds,$$

$$u(t) = V(t; A, B) y(A, B),$$

$$y(A, B) = S^{-1}(A, B)[\bar{x}_T - X(T; A)\bar{x}_0] - S^{-1}(A, B) \int_0^T X(T; A) X^{-1}(s; A) Q(s)\, ds,$$

$t \in [0, T]$ for $Q \in L_1^n$. The following lemma establishes the continuity of the operator F on certain subsets of $L_1^{n \times n} \times L_1^{n \times p} \times L_1^n$.

LEMMA 2. *Assume that* $L \subset L_1^{n\times n}$, $M \subset L_1^{n\times p}$ *are such that*

(i) *there exists a constant* $\alpha > 0$ *such that* $|||A|||_n \leqslant \alpha$, $A \in L$;

(ii) *there exists a constant* $\beta > 0$ *such that* $|||B|||_p \leqslant \beta$, $B \in M$;

(iii) $S(A, B)$ *is invertible for all* $(A, B) \in L \times M$.

Then $F: L \times M \times L_1^n \ni (A, B, Q) \to (x, u) \in C^n[0, T] \times C^p[0, T]$ *is continuous. The space* $L_1^{n\times n} \times L_1^{n\times p} \times L_1^n$ *is associated with the norm* $\|\cdot\|_n + \|\cdot\|_p + |||\cdot|||$ *and the norm of the space* $C^n[0, T] \times C^p[0, T]$ *equals the sup-norm of* $C^n[0, T]$ *plus the sup-norm on* $C^p[0, T]$.

Proof. Let $(A_m, B_m, Q_m) \in L \times M \times L_1^n$ be given with $(A_m, B_m, Q_m) \to (A, B, Q)$ as $m \to \infty$. Let $F(A_m, B_m, Q_m) = (x_m, u_m)$. We first notice that if $X_m(t) \equiv X_m(t; A_m)$, then $X_m \to X(\cdot\,; A)$ uniformly on D as $m \to \infty$. This has been shown by Opial in [11]. From Corollary 1 of the same paper we also obtain that $X_m^{-1} \to X^{-1}(\cdot\,; A)$ uniformly on D and that $|\det X_m(t)| \geqslant d > 0$ for all $m = 1, 2, \ldots$ and all $t \in D$. It is now evident that $U(\cdot\,; A_m, B_m) \to U(\cdot\,; A, B)$ and that $S(A_m, B_m) \to S(A, B)$. This implies that $\det S(A, B)$ is a continuous function on $L \times M$ which is a compact subset of $L_1^{n\times n} \times L_1^{n\times p}$. Since $\det S(A, B) > 0$ on $L \times M$ ($S(A, B)$ is symmetric and positive semidefinite), there exists a constant $d_1 > 0$ such that $\det S(A, B) \geqslant d_1$, $(A, B) \in L \times M$. This in turn implies (see Kartsatos [5], p. 143) that $S^{-1}(A_m, B_m) \to S^{-1}(A, B)$. Since $Y_m \to Y$ uniformly on D, where

$$Y_m(t) = \int_0^t X_m(t) X_m^{-1}(s) Q_m(s)\, ds,$$

$$Y(t) = \int_0^t X(t; A) X^{-1}(s; A) Q(s)\, ds,$$

$t \in D$, it is easy to see that $(x_m, u_m) \to (x, u)$ which completes the proof.

3. The existence result. For each $(f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]$, we denote by $X(t; f, g)$ the fundamental matrix of the system

$$x' = A\big(t, f(t), g(t), f_t, g_t\big)x$$

with $X(0; f, g) = I$. As in Section 2, we define in the obvious way the symbols $U(t; f, g)$, $S(f, g)$, $V(t; f, g)$, $W(t; f, g)$, $y(f, g)$, in which $B(t) \equiv B\big(t, f(t), g(t), f_t, g_t\big)$, $Q(t) \equiv Q\big(t, f(t), g(t), f_t, g_t\big)$. We are planning to show that the operator $G: (f, g) \in C^n[-r_1, T] \times C^p[-r_2, T] \to (x, u) \in C^n[-r_1, T] \times C^p[-r_2, T]$ with

$$x(t) = X(t; f, g) + W(t; f, g)\, y(f, g) + \int_0^t X(t; f, g) X^{-1}(s; f, g) Q\big(s, f(s), g(s), f_s, g_s\big)\, ds, \quad t \in D,$$

$$x(t) = \varphi(t), \quad t \in [-r_1, 0],$$

$$u(t) = V(t; f, g)\, y(f, g), \qquad t \in D,$$
$$u(t) = \psi(t), \qquad t \in [-r_2, 0]$$

(for a fixed pair $(\varphi, \psi) \in C^n[-r_1, 0] \times C^p[-r_2, 0]$ with $\varphi(0) = \bar{x}_0$ and $\psi(0) = 0$ and fixed $\bar{x}_0$, $\bar{x}_T \in \boldsymbol{R}^n$), has a fixed point in a certain ball of its domain under certain assumptions. As in Kartsatos [5], it is easy to see that for each (f, g) the function $x(t)$ is continuous on $[-r_1, T]$ and satisfies system $(\mathrm{S})_{f,g}$ and $x(T) = \bar{x}_T$. Moreover, $u(t)$ is continuous on $[-r_2, T]$ and such that $u(0) = 0$.

Our existence result is contained in the next theorem.

THEOREM 1. *Let L, M be a closed subsets of* $L_1^{n \times n}$, $L_1^{n \times p}$, *respectively. Assume that the mappings*

$$A^0 = A^0(f, g)\colon\ t \to A\big(t, f(t), g(t), f_t, g_t\big),$$
$$B^0 = B^0(f, g)\colon\ t \to B\big(t, f(t), g(t), f_t, g_t\big)$$

are such that $A^0 \in L$, $B^0 \in M$ *for every* $(f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]$. *Moreover,* $\det S(A, B) > 0$ *for* $(A, B) \in L \times M$. *Then system* (S) *is controllable provided that*

(i) *the functions*

$$\alpha(t) = \sup\{\|A(t, x_1, x_2, \varphi, \psi)\|;\ (x_1, x_2, \varphi, \psi) \in \boldsymbol{R}^n \times \boldsymbol{R}^p \times C^n[-r_1, 0] \times C^p[-r_2, 0]\},$$

$$\beta(t) = \sup\{\|B(t, x_1, x_2, \varphi, \psi)\|;\ (x_1, x_2, \varphi, \psi) \in \boldsymbol{R}^n \times \boldsymbol{R}^p \times C^n[-r_1, 0] \times C^p[-r_2, 0]\}$$

are Lebesgue integrable on D;

(ii) *for each integer* $k > 0$, *the function*

$$q_k(t) = \sup\{\|Q(t, x_1, x_2, \varphi, \psi)\|;\ \|x_1\|, \|x_2\| \leqslant k,\ \|\varphi\|_\infty, \|\psi\|_\infty \leqslant k\}$$

is Lebesgue integrable on D and such that

$$\liminf_{k \to \infty} (1/k) \int_D q_k(t)\, dt = 0.$$

Proof. Let

$$L^0 = \{A^0(f, g);\ (f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]\},$$
$$M^0 = \{B^0(f, g);\ (f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]\},$$

and let $\overline{L^0}$, $\overline{M^0}$ be the closures of L^0, M^0 in $L_1^{n \times n}$, $L_1^{n \times p}$, respectively. Then from the proof of Lemma 1 of Opial [12] we obtain that $\|A(t)\| \leqslant \alpha(t)$, $\|B(t)\| \leqslant \beta(t)$ for every $(A, B) \in \overline{L^0} \times \overline{M^0}$. Thus, Lemma 1 in Section 1 implies that the set $\overline{L^0} \times \overline{M^0}$ is a compact subset of $L_1^{n \times n} \times L_1^{n \times p}$, hence a compact subset of $L \times M$. Without loss of generality we set $\overline{L^0} \times \overline{M^0} \equiv L \times M$. We also have

$$|||A|||_n \leqslant \int_D \alpha(t)\,dt, \qquad |||B|||_p \leqslant \int_D \beta(t)\,dt$$

for all $(A, B) \in L \times M$. In view of Lemma 2 and its proof, in order to show the continuity of the operator G (preceding the statement of the theorem), in the sum-norm of $C^n[-r_1, T] \times C^p[-r_2, T]$, it suffices to show that $(f_m, g_m) \in C^n[-r_1, T] \times C^p[-r_2, T]$ with $\|f_m - f\|_\infty, \|g_m - g\|_\infty \to 0$ as $m \to \infty$ implies that

$$\|A^0(f_m, g_m) - A^0(f, g)\|_n \to 0 \quad \text{as } m \to \infty,$$

$$\|B^0(f_m, g_m) - B^0(f, g)\|_p \to 0 \quad \text{as } m \to \infty,$$

$$|||Q^0(f_m, g_m) - Q^0(f, g)||| \to 0 \quad \text{as } m \to \infty,$$

where $Q^0 \equiv Q^0(f, g)\colon t \to Q(t, f(t), g(t), f_t, g_t)$.

To this end, we first note that

$$\|f_{m_t} - f_t\|_\infty \to 0 \quad \text{and} \quad \|g_{m_t} - g_t\|_\infty \to 0 \quad \text{as } m \to \infty.$$

Thus,

$$A(t, f_m(t), g_m(t), f_{m_t}, g_{m_t}) \to A(t, f(t), g(t), f_t, g_t),$$

$$B(t, f_m(t), g_m(t), f_{m_t}, g_{m_t}) \to B(t, f(t), g(t), f_t, g_t),$$

$$Q(t, f_m(t), g_m(t), f_{m_t}, g_{m_t}) \to Q(t, f(t), g(t), f_t, g_t)$$

a.e. on D as $m \to \infty$. By Lebesgue's dominated convergence theorem, we have

$$\|A^0(f_m, g_m) - A^0(f, g)\|_n \leqslant \int_D \left\| A(t, f_m(t), g_m(t), f_{m_t}, g_{m_t}) - A(t, f(t), g(t), f_{m_t}, g_{m_t}) \right\| dt \to 0$$

as $m \to \infty$. Similar inequalities prove our assertion for B^0 and Q^0.

Now, following the proof of the main result of Kartsatos [5], we see that there exists an integer $k \in \{1, 2, \ldots\}$ such that the ball $E_k = \{(f, g) \in C^n[-r_1, T] \times C^p[-r_2, T];\ \|(f, g)\| \leqslant k\}$ is mapped into itself by the operator G. It also follows as in [5] that the set $\{(x_{f,g}, u_{f,g});\ (f, g) \in E_k\}$ is equicontinuous and uniformly bounded. This implies that the operator G is compact on E_k. The Schauder–Tychonov theorem implies now that G has a fixed point in E_k, which completes the proof of the theorem.

The compactness of the set $\overline{L^0} \times \overline{M^0}$ in $L_1^{n \times n} \times L_1^{n \times p}$ in the above theorem is crucial for the proof because it ensures that $S^{-1}(f, g)$ exists for all $(f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]$ and that $\det S(f, g)$ is bounded below by a positive constant for all such (f, g). If this last condition is assumed to hold, then we have, as a corollary to Theorem 1, the following result ($S(A, B)$ has the obvious meaning).

THEOREM 2. *Let assumptions* (i), (ii) *of Theorem* 1 *hold. Assume in addition that there exists a positive constant* d *such that* $\det S(A, B) \geqslant d$ *for all* $(A, B) \in P = \{A^0(f, g);\ (f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]\} \times \{B^0(f, g);\ (f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]\}$. *Then system* (S) *is controllable.*

Proof. We only note that the closure $\bar{P}$ of the set P in $L_1^{n\times n} \times L_1^{n\times p}$ satisfies the condition:

$$|||A|||_n \leqslant \int_D \alpha(t)\,dt, \qquad |||B|||_p \leqslant \int_D \beta(t)\,dt$$

for every $(A, B) \in \bar{P}$. The rest of the proof follows as in Theorem 1. It is therefore omitted.

It is, in general, a pretty tedious task to check the condition on $S(A, B)$ in Theorem 2. However, if the matrices $A^0(f, g)$, $B^0(f, g)$ are sufficiently close to the matrices A, B, which are independent of f, g and for which the associated system satisfies the above condition, then system (S) is controllable. This fact is established in the following theorem.

THEOREM 3. *Let assumptions* (i), (ii) *of Theorem* 1 *be satisfied. Let* $(A, B) \in L_1^{n\times n} \times L_1^{n\times p}$ *be such that the system*

$$(\mathrm{S}_1) \qquad x' = A(t)x + B(t)u$$

has $\det S(A, B) > 0$. *Then there exists* $\eta > 0$ *such that whenever* $(A', B') \in L_1^{n\times n} \times L_1^{n\times p}$ *with*

$$\|A - A'\|_n < \eta \quad \text{and} \quad \|B - B'\|_p < \eta$$

then $\det S(A', B') > 0$. *Also, if* $A^0(f, g)$, $B^0(f, g)$ *are such that, for every* $(f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]$,

$$\|A^0(f, g) - A\|_n < \eta \quad \text{and} \quad \|B^0(f, g) - B\|_p < \eta,$$

then system (S) *is controllable.*

Proof. The first assertion follows from the continuity of $S(A, B)$ in (A, B). In order to show the second assertion, recall that given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\|A^0(f, g) - A\|_n < \delta(\varepsilon)$, $\|B^0(f, g) - B\|_p < \delta(\varepsilon)$ imply that $|\det S(f, g) - \det S(A, B)| < \varepsilon$.

If we fix $\varepsilon < \det S(A, B)$ and $\eta = \delta(\varepsilon)$, then we have that

$$0 < \det S(A, B) - \varepsilon < \det S(f, g)$$

for all $(f, g) \in C^n[-r_1, T] \times C^p[-r_2, T]$. The proof now follows from Theorem 2.

4. A necessary and sufficient condition for time-dependent controllability. In what follows, $(A, B) \in L_1^{n\times n} \times L_1^{n\times p}$, We say that the system

$$(2) \qquad x' = A(t)x + B(t)u$$

is *K-controllable* on $[0, T]$ if $S^{-1}(A, B)$ exists. If (2) is K-controllable, then one control function $u(t)$, $t \in [0, T]$, is given by $u(t) = V(t; A, B)y(A, B)$ with $Q \equiv 0$. The following theorem provides a necessary and sufficient condition for K-controllability in case $p = n$, which is simpler than the existence of $S^{-1}(A, B)$.

THEOREM 4. *If $p = n$, then system* (2) *is K-controllable if and only if there exists $t_1 \in (0, T]$ such that*

$$\det\left[\int_{T-t_1}^{T} X^{-1}(s; A)B(s)\,ds\right] \neq 0. \tag{3}$$

Proof. System (2) is not K-controllable if and only if $S \equiv S(A, B)$ is singular. Since S is positive semi-definite, this happens if and only if $\langle Su, u\rangle = 0$ for some nonzero vector $u \in \mathbf{R}^n$ (the symbol $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^n$). This is equivalent to

$$\left\langle \int_0^T U(s; A, B)U^*(s; A, B)\,ds \cdot u, u\right\rangle = 0$$

or

$$\begin{aligned}\int_0^T \langle U(s; A, B)U^*(s; A, B)u, u\rangle\,ds &= \int_0^T \langle U^*(s; A, B)u, U^*(s; A, B)u\rangle\,ds \\ &= \int_0^T \|U^*(s; A, B)u\|_E^2\,ds = 0,\end{aligned}$$

where the subscript E denotes the Euclidean norm on $\mathbf{R}^n$. Since $\|U^*(t; A, B)u\|_E$ is a continuous function on $[0, T]$, the above equality is equivalent to

$$\begin{aligned}\det[U^*(t; A, B)] &\equiv \det\left[X(T; A)\int_{T-t}^{T} X^{-1}(s; A)B(s)\,ds\right]^* \\ &= \det[X(T; A)]^* \det\left[\int_{T-t}^{T} X^{-1}(s; A)B(s)\,ds\right]^* \\ &= \det[X(T; A)]\det\left[\int_{T-t}^{T} X^{-1}(s; A)B(s)\,ds\right] \\ &= 0\end{aligned}$$

for every $t \in [0, T]$. This completes the proof.

It is easy to see that Condition (3) does not necessarily imply that the $n \times n$ matrix $B(t)$ is invertible. In fact, if $n = 2$, $A(t) \equiv 0$ and

$$B(t) \equiv \begin{bmatrix} t^2 & 1/(t^2+1) \\ t^3(t^2+1) & t \end{bmatrix},$$

then Condition (3) becomes, for $t_1 = T$,

$$\det\begin{bmatrix} \frac{1}{3}T^3 & \tan^{-1}(T) \\ \frac{1}{6}T^6+\frac{1}{4}T^4 & \frac{1}{2}T^2 \end{bmatrix} < 0 \quad \text{for every } T>0,$$

although $\det B(t) \equiv 0$ for all $t \geqslant 0$.

5. An example. Consider the system

$$x' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u. \tag{4}$$

Here, we have

$$X(T;\, A)X^{-1}(t;\, A)B = \begin{bmatrix} \cos(T-t) & \sin(T-t) \\ -\sin(T-t) & \cos(T-t) \end{bmatrix}.$$

We let $T = 2\pi$. It is easily seen that $U(2\pi;\, A,\, B) = 0$. Thus (cf. Conti [4], p. 103), the system is not controllable with respect to constant controls u. However, it is K-controllable, according to Theorem 4, because (3) holds for $t_1 = 1$. Of course this fact can be checked by previous results.

We can now apply the result of Theorem 3 to ensure the controllability of

$$x' = \begin{bmatrix} \varepsilon\sin t & 1+\varepsilon\,|\sin(u(t-r_2))| \\ -1 & \varepsilon\cos(x(t-r_1)) \end{bmatrix} x + \begin{bmatrix} 1+[\varepsilon/(1+u^2(t))] & 0 \\ 0 & 1-\varepsilon e^{\cos(x(t-r_1))} \end{bmatrix} u + \begin{bmatrix} u^{1/3}(t-r_2) \\ x^{3/5}(t-r_1) \end{bmatrix}$$

on a fixed interval $[0,\, T]$ for all sufficiently small $\varepsilon \in (0,\, \infty)$, where $r_1,\, r_2$ are fixed positive constants. It is easy to see that in this example we have

$$\liminf_{k\to\infty}(1/k)\int_D q_k(t)\,dt = \lim_{k\to\infty}(1/k)\int_D q_k(t)\,dt = 0.$$

6. Remarks. It is rather important to note that Condition (3) not only ensures the K-controllability of system (2), which in turn guarantees the existence of controls $u \in C[0,\, T]$ with $u(0) = 0$, but also it allows us to pick a control $u(t) \equiv V_T(t;\, A,\, B)\,y(A,\, B)$ $(Q \equiv 0)$. This form of $u(t)$ is quite susceptible to numerical approximation – a fact that is of considerable importance in applications.

Condition (3) complements the various conditions of Conti [4], 90–130, concerning the controllability of linear systems.

It would be interesting to extend these results to the case of Anichini [1], where controls with given boundary conditions are considered.

References

[1] G. Anichini, *Global controllability of nonlinear control processes with prescribed controls*, J. Optim. Th. Appl. 32 (1980), 183–199.

[2] R. J. Becker, *Periodic solutions of semilinear equations of evolution of compact type*, J. Math. Anal. Appl. 82 (1981), 33–48.

[3] R. Conti, *On global controllability*, Intern. Conf. Differential Equations, Edit. H. A. Antosiewicz, Academic Press, New York 1975.

[4] –, *Linear Differential Equations and Control*, Instit. Math., Academic Press, New York 1976.

[5] A. G. Kartsatos, *Global controllability of perturbed quasilinear systems*, Probl. Control Inf. Th. 3 (1974), 137–145.

[6] –, *Stability via Tychonov's theorem*, Int. J. Sys. Sci. 5 (1974), 933–937.

[7] –, *Nonzero solutions to boundary value problems for nonlinear systems*, Pacific J. Math. 53 (1974), 425–433.

[8] –, *Perturbations of m-accretive operators and quasi-linear evolution equations*, J. Math. Soc. Japan 30 (1978), 75–84.

[9] –, M. E. Parrott, *Existence of solutions and Galerkin approximations for nonlinear functional evolution equations*, Tôhoku Math. J. 34 (1982), 509–523.

[10] –, –, *On a class of nonlinear functional pseudoparabolic problems*, Funkc. Ekvacioj 25 (1982), 207–221.

[11] Z. Opial, *Continuous parameter dependence in linear systems of differential equations*, J. Differ. Eqs. 3 (1967), 571–579.

[12] –, *Linear problems for systems of nonlinear differential equations*, ibidem 3 (1967), 580–594.

[13] H. Sager, *Boundary value problems for semilinear evolution equations of compact type*, Doct. Dissert., Univ. Cape Town, Cape Town 1982.

WINTHROP COLLEGE
ROCK HILL, SOUTH CAROLINA 29733, USA
and
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH FLORIDA
TAMPA, FLORIDA 33620, USA

Reçu par la Rédaction le 1985.04.20