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On the extension of holomorphic maps with values in a complex Lie group

by NGUYEN VAN KHUE (Warszawa)

Abstract. The aim of this note is to study the Hartogs and Riemann extension theorem for holomorphic maps on open subsets of locally convex spaces with values in complex Lie groups.

The Hartogs extension theorem for holomorphic maps on a Riemann domain over a Stein manifold with values in a complex Lie group has been proved by Adachi Suzuki and Yoshida [1]. The aim of this note is to study the Hartogs and Riemann extension theorem for holomorphic maps on open subsets of locally convex spaces with values in complex Lie groups.

Let G be a complex Lie group and Ω be an open set in a Fréchet space F. Let K be a closed set in F. The aim of this note is to study the extension of holomorphic functions on $\Omega\backslash K$ with values in G to holomorhic functions on Ω.

For every complex Lie group G and for every open set Ω in a Fréchet space F, let $\mathcal{O}(\Omega, G)$ denote the set of all holomorphic maps from Ω onto G.

THEOREM 1. *Let Ω be a connected neighbourhood of the sphere $S = \{x \in B\colon \|x\| = 1\}$ in a Banach space B such that*

$$\operatorname{dist}(S, \partial\Omega) = \delta > 0 \quad \textit{and} \quad \dim B \geqslant 2.$$

Then the restriction map $\mathcal{O}(\Omega \cup W, G) \to \mathcal{O}(\Omega, G)$ is surjective, where

$$W = \{x \in B\colon \|x\| \leqslant 1\}.$$

Proof. Let $0 < \varepsilon < \min(\delta, 1)$. Put $U = \{x \in B\colon 1-\varepsilon < \|x\| < 1+\varepsilon\}$. Then U is an open neighbourhood of S. Since $\dim B \geqslant 2$, U is connected and simply connected [2]. Let us show that $U \subseteq \Omega$. Given $x \in U$ and let $1 < \|x\| < 1+\varepsilon$. Put

$$\lambda_0 = \sup\{\lambda > 0\colon \xi x \in \Omega \ \ \forall 1/\|x\| \leqslant \xi \leqslant \lambda\}.$$

If $\lambda_0 = \infty$, then $x \in \Omega$ and if $\lambda_0 < \infty$, then $\lambda_0 x \in \partial\Omega$. Since $x/\|x\| \in S$ we have

$$\varepsilon \leqslant \big\|\lambda_0 x - x/\|x\|\,\big\| = \|x\|(\lambda_0 - 1/\|x\|) = \lambda_0\|x\| - 1.$$

Thus $\lambda_0 = 1+\varepsilon/\|x\|$. Hence $x \in \Omega$. Similarly we have $\Omega \supset \{x \in B: 1-\varepsilon < \|x\| < 1\}$.

(a) First we assume that G is commutative. Consider the map exp from the Lie algebra L of G into G. Since G is commutative exp is a covering map from L onto the component G_e of G containing the unit element e. Let $\sigma \in \mathcal{O}(\Omega, G)$. Considering the map σg_0^{-1}, where $g_0 \in \sigma(\Omega)$, because of the connectedness of Ω we can assume that $\sigma(\Omega) \subseteq G_e$. Since U is connected and simply connected, there exists a continuous map $\tilde{\sigma}: U \to L$ such that $\sigma|_U = \exp \tilde{\sigma}$ [9]. Obviously $\tilde{\sigma}$ is holomorphic, because exp is an analytic covering map. By [6], $\tilde{\sigma}$ has a holomorphic extension $\tilde{\beta}: \Omega \cup W \to L$. By the connectedness of Ω and since $\sigma|_U = \exp \tilde{\beta}|_U$ and U is not empty open, it is easy to see that $\sigma = \exp \tilde{\beta}\Omega$.

(b) Now let G be an arbitrary complex Lie group. Then $G = Z \times X$, where Z is a complex commutative Lie group and X is a Stein manifold [5]. Let $\sigma = (\sigma_1, \sigma_2) \in \mathcal{O}(\Omega, G)$. By (a), it suffices to check that σ_2 has a holomorphic extension β_2: $\Omega \cup W \to X$. We can assume that X is a closed submanifold of C^n for some n. By (a), σ_2 has a holomorphic extension β: $\Omega \cup W \to C^n$. Put $V = \{x \in \Omega \cup W: \beta(x) \in X\}$. Then $\operatorname{Int} V \neq \emptyset$. Let $x_0 \in \Omega \cup W \cap \operatorname{Int} V$ and $x_0 \in \partial \operatorname{Int} V$. Take a neighbourhood V_0 of $\beta(x_0)$ in C^n and a connected neighbourhood U_0 of x_0 in $\Omega \cup W$ such that

$$V_0 \cap X = \{z \in V_0: f_i(z) = 0,\ i = 1, \ldots, n\}$$

and $\beta(U_0) \subseteq V_0$, where f_i are holomorphic functions on V_0. Because of the connectedness of U_0 and since $f_i \beta(U_0 \cap \operatorname{Int} V) = 0$ and $U_0 \cap \operatorname{Int} V \neq \emptyset$ we infer that $f_i \beta = 0$ on U_0. Hence $\beta(U_0) \subseteq X$. Thus $\operatorname{Int} V$ is not empty closed-open in $\Omega \cup W$ and therefore $\operatorname{Int} V = \Omega \cup W$, because $\Omega \cup W$ is connected. This statement completes the proof of Theorem 1.

THEOREM 2. *Let T be a finite-dimensional analytic set in a Fréchet space F with* $\operatorname{codim}_x T \geq 2$ *for all $x \in T$. Then the restriction map $\mathcal{O}(\Omega, G) \to \mathcal{O}(\Omega \backslash T, G)$ is surjective.*

The proof of Theorem 2 is based on the following

LEMMA 1. *Let T be a submanifold of a complex manifold X such that* $\operatorname{codim}_x T \geq 2$ *for all $x \in T$. Then for every $x \in T$ there exists a neighbourhood V of x such that $V \backslash V \cap T$ is connected and simply connected.*

Proof. Take a chart (V, θ) at x such that $\theta(V) = V_1 \times V_2$, $\theta(V \cap T) = V_1 \times 0$, V_1 and V_2 are balls in C^n and C^{n-p} respectively, where $p = \dim_x T$ and $n = \dim_x X$. Then $V \backslash V \cap T$ is connected, because $V \cap T$ is a proper analytic subset of the connected open set V. Since $\dim V_2 \geq 2$, $V_2 \backslash 0$ is simply connected. Hence $V \backslash V \cap T$ is connected.

LEMMA 2. *Let π: $Z \to W$ be a covering map and let σ be a map from a convex set X in a topological linear space F into W which is continuous on every* *finite-dimensional subspace of F. Let σ be continuous on an open convex subset V of X. Then there exists a map β: $X \to Z$ such that $\pi\beta = \sigma$, β is continuous on every finite-dimensional subspace of F and $\beta|_V$ is continuous.*

Proof. We can assume that $V \neq \emptyset$. Let $x_0 \in V$ and $z_0 \in \pi^{-1}(\sigma(x_0))$. Since V is connected and simply connected there exists a unique continuous map γ: $V \to Z$ such that $\pi\gamma = \sigma|_V$ and $\gamma(x_0) = z_0$. Consider the family $\mathscr{B}$ of all finite-dimensional subspaces B of F containing x_0. Then for every $B \in \mathscr{B}$ there exists a unique continuous map β_B: $X \cap B \to Z$ such that $\pi\beta_B = \sigma|_{X \cap B}$ and $\beta_B(x_0) = z_0$. Thus the formula $\beta|_{X \cap B} = \beta_B$ defines a map β: $X \to Z$ such that $\pi\beta = \sigma$ and β is continuous on every finite-dimensional subspace of F. Since $\beta|_{V \cap B} = \gamma|_{V \cap B}$ for $B \in \mathscr{B}$, it follows that $\beta|_V$ is continuous.

Proof of Theorem 2. Since $G \cong Z \times X$, where Z is a complex commutative Lie group and X is a Stein manifold, and $\Omega \setminus T$ is connected, as in the proof of Theorem 1 we can assume that G is commutative.

(a) Assume that $\dim F = n$. We proved Theorem 2 by decreasing induction on $d(T) = \min\{\operatorname{codim}_x T: x \in T\}$. The case $d(T) = n+1$ is trivial. Suppose that Theorem 2 has been proved for all analytic sets V with $d(V) > q \geqslant 2$. Let T be an analytic set in Ω with $d(T) \geqslant q \geqslant 2$ and $\sigma \in \mathcal{O}(\Omega \setminus T, G)$. Let $R(T)$ denote regular part of T and $S(T) = T \setminus R(T)$. Then $R(T)$ is a submanifold of an open set W in F and $R(T) = T \cap W$. Lemma 1 implies that for every $x \in R(T)$ there exists a convex neighbourhood W_x of x such that $W_x \setminus W_x \cap R(T)$ is connected and simply connected and $W_x \subseteq \Omega$. From the proof of Theorem 1 it follows that $\sigma|_{W_x \setminus W_x \cap R(T)}$ has a holomorphic extension $\sigma_x \in \mathcal{O}(W_x, G)$. Since

$$W_x \cap W_y \setminus T = (W_x \setminus T) \cap (W_y \setminus T),$$

we have

$$\sigma_x|_{W_x \cap W_y \setminus T} = \sigma|_{W_x \cap W_y \setminus T} = \sigma_y|_{W_x \cap W_y \setminus T}.$$

Hence $\sigma_x|_{W_x \cap W_y} = \sigma_y|_{W_x \cap W_y}$ for all $x, y \in T$. Thus the maps $\{\sigma_x: x \in T\}$ define an element $\sigma_1 \in \mathcal{O}(\Omega \setminus S(T), G)$ such that $\sigma_1|_{\Omega \setminus T} = \sigma$. Since $d(S(T)) > q \geqslant 2$, by the inductive hypothesis σ_1 can be extended to a holomorphic map β from Ω into G.

(b) Let F be an infinite-dimensional Fréchet space and $\sigma \in \mathcal{O}(\Omega \setminus T, G)$. Let $x \in T$ and let W_x be a convex neighbourhood of x contained in Ω. In view of (a) for every finite-dimensional subspace B of F for which $\dim B \geqslant d = \sup\{\dim_x T: x \in T\} + 2$ there exists a holomorphic extension $\sigma_{x,B}$: $W_x \cap B \to G$ of $\sigma|_{W_x \cap B \setminus T}$. Obviously $\sigma_{x,B_1} = \sigma_{x,B_2}|_{W_x \cap B_1}$ for every $B_1 \subseteq B_2$, where $\dim B_1 \geqslant d$. Hence the maps $\{\sigma_{x,B}: B \subseteq F, \dim B \geqslant d\}$ define a map σ_x: $W_x \to G$ such that $\sigma_x|_{W_x \setminus T} = \sigma|_{W_x \setminus T}$ and σ_x is holomorphic on every finite-dimensional subspace of F. By Lemma 2 and by [7] we infer that σ_x is holomorphic. Hence, by the proof of (a) we infer that σ can be extended to a holomorphic map from Ω into G.

THEOREM 3. *Let K be a bounded closed set in a Fréchet space F of dimension*

$\geqslant 2$ *such that* $F\backslash K$ *is connected. Then the restriction map* $\mathcal{O}(F,G)\to\mathcal{O}(F\backslash K,G)$ *is surjective.*

Proof. Since $F\backslash\overline{\operatorname{conv} K}$ is connected we can assume that K is convex. Whence, by Theorem 1, it is easy to see that the restriction map $\mathcal{O}(F,G)\to\mathcal{O}(F\backslash K,G)$ is surjective when F is a Banach space. Hence we can assume that the space F does not have a bounded neighbourhood of zero. In this case Theorem 3 is an immediate consequence of the following.

LEMMA 3. *Theorem* 3 *is true when* $G = C$.

PROOF. Consider the family $\mathfrak{F}$ of all absolutely convex bounded sets in F. For every $B\in\mathfrak{F}$ by $F[B]$ we denote linear subspace of F generated by B. This space becomes a Banach space with respect to the norm generated by B. Theorem 1 implies that for every $B\in\mathfrak{F}$ there exists a unique holomorphic map σ_B: $F[B]\to C$ which is an extension of $\sigma|F[B]\backslash K$, where $\sigma\in\mathcal{O}(F\backslash K,C)$. Obviously $\sigma_{B_1} = \sigma_{B_2}|_{F[B_1]}$ for $B_1,B_2\in\mathfrak{F}$ and $B_1\subseteq B_2$. Thus the maps σ_B: $B\in\mathfrak{F}$ define a map β: $F\to C$ such that $\beta|_{F\backslash K} = \sigma$ and β is holomorphic in every finite-dimensional subspace of F. By [7], β is holomorphic.

LEMMA 4. *Let* K *be a bounded closed convex set in a Fréchet space* F *which does not have a bounded neighbourhood of zero. Then* $F\backslash K$ *is connected and simply connected.*

Proof. Obviously $F\backslash K$ is connected, because K is bounded convex. Let $x_0\in F\backslash K$. Let us show that the fundamental group $\pi_1(F\backslash K,x_0)$ is trivial. Let σ: $I\to F\backslash K$ be a continuous map such that $\sigma(0)=\sigma(1)=x_0$, where $I=[0,1]$. Since F is a Fréchet space which does not have a bounded neighbourhood of zero there exists $x_1\in F\backslash\operatorname{span}(\sigma(I)+K)$. By the connectedness of $F\backslash K$ we can assume that $x_1 = x_0$ ([9]). For every $(s,t)\in I\times I$ put

$$H(s,t) = x_0 - t(x_0-\sigma(s)).$$

Then $H(S,t)\in F\backslash K$, since $x_0\notin\operatorname{span}(\sigma(I)+K)$. Obviously

$$H(s,0) = H(0,t) = x_0 \quad \text{for } s,t\in I$$

and

$$H(s,1) = \sigma(s) \quad \text{for } s\in I.$$

Hence the group $\pi_1(F\backslash K,x_0)$ is trivial.

THEOREM 4. *Let* X *be a connected Stein manifold. Then the restriction map* $\mathcal{O}(Y,G)\to\mathcal{O}(X,G)$ *is surjective for every Stein manifold* Y *containing* X *as a closed submanifold and for every nilpotent Lie group* G *if and only if* $H^1(X,Z)=0$, *where* Z *denotes the group of integral numbers.*

Proof. We can assume that G is connected.

(a) Let G be commutative. Then the sequence

$$0 \to N \to L \underset{\exp}{\to} G \to e$$

is exact. Since exp is an analytic covering map the sequence

$$0 \to \mathcal{O}^N \to \mathcal{O}^L \to \mathcal{O}^G \to e$$

is exact, where by $\mathcal{O}^N$, $\mathcal{O}^L$ and $\mathcal{O}^G$ we denote the sheaves of germs of holomorphic maps on X with values in N, L and G respectively. Since $N = Z^p$ for some p we have $H^1(X, \mathcal{O}^N) = 0$. Hence, since the restriction $\mathcal{O}(Y, L) \to \mathcal{O}(X, L)$ is surjective for every Stein manifold Y containing X as a closed submanifold [4] we infer that the restriction map $\mathcal{O}(Y, G) \to \mathcal{O}(X, G)$ is surjective.

(b) Now suppose G is nilpotent. We shall prove the *if* part by induction on dim G. Let the restriction map $\mathcal{O}(Z, \Gamma) \to \mathcal{O}(X, \Gamma)$ be surjective for every Stein manifold Z containing X as a closed submanifold and for every nilpotent group of dimension $< n$. Consider a nilpotent group G of dimension $\leqslant n$, a Stein manifold Y containing X as a closed submanifold and a holomorphic map σ: $X \to G$. We may assume that $Y = G^m$ for some m. By $Z(G)$ we denote the centre of G. Since G is nilpotent and dim $Z(G) > 0$, $G/Z(G)$ is also a nilpotent group of dimension $< n$. Hence, by the inductive hypothesis there exists a holomorphic map $\tilde{\sigma}: Y \to G/Z(G)$ such that $\tilde{\sigma}|_X = \eta\sigma$, where η: $G \to G/Z(G)$ denotes the canonical map and $\sigma \in \mathcal{O}(X, G)$. Consider the commutative diagram:

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\pi_2} & G \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \eta} \\ Y & \xrightarrow[\tilde{\sigma}]{} & G/Z(G) \end{array}$$

where $\tilde{G} = \{(y, g) \in Y \times G\colon \sigma(y) = \eta g\}$, π_1 and π_2 are canonical projections on Y and G respectively. Since Y is contractible there exists a holomorphic section γ of the main bundle (G, π_1, Y) [3]. Obviously $\sigma(\pi_2\gamma)^{-1} \in Z(G)$. Hence by (a), $\sigma(\pi_2\gamma)^{-1}$ has a holomorphic extension α: $Y \to Z(G)$. Setting $\tilde{\alpha} = \alpha(\pi_2\gamma)$ we get a holomorphic extension of σ.

The *only if* part. We can assume that X is closed submanifold of C^n for some n. Consider the Lie group $G = C/Z + iZ$. Since the canonical map η: $C \to G$ is an analytic covering map and X is a Stein manifold, the sequence

$$\mathcal{O}(X, C) \to \mathcal{O}(X, G) \to H^1(X, \mathcal{O}^{Z+iZ}) \to 0$$

is exact. Whence, since the maps $\mathcal{O}(C^n, G) \to \mathcal{O}(X, G)$ and $\mathcal{O}(C^n, C) \to \mathcal{O}(C^n, G)$ are surjective we infer that the map $\tilde{\eta}$ is surjective. Hence

$$H^1(X, Z) + H^1(X, Z) = H^1(X, \mathcal{O}^{Z+iZ}) = 0.$$

Consequently $H^1(X, Z) = 0$.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES, WARSZAWA

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