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On certain generalized close-to-star functions in the unit disc

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Abstract. A new class W_k^* of functions $f(z) = z + a_2 z^2 + \ldots$ analytic in the unit disc D is introduced in this paper. This generalizes the class of close-to-star functions of M. O. Reade in the same way as the class of functions of bounded boundary rotation generalizes the class of convex functions. The radius of starlikeness and radius of convexity of W_k^* are determined. A subclass of W_k^* is also considered and similar problems are solved.

1. Introduction. Let M_k denote the class of real valued functions $m(t)$ of bounded variation on $[-\lambda, \lambda]$ which satisfy the conditions,

$$\int_{-\pi}^{\pi} dm(t) = 2, \qquad \int_{-\pi}^{\pi} |dm(t)| \leqslant k. \tag{1.1}$$

A function $f(z)$ is said to be in the *class* U_k if $f(z)$ is analytic in $D = \{z\colon |z| < 1\}$ and

$$f(z) = z\exp \int_{-\pi}^{\pi} -\log(1-ze^{-it})\,dm(t) \tag{1.2}$$

for some $m(t) \in M_k$. For $k = 2$, U_k reduces to the familiar class of univalent starlike functions. Let P_k denote the class of functions which are analytic in D and have the representation

$$p(z) = \tfrac{1}{2} \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\,dm(t), \qquad \text{where } m(t) \in M_k. \tag{1.3}$$

In this paper a new class of analytic functions W_k^* is defined as follows.

DEFINITION. A function $f(z)$ analytic in D belongs to W_k^* if and only if there exists a $g(z)$ belonging to U_k such that

$$\frac{f(z)}{g(z)} = p(z), \qquad \text{where } p(z) \in P_k. \tag{1.4}$$

A function $f(z) \in W_k$ shall be called a *generalized close-to-star function.*

This is justified by the observation that the class W_k^* reduces to the class of close-to-star functions introduced by M. O. Reade in Michigan Mathematical Journal for $k = 2$. The principal aim of this paper is to determine the radius of convexity for this class for all $k \geqslant 2$ and precisely determine the radius of starlikeness (which is also the radius of univalence) for this class for all $k \geqslant 4$. When $2 \leqslant k < 4$ an estimate is obtained for the radius of starlikeness is obtained which is not sharp.

Further a subclass of W_k^* consisting of functions $f(z)$ which satisfy

$$\frac{f(z)}{z} \in P_k \tag{1.5}$$

is also considered, the radius of convexity is determined for all $k \geqslant 2$ and the radius of starlikeness (same as radius of univalence) is also determined for all $k \geqslant 4$. When $2 \leqslant k < 4$ an estimate is obtained for the radius of starlikeness which is not sharp. The problem of finding coefficient estimates for the class W_k^* is open.

2. Lemmas. We need the following lemmas for our discussion.

LEMMA 1. *Suppose $p(z) \in P_k$; then*

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant \frac{-r(k-4r+kr^2)}{(1-r^2)(1-kr+r^2)}, \tag{2.1}$$

$k \geqslant 4$ *and* $|z| = r < R_0 = \dfrac{k-\sqrt{k^2-4}}{2}$. *The above inequality is sharp.*

For $2 \leqslant k \leqslant 4$ *and* $|z| = r < R_0 = \dfrac{k-\sqrt{k^2-4}}{2}$,

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant \frac{-2kr+(8-4k+k^2)r^2-2kr^3}{2(1-r^2)(1-kr+r^2)}. \tag{2.2}$$

However, inequality (2.2) *is not sharp.*

Proof. Suppose $p(z) \in P_k$. Then there exists a function $f(z) \in V_k$ [4] such that $p(z) = 1+zf''(z)/f'(z)$ which gives,

$$\left\{\frac{f''(z)}{f'(z)}\right\}' = \frac{zp'(z)-(p(z)-1)}{z^2}; \qquad \left\{\frac{f''(z)}{f'(z)}\right\}^2 = \frac{(p(z)-1)^2}{z^2}. \tag{2.3}$$

Also it is known [2] when $f(z) \in V_k$

$$|\{f,z\}| = \left|\left\{\frac{f''(z)}{f'(z)}\right\}' - \frac{1}{2}\left\{\frac{f''(z)}{f'(z)}\right\}^2\right| \leqslant \begin{cases} \dfrac{k^2-4}{2(1-|z|^2)^2}, & k \geqslant 4, \\[2ex] \dfrac{2(k-1)}{(1-|z|^2)^2}, & 2 \leqslant k \leqslant 4. \end{cases} \tag{2.4}$$

First let us consider the case $k \geqslant 4$. From (2.3) and (2.4) we get

$$\left|\frac{zp'(z)-(p(z)-1)}{z^2}-\frac{(p(z)-1)^2}{2z^2}\right| \leqslant \frac{k^2-4}{2(1-|z|^2)^2},$$

whence we get

$$|zp'(z)+\tfrac{1}{2}(1-p^2(z))| \leqslant \frac{(k^2-4)|z|^2}{2(1-|z|^2)^2}. \tag{2.5}$$

Since $\operatorname{Re} p(z) > 0$ for $|z| < R_0 = \frac{1}{2}(k-\sqrt{k^2-4})$, [4], $p(z) \neq 0$ for $|z| < R_0$. Therefore, for $|z| < R_0$, we have from (2.5)

$$\left|z\frac{p'(z)}{p(z)}+\tfrac{1}{2}\left(\frac{1}{p(z)}-p(z)\right)\right| \leqslant \frac{(k^2-4)|z|^2}{2(1-|z|^2)^2|p(z)|}$$

and so

$$\operatorname{Re}\left\{z\frac{p'(z)}{p(z)}+\tfrac{1}{2}\left(\frac{1}{p(z)}-p(z)\right)\right\} \geqslant \frac{-(k^2-4)|z|^2}{2(1-|z|^2)^2|p(z)|};$$

that is,

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant \tfrac{1}{2}\operatorname{Re} p(z)-\frac{1}{2\operatorname{Re} p(z)}-\frac{(k^2-4)|z|^2}{2(1-|z|^2)^2|p(z)|}, \tag{2.6}$$

where we have used the fact that $\operatorname{Re} 1/p(z) \leqslant 1/\operatorname{Re} p(z)$ when $\operatorname{Re} p(z) > 0$. Also since $p(z) \in P_k$ we have [4]

$$\frac{1-kr+r^2}{1-r^2} \leqslant \operatorname{Re} p(z) \leqslant \frac{1+kr+r^2}{1-r^2}, \quad \text{where } |z| = r. \tag{2.7}$$

Inequalities (2.6) and (2.7) lead to

$$\operatorname{Re}\left\{\frac{zp'(z)}{p(z)}\right\} \geqslant \frac{1-kr+r^2}{1-r^2}-\frac{1-r^2}{2(1-kr+r^2)}-\frac{(k^2-4)r^2}{2(1-r^2)(1-kr+r^2)}$$

that is,

$$\operatorname{Re}\left\{z\frac{p'(z)}{p(z)}\right\} \geqslant \frac{-r(k-4r+kr^2)}{(1-r^2)(1-kr+r^2)}, \quad \text{where } r < R_0 = \frac{k-\sqrt{k^2-4}}{2}.$$

This is sharp for the function

$$p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{1-z}{1+z}\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{1+z}{1-z}\right).$$

For $2 \leqslant k \leqslant 4$, using the other estimate in (2.4) and arguing similarly we get inequality (2.2). For $2 < k < 4$ inequality (2.2) is not sharp because the estimate in (2.4) for $2 < k < 4$ is not sharp. For $k = 2$, though the estimates in (2.4) and (2.7) are sharp, individually, equality is not attained simultaneously in both of them. Hence the lemma fails to be sharp in the case $2 \leqslant k < 4$. The proof of the lemma is complete.

LEMMA 2. [7]. *Let* $h(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ *be regular and single valued on* $|z| = 1$ *and let* $\operatorname{Re} h(z) > 0$ *for* $|z| = 1$. *Then* $|c_n + c_{-n}| \leqslant 2\operatorname{Re} c_0$, $n = 1, 2 \ldots$

LEMMA 3. *Suppose* $f(z) \in W_k^*$. *Then we have*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{R_0^2 - 4R_0 r + r^2}{R_0^2 - r^2}, \quad \text{where } R_0 = \frac{k - \sqrt{k^2-4}}{2}.$$

Proof. Since $f(z) \in W_k^*$, there exists a $g(z) \in U_k$ such that $f(z)/g(z) = p(z) \in P_k$. Also $\operatorname{Re} p(z) > 0$ for $|z| < R_0 = \frac{1}{2}(k - \sqrt{k^2-4})$, [4] Let a be any complex number such that $|a| < R_0$. Then

$$P(z) = p\left(\frac{R_0^2(z+a)}{R_0^2 + \bar{a}z}\right) = p(a) + p'(a)\left(1 - \frac{|a|^2}{R_0^2}\right) z + \ldots$$

is regular in $|z| < R_0$ and $\operatorname{Re} P(z) \geqslant 0$ in $|z| < R_0$. Hence, [3], p. 170, we have

$$\left|p'(a)\left(1 - \frac{|a|^2}{R_0^2}\right)\right| \leqslant \frac{2|p(a)|}{R_0}$$

which implies

$$\left|\frac{p'(a)}{p(a)|}\right| \leqslant \frac{2R_0}{R_0^2 - |a|^2}.$$

Since a is any complex number such that $|a| < R_0$, we can rewrite this as

$$\left|\frac{zp'(z)}{p(z)}\right| \leqslant \frac{2rR_0}{R_0^2 - r^2}, \quad \text{where } |z| = r < R_0.$$

Hence we get

$$\left|\frac{zf'(z)}{f(z)} - z\frac{g'(z)}{g(z)}\right| \leqslant \frac{2rR_0}{R_0^2 - r^2},$$

whence we have,

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geqslant \operatorname{Re}\left\{z\frac{g'(z)}{g(z)}\right\} - \frac{2rR_0}{R_0^2 - r^2}.$$

Since

$$\operatorname{Re}\left\{z\frac{g'(z)}{g(z)}\right\} \geqslant 0 \quad \text{and} \quad g(0) = 1 \quad \text{for } |z| \leqslant R_0, \text{ [4]}.$$

we have by [3], p. 173,

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \geqslant \frac{R_0 - r}{R_0 + r}.$$

Hence we have

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geqslant \frac{R_0^2-4rR_0+r^2}{R_0^2-r^2}$$

completing the proof of the lemma.

LEMMA 4. *If $f(z)$ belongs to the subclass of W_k^* satisfying* (1.5), *then*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{R_0^2-2rR_0-r^2}{R_0^2-r^2}, \quad \textit{where } |z| = r < R_0 = \frac{k-\sqrt{k^2-4}}{2}.$$

The proof of this lemma is similar to that of Lemma 3 and is hence omitted.

3. Theorems.

THEOREM 1. *Let $f(z) \in W_k^*$. Then $f(z)$ is starlike univalent in $|z| < r_0$, where r_0 is the least positive root of the equation*

(3.1) $$1-3kr+(6+k^2)r^2-3kr^3+r^4 = 0 \quad \textit{for } k \geqslant 4$$

and this bound r_0 is sharp. For $2 \leqslant k \leqslant 4$, $f(z)$ is starlike univalent in $|z| < r_1$, where r_1 is the least positive root of the equation

(3.2) $$2-6kr+(12-4k+3k^2)r^2-6kr^3+2r^4 = 0.$$

However, this bound r_1 is not sharp.

Proof. Since $f(z) \in W_k^*$, $\exists g(z) \in U_k$ such that

$$\frac{f(z)}{g(z)} = p(z) \in P_k.$$

Therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\frac{zg'(z)}{g(z)} + \operatorname{Re}\frac{zp'(z)}{p(z)}.$$

For $k \geqslant 4$ by Lemma 1 and [4] we get

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \frac{1-kr+r^2}{1-r^2} - \frac{r(k-4r+r^2)}{(1-r^2)(1-kr+r^2)},$$

where $|z| = r < R_0 = \dfrac{k-\sqrt{k^2-4}}{2}$.

Hence

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geqslant \frac{(1-kr+r^2)^2-r(k-4r+kr^2)}{(1-r^2)(1-kr+r^2)} > 0,$$

provided $T(r) \equiv 1-3kr+(6+k^2)r^2-3kr^3+r^4 > 0$. $T(0) > 0$ and $T(R_0) < 0$ and hence $\operatorname{Re}\left\{z\dfrac{f'(z)}{f(z)}\right\} > 0$ provided $r < r_0$, where r_0 is the least positive root of the equation $T(r) = 0$, lying in $(0, R_0)$. For

$$g(z) = \frac{z(1-z)^{k/2-1}}{(1+z)^{k/2+1}} \quad \text{and} \quad p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{1-z}{1+z}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{1+z}{1-z}\right)$$

and choosing $f(z) = g(z)p(z)$ we have

$$\frac{zf'(z)}{f(z)} = \frac{1-3kz+(6+k^2)z^2-3kz^3+z^4}{(1-z^2)(1-kz+z^2)} = 0$$

for $z = r_0$. Hence this result is sharp.

If $2 \leqslant k \leqslant 4$ and $|z| < R_0 = \left(k-\sqrt{k^2-4}\right)/2$, then by [4] and Lemma 1 we have

$$\begin{aligned}\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} &\geqslant \frac{1-kr+r^2}{1-r^2} + \frac{-2kr+(8-4k+k^2)r^2-2kr^3}{2(1-r^2)(1-kr+r^2)} \\ &= \frac{2-6kr+(12-4k+3k^2)r^2-6kr^3+2r^4}{2(1-r^2)(1-kr+r^2)} > 0\end{aligned}$$

for $|z| = r < r_1$, where r_1 is the least positive root of the equation,

$$2-6kr+(12-4k+3k^2)r^2-6kr^3+2r^4 = 0.$$

The bound r_1 is not sharp since inequality (2.2) in Lemma 1 is not sharp. For $k = 2$, r_1 turns out to be the smallest positive root of the equation $\Psi(r) \equiv 1-6r+8r^2-6r^3+r^4 = 0$.

$\Psi(r)$ decreases with r in $(0, \frac{1}{2})$ and vanishes at $r = r_1$, where $0 < r_1 < \frac{1}{2}$. And we find by actual computation $\Psi(2-\sqrt{3}) < 0$. Also $(2-\sqrt{3}) < \frac{1}{2}$. This implies $r < 2-\sqrt{3}$ which is the sharp estimate for the radius of starlikeness of the class of close-to-star functions W_2^*, [1].

THEOREM 2. *Suppose $f(z) \in W_k^*$. Then $f(z)$ is convex in $|z| < (5-2\sqrt{6})R_0$, where $R_0 = \left(k-\sqrt{k^2-4}\right)/2$ for all $k \geqslant 2$.*

Proof. Since $p(z) \in P_k$, $\operatorname{Re} p(z) > 0$ for $|z| < R_0$, [4]. Consider, for any complex number a such that $|a| < R_0$ functions f_1, g_1 and p_1 defined as follows

$$f_1(z) = f\left(\frac{R_0^2(z+a)}{R_0^2+\bar{a}z}\right), \quad g_1(z) = g\left(\frac{R_0^2(z+a)}{R_0^2+\bar{a}z}\right)$$

and

$$p_1(z) = p\left(\frac{R_0^2(z+a)}{R_0^2+\bar{a}z}\right).$$

Then

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} = \operatorname{Re}\left\{\frac{zg'\left(\frac{R_0^2(z+a)}{R_0^2+\bar{a}z}\right)}{g\left(\frac{R_0^2(z+a)}{R_0^2+\bar{a}z}\right)}\,\frac{R_0^2(R_0^2-|a|^2)}{(R_0^2+\bar{a}z)^2}\right\}.$$

Putting $\zeta = R_0^2(z+a)/(R_0^2+\bar{a}z)$, we have

$$\operatorname{Re}\left\{\frac{zg_1'(z)}{g_1(z)}\right\} = (R_0^2-|a|^2)\operatorname{Re}\left\{\frac{\zeta g'(\zeta)}{g(\zeta)}\frac{z}{(z+a)(R_0^2+\bar{a}z)}\right\}$$

$$= \frac{R_0^2-|a|^2}{R_0^2+2R_0\operatorname{Re}(ae^{-i\theta})+|a|^2}\operatorname{Re}\left\{\zeta\frac{g'(\zeta)}{g(\zeta)}\right\} \quad \text{on } |z| = R_0$$

$$> 0.$$

Now define

$$g_2(z) = zg_1(z)/(a+z)(R_0^2+\bar{a}z).$$

Hence

$$\frac{zg_2'(z)}{g_2(z)} = \frac{zg_1'(z)}{g_1(z)} + \frac{(aR_0^2-\bar{a}z^2)}{(R_0^2+\bar{a}z)(a+z)}$$

and

$$\operatorname{Re}\left\{\frac{zg_2'(z)}{g_2(z)}\right\} = \operatorname{Re}\left\{\frac{zg_1'(z)}{g_1(z)}\right\} \geqslant 0 \quad \text{for } |z| \leqslant R_0.$$

Since $f(z) \in W_k^*$, $\exists a g(z) \in U_k$ such that $f(z)/g(z) = p(z)$, where $p(z) \in P_k$.

Let $f_1(z) = a(1+a_1z+\ldots)$ and $g_2(z) = b(z+b_2z^2+\ldots)$. We have

$$\frac{(R_0^2+\bar{a}z)(a+z)}{z}p_1(z) = \frac{f_1(z)}{g_2(z)} = \frac{a}{b}\left\{\frac{1}{z}+(a_1-b_2)+(b_2^2-b_3-b_2a_1+a_2)z+\ldots\right\}.$$

Also $\operatorname{Re}p_1(z) > 0$ for $|z| < R_0$. Since $(R_0^2+\bar{a}z)(a+z)/z$ is real and positive for $|z| = R_0$, we have $\operatorname{Re}\{f_1(z)/g_2(z)\} > 0$ for $|z| \leqslant R_0$. Putting $\xi = z/R_0$,

$$\text{(3.3)} \qquad \frac{f_1(\xi)}{g_2(\xi)} = \frac{a}{b}\left\{\frac{1}{R_0\xi}+(a_1-b_2)+(b_2^2-b_3-b_2a_1+a_2)R_0\xi+\ldots\right\}$$

and

$$\operatorname{Re}\left\{\frac{f_1(\xi)}{g_2(\xi)}\right\} > 0 \quad \text{for } |\xi| < 1.$$

Also $g_2(R_0\xi)/\xi = b\{\xi+b_2R_0\xi^2+\ldots\}$ is starlike in $|\xi| < 1$. Hence

$$\text{(3.4)} \qquad |b_2R_0| \leqslant 2$$

and by Carathéodory–Toeplitz theorem

$$\text{(3.5)} \qquad |b_3R_0^2-b_2^2R_0^2| \leqslant 1.$$

Applying Lemma 2 to (3.3) we have

$$|R_0(a_2-b_2a_1-b_3+b_2^2)+1/R_0| \leqslant 2|a_1-b_2|.$$

From (3.4) and (3.5) we get

$$R_0^2|a_2| \leqslant 4R_0|a_1|+6. \tag{3.6}$$

We have

$$a_1 = \left(\frac{R_0^2-|a|^2}{R_0^2}\right)\frac{f'(a)}{f(a)},$$

$$a_2 = \left\{\left(\frac{R_0^2-|a|^2}{R_0^2}\right)^2 f''(a)-2\bar{a}\left(\frac{R_0^2-|a|^2}{R_0^2}\right)f'(a)\right\}/(2f(a)).$$

Therefore from (3.6) we get

$$\left|\frac{\left(\frac{R_0^2-|a|^2}{R_0^2}\right)^2 f''(a)-2\bar{a}\left(\frac{R_0^2-|a|^2}{R_0^2}\right)f'(a)}{2f(a)}\right| \leqslant \frac{4}{R_0}\frac{f'(a)}{f(a)}\left(\frac{R_0^2-|a|^2}{R_0^2}\right)+\frac{6}{R_0^2}.$$

Since a is any complex number such that $|a| < R_0$, we can replace a by z and rewrite the above inequality as follows:

$$\left|z\frac{f''(z)}{f'(z)}-\frac{2r^2}{R_0^2-r^2}\right| \leqslant \frac{2R_0^4}{(R_0^2-r^2)^2}\left\{\frac{4r(R_0^2-r^2)}{R_0^3}+\frac{6r^2}{R_0^2}\left|\frac{f(z)}{zf'(z)}\right|\right\},$$

where $|z| = r < R_0$. Using Lemma 3 we now have, for $|z| = r < R_0$,

$$\left|z\frac{f''(z)}{f'(z)}-\frac{2r^2}{R_0^2-r^2}\right| \leqslant \frac{2R_0^4}{(R_0^2-r^2)^2}\left\{\frac{4r(R_0^2-r^2)}{R_0^3}+\frac{6r^2}{R_0^2}\frac{R_0^2-r^2}{R_0^2-4R_0r+r^2}\right\}$$

$$= \frac{2R_0}{(R_0^2-r^2)}\left\{\frac{4R_0^2r-10R_0r^2+4r^3}{R_0^2-4R_0r+r^2}\right\}, \quad \text{if } R_0^2-4R_0r+r^2 > 0.$$

Therefore

$$\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geqslant \frac{R_0^2+r^2}{R_0^2-r^2}-\frac{2R_0}{R_0^2-r^2}\cdot\frac{4R_0^2r-10R_0r^2+4r^3}{R_0^2-4R_0r+r^2}$$

$$= \frac{R_0^4-12R_0^3r+22R_0^2r^2-12R_0r^3+r^4}{(R_0^2-r^2)(R_0^2-4R_0r+r^2)} > 0$$

provided $R_0^2-10R_0r+r^2 > 0$, i.e. $r < (5-2\sqrt{6})R_0$.

When $k = 2$, $R_0 = 1$ we obtain as a special case the result of Sakaguchi which is sharp. Therefore the constant $(5-2\sqrt{6})$ cannot be improved.

THEOREM 3. *Suppose $f(z)$ is analytic in D and $f(z)/z \in P_k$. Then $f(z)$ is starlike univalent in $|z| < r_0$, where r_0 is the least positive root of the equation $1-2kr+4r^2-r^4 = 0$, for $k \geqslant 4$.*

The bound r_0 is sharp. For $2 \leqslant k \leqslant 4$, $f(z)$ is starlike univalent in $|z| < r_1$, where r_1 is the least positive root of the equation

$$2-4kr+(8-4k+k^2)r^2-2r^4 = 0.$$

However, the bound r_1 is not sharp.

Proof. Since $f(z)/z = p(z) \in P_k$, we have

$$\mathrm{Re}\left\{z\frac{f'(z)}{f(z)}\right\} = 1+\mathrm{Re}\left\{\frac{zp'(z)}{p(z)}\right\}.$$

For $k \geqslant 4$, applying Lemma 1 we have, for $|z| = r < R_0 = (k-\sqrt{k^2-4})/2$,

$$\mathrm{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geqslant 1-\frac{r(k-4r+kr^2)}{(1-r^2)(1-kr+r^2)} = \frac{1-2kr+4r^2-r^4}{(1-r^2)(1-kr+r^2)} > 0$$

provided $T(r) \equiv 1-2kr+4r^2-r^4 > 0$. $T(0) > 0$ and $T(R_0) < 0$. Let r_0 be the least positive root of $T(r) = 0$ lying in $(0, R_0)$. Then $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ provided $r < r_0$. Consider the function

$$f(z) = z\left\{\left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{1-z}{1+z}\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{1+z}{1-z}\right)\right\}.$$

Then $zf'(z)/f(z) = 0$ for $z = r_0$. Hence the result is sharp for $k \geqslant 4$. For $2 \leqslant k \leqslant 4$, by applying Lemma 1 we get

$$\mathrm{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geqslant 1+\frac{-2kr+(8-4k+k^2)r^2-2kr^3}{2(1-r^2)(1-kr+r^2)}$$
$$= \frac{2-4kr+(8-4k+k^2)r^2-2r^4}{2(1-r^2)(1-kr+r^2)} > 0$$

provided $T_1(r) = 2-4kr+(8-4k+k^2)r^2-2r^4 > 0$ and $r < R_0$. $T_1(0) > 0$, $T_1(R_0) < 0$ and hence $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ for $|z| < r_1$, where r_1 is the smallest positive root of the equation $T_1(r) = 0$. The bound r_1 is not sharp since inequality (2.2) in Lemma 1 is not sharp. For $k = 2$, r_1 turns out to be the smallest positive root of the equation $\Psi_1(r) \equiv 1-4r+2r^2-r^4 = 0$.

$\Psi_1(r)$ decreases with r in $(0, 1)$ and vanishes at $r = r_1$, where $0 < r_1 < \frac{1}{2}$. And we find by actual computation $\Psi_1(\sqrt{2}-1) < 0$ and $(\sqrt{2}-1) < \frac{1}{2}$. This implies $r_1 < (\sqrt{2}-1)$, which is known to be the sharp estimate for the radius of starlikeness of the class of functions $f(z)$ such that $f(z)/z \in P_2$ [1].

THEOREM 4. *Let $f(z)$ be as in Theorem 3. Then $f(z)$ is convex for $|z| < r_0 = (0.179\ldots)R_0$, where r_0 is the least positive root of*

$$R_0^3-5R_0^2r-3R_0r^2-r^3 = 0$$

and $R_0 = (k-\sqrt{k^2-4})/2$.

Proof. Since $f(z)/z \in P_k$, we have $\operatorname{Re} f(z)/z > 0$ for $|z| < R_0$, [4]. Now consider the function f_1 for any complex number a satisfying $|a| < R_0$

$$f_1(z) = \frac{f(\zeta)}{\zeta}\left\{\frac{R_0^2(z+a)(R_0^2+\bar{a}z)}{z}\right\},$$

where $\zeta = R_0^2(z+a)/(R_0^2+\bar{a}z)$. Let $G(z) = zg_1(z) = f(\zeta)(R_0^2+\bar{a}z)^2$. Hence

$$f_1(z) = \frac{G(z)}{z} = \frac{a_{-1}}{z}+a_0+a_1z+\ldots$$

Since $f(z)/z \in P_k$, we have $\operatorname{Re}\{f(\zeta)/\zeta\} > 0$ for $|\zeta| < R_0$ and since $(z+a)(R_0^2+\bar{a}z)/z$ is positive real for $|z| = R_0$, it follows that $\operatorname{Re} f_1(z) > 0$ for $|z| \leqslant R_0$. Putting $\xi R_0 = z$, for $|\xi| \leqslant 1$, we have $\operatorname{Re} f_1(R_0\xi) > 0$. Also $f_1(R_0\xi) = a_{-1}/R_0\xi + a_0 + a_1R_0\xi + \ldots$ By Lemma 2 we have $|a_1+\bar{a}_{-1}| \leqslant 2\operatorname{Re} a_0$. Also

$$a_{-1} = R_0^3 f(a), \qquad a_0 = R_0^2[(R_0^2-|a|^2)f'(a)+2\bar{a}f(a)]$$

and

$$a_1 = R_0\left\{\frac{(R_0^2-|a|^2)^2}{2}f''(a)+\bar{a}(R_0^2-|a|^2)f'(a)+\bar{a}^2f(a)\right\}.$$

Hence we have

$$\left|\frac{f''(a)}{f'(a)}+\frac{2\bar{a}}{R_0^2-|a|^2}\right| \leqslant \frac{4R_0}{R_0^2-|a|^2}+\frac{8|a|R_0}{(R_0^2-|a|^2)^2}\left|\frac{f(a)}{f'(a)}\right|+ \\ +\frac{2|a|^2}{(R_0^2-|a|^2)^2}\left|\frac{f(a)}{f'(a)}\right|+\frac{2R_0^2}{(R_0^2-|a|^2)^2}\left|\frac{f(a)}{f'(a)}\right|.$$

Since a is any complex number such that $|a| < R_0$, we can replace a by z and rewrite the above and using Lemma 4 we obtain for $|z| < R_0$

$$\left|z\frac{f''(z)}{f'(z)}+\frac{2r^2}{R_0^2-r^2}\right| \leqslant \frac{4R_0r}{R_0^2-r^2}+\frac{2(R_0^2r^2+4R_0r^3+r^4)}{(R_0^2-r^2)(R_0^2-2R_0r-r^2)},$$

where $|z| = r < (\sqrt{2}-1)R_0$. Hence

$$\operatorname{Re}\left\{1+z\frac{f''(z)}{f'(z)}\right\} \geqslant \frac{R_0^2+r^2}{R_0^2-r^2}-\frac{4R_0^3r-6R_0^2r^2+4R_0r^3+2r^4}{(R_0^2-r^2)(R_0^2-2R_0r-r^2)} > 0$$

provided $T(r) = R_0^3-5R_0^2r-3R_0r^2-r^3 > 0$ and $r < (\sqrt{2}-1)R_0$. Consider the equation $1-5r/R_0-3r^2/R_0^2-r^3/R_0^3 = 0$.

Let r_0 be the smallest positive root of this equation. Then $r_0 = (0.179\ldots)R_0$. Hence for $|z| < r_0$, $f(z)$ is convex. For $k = 2$ this reduces to a result of M. O. Reade and others [6] which is sharp. Hence the constant $(0.179\ldots)$ cannot be improved.

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