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QUOTIENT STRUCTURES FOR QUASI-UNIFORM SPACES

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1. Introduction. In this paper we are interested in the quotient quasi-uniform structure. It is well-known that the quotient topology generated by a surjective map of a uniform space need not be uniformizable. Isbell, in [1], provides such examples. Considering the problem at the quasi-uniform level, we show that the difficulty is even more fundamental. Since every topological space admits a compatible quasi-uniform structure, the generated quotient topology must also admit such a structure. However, we show that it may not be a structure for which the map is quasi-uniformly continuous. The problem is then approached from the other direction in that we consider a natural definition of a quotient structure in the category of quasi-uniform spaces and quasi-uniformly continuous maps. We show that such a structure does exist.

It is known that the direct image of a quasi-uniform space need not be a quasi-uniform structure. We give necessary and sufficient conditions for which it is and show that in this case it is the quotient structure. q-maps are defined in a natural way and used to characterize the direct image structure. Conditions are given for which the direct image structure yields the direct image or quotient topology.

Definition 1.1. Let X be a non-empty set. A *quasi-uniform structure* for X is a filter $\mathscr{U}$ of subsets of $X\times X$ such that:

(1) $\Delta = \{(x, x): x\in X\} \subset U$ for each $U\in \mathscr{U}$,

(2) for each U in $\mathscr{U}$ there exists a V in $\mathscr{U}$ with $V\circ V \subset U$.

Definition 1.2. If $\mathscr{U}$ is a quasi-uniform structure for a set X, let

$$t_{\mathscr{U}} = \{O \subset X: \text{ if } x\in O, \text{ then there exists } U \text{ in } \mathscr{U} \text{ such that } U[x] \subset O\}.$$

Then $t_{\mathscr{U}}$ is the quasi-uniform topology on X generated by $\mathscr{U}$. A quasi-uniform structure $\mathscr{U}$ is said to be *compatible with a topology* provided $t = t_{\mathscr{U}}$.

An excellent introduction to quasi-uniform spaces may be found in [3].

2. Quotient structures. Let $(X, \mathscr{U})$ be a quasi-uniform space and $f\colon X \to Y$ with f surjective. It is natural to ask if there exists a quasi-uniform structure $\mathscr{V}$ for Y such that:

(a) $f\colon (X, \mathscr{U}) \to (Y, \mathscr{V})$ is quasi-uniformly continuous,

(b) $t_{\mathscr{V}}$ is the quotient topology on Y.

The following example shows that it is not always possible to find such a structure $\mathscr{V}$ for Y.

Example 2.1. Let $f\colon N \to Y$, where N denotes the natural numbers and $Y = \{1, 2\}$. Define f by $f(n) = 1$ if n is odd and $f(n) = 2$ if n is even. Let $\mathscr{U}$ be the quasi-uniform structure on N generated by the base $\mathscr{B} = \{U_n\colon n \in N\}$, where

$$U_n = \{(x, y)\colon x = y \text{ or } x > n\}.$$

$\mathscr{U}$ generates the discrete topology on N and the quotient topology on Y is thus the discrete topology. Let $\mathscr{V}$ be a quasi-uniform structure on Y for which f is quasi-uniformly continuous. Then if $V \in \mathscr{V}$, $f^{-1}(V) \in \mathscr{U}$ and there exists a U_n such that $U_n \subset f^{-1}(V)$. (By $f^{-1}(V)$ we mean, where $V \subset Y \times Y$, the set $\{(a, b) \in X \times X\colon (f(a), (fb)) \in V\}$.) Then $f(U_n) \subset f(f^{-1}(V)) = V$. But $f(U_n) = Y \times Y$, therefore $\mathscr{V} = \{Y \times Y\}$ and $t_{\mathscr{V}}$ is the trivial topology on Y.

Let $\mathscr{Q}$ be the category of quasi-uniform spaces and quasi-uniformly continuous maps. We say that $(Y, \mathscr{W})$ is a *quotient* of $(X, \mathscr{U})$ if there exists an onto map $f\colon X \to Y$ with the property that if $f = gh$, where g is one-to-one and onto, then g is an isomorphism.

Definition 2.1. Let $f\colon (X, \mathscr{U}) \to Y$. Set $\mathscr{W} = \bigvee\{\mathscr{V}\colon f\colon (X, \mathscr{U}) \to (Y, \mathscr{V})$ is quasi-uniformly continuous$\}$. $\mathscr{W}$ is called the *quotient structure* for Y.

The above collection of quasi-uniform structures is non-empty since f is quasi-uniformly continuous with $\mathscr{V} = \{Y \times Y\}$. It is shown in [3] that the least upper bound of a non-empty collection of quasi-uniform structures is a quasi-uniform structure. f is clearly quasi-uniformly continuous with respect to $\mathscr{W}$. However, as Example 2.1 shows, $t_{\mathscr{W}}$ need not be the quotient topology. We return to this point later. $\mathscr{W}$ is the strongest quasi-uniform structure on Y for which f is quasi-uniformly continuous.

THEOREM 2.1. *Let $f\colon (X, \mathscr{U}) \to Y$ be surjective. Let $\mathscr{W}$ be the quotient structure on Y. Then $(Y, \mathscr{W})$ is a quotient object in the category $\mathscr{Q}$.*

Proof. Suppose that $f = gh$, where g is one-to-one and onto. Now $h\colon (X, \mathscr{U}) \to (Y', \mathscr{S})$ and $g\colon (Y', \mathscr{S}) \to (Y, \mathscr{W})$. We must show that if $S \in \mathscr{S}$, then $g(S) \in \mathscr{W}$. Since g is one-to-one and onto, $\{g(S)\colon S \in \mathscr{S}\}$ forms a quasi-uniform structure on Y. Denote it by $g(\mathscr{S})$. Now for each $S \in \mathscr{S}$ we have $f^{-1}(g(S)) = h^{-1}(S) \in \mathscr{U}$. Thus f is quasi-uniformly continuous with respect to $g(\mathscr{S})$ and hence $g(\mathscr{S}) \leqslant \mathscr{W}$.

If f is a surjective mapping from a topological space X to a set Y, the quotient topology on Y is defined as the strongest topology on Y

for which f is continuous. It can also be characterized by *O is open in Y if and only if $f^{-1}(O)$ is open in X*. This concept of the direct image topology leads us to the following definition:

Definition 2.2. Let $f: (X, \mathscr{U}) \to Y$ be surjective. Set $\mathscr{V} = \{V \subset Y \times Y: f^{-1}(V) \in \mathscr{U}\}$. $\mathscr{V}$ is called the *direct image* of $\mathscr{U}$.

It is well-known that the direct image need not be a quasi-uniform structure. However, we give a necessary and sufficient condition for $\mathscr{V}$ to be a quasi-uniform structure and we prove that if it is a quasi-uniform structure, then it is the quotient structure. First, for the convenience of the reader, we provide an example to show that $\mathscr{V}$ need not be a quasi-uniform structure.

Example 2.2. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3\}$ and let f be defined by $f(1) = 1$, $f(2) = f(3) = 2$, and $f(4) = 3$. Let $\mathscr{U}$ be the quasi-uniform structure generated by the base consisting of the single set $U = \Delta_X \cup \{(1, 2), (2, 3)\}$. Now $T = \Delta_Y \cup \{(1, 2), (2, 3)\} \in \mathscr{V}$, since $f^{-1}(T) = U$. Suppose there exists an $S \in \mathscr{V}$ with $S \circ S \subset T$. Then $f^{-1}(S) \supset U$ and $T = f(U) \subset f(f^{-1}(S)) = S$. Hence $T \circ T \subset T$, but this is impossible. Therefore $\mathscr{V}$ is not a quasi-uniform structure on Y.

There is another very natural reason to consider the direct image of a quasi-uniform structure. Let $(X, \mathscr{U})$ be a quasi-uniform space and $\mathscr{R}$ an equivalence relation on X. Let $[x]$ denote the equivalence class containing x. Set

$$\tilde{U} = \{([x], [y]): \text{there exists } x' \in [x] \text{ and } y' \in [y] \text{ with } (x', y') \in U\}.$$

Let $\tilde{\mathscr{U}} = \{\tilde{U}: U \in \mathscr{U}\}$. Let $p: X \to X/\mathscr{R}$ be the canonical map. Then $\tilde{\mathscr{U}}$ is the direct image of $\mathscr{U}$ under the map p.

THEOREM 2.2. *Let $f: (X, \mathscr{U}) \to Y$ be surjective. If the direct image $\mathscr{V}$ is a quasi-uniform structure, then $\mathscr{V}$ is the quotient structure.*

Proof. Let $\mathscr{W}$ denote the quotient structure on Y. If $\mathscr{V}$ is a quasi-uniform structure, then f is quasi-uniformly continuous with respect to $\mathscr{V}$ and, since $\mathscr{W}$ is the strongest such structure, we have $\mathscr{V} \leqslant \mathscr{W}$. Now suppose $W \in \mathscr{W}$; then $f^{-1}(W) \in \mathscr{U}$ and $W \in \mathscr{V}$. Therefore $\mathscr{V} = \mathscr{W}$.

It is easy to show that the direct image $\mathscr{V}$ satisfies the definition of a quasi-uniform structure except for condition (2). What we essentially need is that f preserves composition, that is if $U \in \mathscr{U}$, then $f(U) \circ f(U) = f(U \circ U)$. However, this is slightly stronger than what we must have. The following theorem gives a necessary and sufficient condition for $\mathscr{V}$ to be a quasi-uniform structure.

THEOREM 2.3. *Let $f: (X, \mathscr{U}) \to Y$ be surjective. Set $\mathscr{V} = \{V \subset Y \times Y: f^{-1}(V) \in \mathscr{U}\}$. $\mathscr{V}$ is a quasi-uniform structure on X if and only if for each $U \in \mathscr{U}$ there exists a $V \in \mathscr{U}$ with $f(V) \circ f(V) \subset f(U \circ U)$.*

Proof. Necessity. Let $U \in \mathscr{U}$. Then $U \circ U \in \mathscr{U}$ and $f(U \circ U) \in \mathscr{V}$. Now there exists $T \in \mathscr{V}$ with $T \circ T \subset f(U \circ U)$. Now $f^{-1}(T) \in \mathscr{U}$ and there exists $V \in \mathscr{U}$ with $V = f^{-1}(T)$. Therefore $f(V) \subset f(f^{-1}(T)) = T$ since f is surjective and we have

$$f(V) \circ f(V) \subset T \circ T \subset f(U \circ U).$$

Sufficiency. Let $W \in \mathscr{V}$. Then $f^{-1}(W) \in \mathscr{U}$ and there exists $U \in \mathscr{U}$ with $U \circ U \subset f^{-1}(W)$. There exists $V \in \mathscr{U}$ with $f(V) \circ f(V) \subset f(U \circ U)$. Therefore

$$f(V) \circ f(V) \subset f(U \circ U) \subset f(f^{-1}(W)) = W.$$

Since $f(V) \in \mathscr{V}$, we are through.

If $f\colon (X, \mathscr{U}) \to Y$ is surjective, set $\mathscr{R} = \{(x, y)\colon f(x) = f(y)\}$. Then $\mathscr{R}$ is an equivalence relation on X. Clearly, Y can be thought of as $X/\mathscr{R}$ and the structure $\tilde{\mathscr{U}}$ is the direct image of $\mathscr{U}$ under f.

THEOREM 2.3. *Let U and V belong to $\mathscr{U}$. $f(V) \circ f(V) \subset f(U \circ U)$ if and only if $V \circ \mathscr{R} \circ V \subset \mathscr{R} \circ U \circ U \circ \mathscr{R}$.*

This theorem together with Theorem 2.2 provides a necessary and sufficient condition in terms of the equivalence relation $\mathscr{R}$ for $\tilde{\mathscr{U}}$ to be a quasi-uniform structure on $X/\mathscr{R}$.

Our next theorem characterizes the quotient structure. We have already noted that the direct image fails to be a structure because it requires that $f(U)$ belong to it for each U in $\mathscr{U}$. The theorem shows that we must be more restrictive.

THEOREM 2.4. *Let $f\colon (X, \mathscr{U}) \to Y$ be surjective. Then*

$$\mathscr{S} = \{f(U)\colon U \in \mathscr{U} \text{ and there exists } \{V_1, V_2, \ldots\} \subset \mathscr{U} \text{ such that } f(V_1) \circ f(V_1) \subset f(U) \text{ and } f(V_{k+1}) \circ f(V_{k+1}) \subset f(V_k) \text{ for } k = 1, 2, \ldots\}$$

is a subbase for the quotient structure $\mathscr{W}$.

The proof is straightforward and left to the reader.

We now suppose that $\mathscr{V}$, the direct image, is a quasi-uniform structure and consider conditions for which it yields the quotient topology.

THEOREM 2.5. *Let $f\colon (X, \mathscr{U}) \to (Y, \mathscr{V})$, where $\mathscr{V}$ is the direct image quasi-uniform structure. Each of the following conditions are sufficient for $t_{\mathscr{V}}$ to be the quotient topology on Y:*

(a) *For each $U \in \mathscr{U}$ there exists a $V \in \mathscr{U}$ with $\mathscr{R} \circ V \subset U$.*

(b) *f is finite to one.* (Elementary identification maps, for example.)

(c) *$\mathscr{U}$ is saturated, that is $\mathscr{U}$ is closed under arbitrary intersections.*

3. q-maps.

Definition 3.1. Let $f\colon (X, \mathscr{U}) \to (Y, \mathscr{P})$. f is a q-map if for each $U \in \mathscr{U}$ we have $f(U) \in \mathscr{P}$. f is called *q-open* if for each $U \in \mathscr{U}$ there exists a $V \in \mathscr{P}$ such that $f(U[t]) \supset V[f(t)]$ for each $t \in X$.

It is clear that a q-map must be surjective.

THEOREM 3.1. *A q-open map is an open map. A surjective q-open map is a q-map.*

Proof. Let $f\colon (X, \mathscr{U}) \to (Y, \mathscr{P})$ be a q-open map. Let $O \in t$, the topology on X, with $y \in f(O)$. Now there exists an $x \in O$ with $f(x) = y$. Since O is open, there exists a $U \in \mathscr{U}$ and $U[x] \subset O$. Since f is q-open, there exists a $V \in \mathscr{P}$ with $f(U[t]) \supset V[f(t)]$ for each $t \in X$. Now $y \in V[f(x)] \subset f(U[x]) \subset f(O)$. Hence $f(O) \in t_{\mathscr{P}}$, and f is an open mapping.

Let $U \in \mathscr{U}$, and suppose that f is surjective. Then there is a $V \in \mathscr{P}$ with $f(U[t]) \supset V[f(t)]$ for each $t \in X$. We show that $f(U) \supset V$. Let $(a, b) \in V$. Since f is onto, there exists an $x \in X$ with $f(x) = a$. Now $b \in V[f(x)] \subset f(U[x])$. Hence there exists $y \in U[x]$ such that $f(y) = b$. Then $(a, b) \in f(U)$. Thus $f(U) \supset V \in \mathscr{P}$, and f is a q-map.

The following example shows that the converse of Theorem 3.1 does not hold.

Example 3.1. Let X denote the reals and Y denote the non-negative reals. Define $U_t = \{(a, b)\colon a = b \text{ or } a \geqslant t\}$. Let $\mathscr{U}$ be the quasi-uniform structure on X generated by the base consisting of all U_t for $t \in X$. Similarly, let $\mathscr{V}$ be the quasi-uniform structure on Y generated by the base consisting of all U_t for $t \in Y$. Let $f(t) = t^2$ for $t \in X$. Then $f\colon (X, \mathscr{U}) \to (Y, \mathscr{V})$ is an open q-map that is not q-open.

Putting the quotient structure $\mathscr{W}$ on the set Y in Example 2.1 provides an example of a q-map that is not open. It is clear that an open map need not be a q-map.

The following theorem shows that we can characterize the direct image structure in terms of q-maps:

THEOREM 3.2. *Let $f\colon (X, \mathscr{U}) \to (Y, \mathscr{P})$ be a quasi-uniformly continuous surjective map. Then $\mathscr{P}$ is the direct image quasi-uniform structure if and only if f is a q-map.*

Proof. Suppose $\mathscr{P}$ is the direct image quasi-uniform structure and $U \in \mathscr{U}$. Then $f^{-1}\big(f(U)\big) \supset U \in \mathscr{U}$ and $f(U) \in \mathscr{P}$. Thus f is a q-map.

Suppose f is a q-map. Let $\mathscr{V} = \{V \subset Y \times Y\colon f^{-1}(V) \in \mathscr{U}\}$. If $P \in \mathscr{P}$, then $f^{-1}(P) \in \mathscr{U}$ since f is quasi-uniformly continuous and $P \in \mathscr{V}$. Let $V \in \mathscr{V}$; then $f^{-1}(V) \in \mathscr{U}$ and, since f is a q-map, $f\big(f^{-1}(V)\big) \in \mathscr{P}$, but $V = f\big(f^{-1}(V)\big)$. Hence $\mathscr{V} = \mathscr{P}$.

The following theorem is analogous to Theorem 9 of Chapter 3 in [2], except we use the direct image structure on the middle space rather than the quotient structure:

THEOREM 3.3. *Let f be a quasi-uniformly continuous map of $(X, \mathscr{U})$ onto $(Y, \mathscr{V})$ such that $\mathscr{V}$ is the direct image structure. Then $g\colon (Y, \mathscr{V}) \to (Z, \mathscr{P})$ is quasi-uniformly continuous if and only if the composition gf is quasi-uniformly continuous.*

Proof. If g is quasi-uniformly continuous, it is clear that gf is quasi-uniformly continuous. Now, let $R \in \mathscr{P}$. Then, if gf is quasi-uniformly continuous, we have $f^{-1}\big(g^{-1}(R)\big) = (gf)^{-1}(R) \in \mathscr{U}$. Therefore $g^{-1}(R) \in \mathscr{V}$ and g is quasi-uniformly continuous.

REFERENCES

[1] J. R. Isbell, *Uniform spaces*, American Mathematical Society Mathematical Surveys 12 (1964).

[2] J. L. Kelly, *General topology*, Van Nostrand, Princeton 1955.

[3] M. G. Murdeshwar and S. A. Naimpally, *Quasi-uniform topological spaces*, Noordhoff Series A (1966).

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