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On a theorem of inversive geometry

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Abstract. This paper proves a metrical theorem of inversive geometry by conformal mapping. Furthermore, this paper gives a proof that the above property characterizes orthogonal pencils of coaxal circles from the standpoint of conformal mapping by solving a functional equation

1. Throughout this paper we include, unless otherwise stated, straight lines and line segments among circles and circular arcs, respectively.

We denote orthogonal pencils of coaxal circles by F_1, F_2. Let two arbitrary members of F_1 and two arbitrary members of F_2 be C_{11}, C_{12} and C_{21}, C_{22}, respectively. Furthermore, let a curvilinear rectangle formed by $C_{11}, C_{12}, C_{21}, C_{22}$ be $P_1P_2P_3P_4$. Here we put $P_1 = C_{11}\cap C_{21}$, $P_2 = C_{11}\cap C_{22}$, $P_3 = C_{12}\cap C_{22}$, $P_4 = C_{12}\cap C_{21}$.

The following theorem of inversive geometry plays an important role in the present note:

Theorem A. (a) *There exists exactly one member K_1 of F_1 such that the opposite sides (circular arcs) P_1P_2, P_4P_3 of the curvilinear rectangle $P_1P_2P_3P_4$ are inverse with respect to K_1.*

(b) *There exists exactly one member K_2 of F_2 such that the opposite sides (circular arcs) P_1P_4, P_2P_3 of $P_1P_2P_3P_4$ are inverse with respect to K_2.*

(c) *If we denote by V the point of intersection of the two orthogonal circles K_1, K_2 which lies inside $P_1P_2P_3P_4$, then*

$$\frac{1}{\overline{P_1V}^2}+\frac{1}{\overline{P_3V}^2}=\frac{1}{\overline{P_2V}^2}+\frac{1}{\overline{P_4V}^2}$$

holds.

In Section 2 we shall give a proof of Theorem A from the standpoint of conformal mapping in analytic function theory (see [3]). Conversely, in Section 3 we shall prove that property (c) in Theorem A characterizes orthogonal pencils of coaxal circles from the standpoint of conformal mapping by solving a functional equation (see [1], [3] and [5], p. 106–112). The purpose of the present note is to give this characterization of orthogonal pencils of coaxal circles.

2. We shall give a proof of Theorem A. We put aside the degenerate cases of orthogonal pencils of coaxal circles, i.e., orthogonal pencils of concentric circles (not straight lines) and concurrent straight lines passing through the common centre, and orthogonal pencils of parallel straight lines (see [6], p. 155). We discuss two cases.

Case 1. Consider orthogonal elliptic and hyperbolic pencils of coaxal circles.

We may assume that the two pencils lie on the w-plane. Furthermore, we may assume that the two limiting points of the two pencils are at 1 and -1. Let C be a curvilinear rectangle formed by any four members arbitrarily chosen from the two pencils. Consider the function $w=f(z) = \tanh z$. Then, there exist a non-empty simply connected domain D and four points A_1, A_2, A_3, A_4 on the z-plane satisfying the following three conditions:

(i) f is regular and univalent in D.

(ii) The four points A_1, A_2, A_3, A_4 form the four vertices of a rectangle which is contained entirely in D and whose sides are parallel to the real and imaginary axes on the z-plane. Here the vertices A_1, A_2, A_3, A_4 are listed consecutively.

(iii) The four points $f(A_1), f(A_2), f(A_3), f(A_4)$ coincide with the four vertices of the curvilinear rectangle C on the w-plane.

The above facts result from the following mapping property of $f(z) = \tanh z$:

The horizontal and vertical lines $\operatorname{Im}(z) = \text{const}$ and $\operatorname{Re}(z) = \text{const}$ on the z-plane are transformed by the function $w = f(z) = \tanh z$ into orthogonal elliptic and hyperbolic pencils of coaxal circles on the w-plane whose two limiting points are at 1 and -1.

Let A_1, A_2, A_3, A_4 represent the complex numbers $x+y, x-\bar{y}, x-y, x+\bar{y}$, respectively, x denoting the centre M of the rectangle $A_1A_2A_3A_4$. Furthermore, let M_1, M_2, M_3, M_4 be the midpoints of the four sides $A_1A_2, A_2A_3, A_3A_4, A_4A_1$, respectively.

(a) The two points A_1, A_4 and the two points A_2, A_3 are inverse, respectively, with respect to the line segment M_2M_4. We denote the straight line M_2M_4 by L_1. Since $w=f(z) = \tanh z$, f carries L_1 into a circle on the w-plane. Let this circle be K_1. Since the images of the two sides A_1A_2, A_4A_3 are the opposite sides P_1P_2, P_4P_3 of $P_1P_2P_3P_4$ on the w-plane, by the Reflection Principle of Analytic Functions (see [4],

p. 221) the opposite sides P_1P_2, P_4P_3 are inverse with respect to K_1. The proof of the uniqueness of K_1 is clear.

(b) The proof is similar to that given for (a).

(c) Since $f(M) = V$ and M represents the complex number x, we have $f(x) = V$. By the identities

$$|\cosh(x+y)|^2+|\cosh(x-y)|^2 = |\cosh(x+\bar{y})|^2+|\cosh(x-\bar{y})|^2$$

and

$$|\sinh y| = |\sinh\bar{y}|, \quad f(z) = \tanh z = \frac{\sinh z}{\cosh z}$$

satisfies

$$\frac{1}{|f(x+y)-f(x)|^2}+\frac{1}{|f(x-y)-f(x)|^2} = \frac{1}{|f(x+\bar{y})-f(x)|^2}+\frac{1}{|f(x-\bar{y})-f(x)|^2}.$$

Hence we have

$$\frac{1}{\overline{P_1V^2}}+\frac{1}{\overline{P_3V^2}} = \frac{1}{\overline{P_2V^2}}+\frac{1}{\overline{P_4V^2}}.$$

Thus Theorem A is proved in this case.

Case 2. Consider orthogonal parabolic pencils of coaxal circles.

If we consider the mapping function $w = f(z) = 1/z$, then we can similarly prove the desired result in this case. To this end we use the following mapping property of $f(z) = 1/z$:

The horizontal and vertical lines $\operatorname{Im}(z) = \text{const}$ and $\operatorname{Re}(z) = \text{const}$ on the z-plane are transformed by the function $w = f(z) = 1/z$ into orthogonal parabolic pencils of coaxal circles on the w-plane whose limiting point is at the origin and whose two common radical axes coincide with the real and imaginary axes. For the proof of (c) we use the identity $|x+y|^2+|x-y|^2 = |x+\bar{y}|^2+|x-\bar{y}|^2$. Thus the theorem is proved.

3. We shall state the main theorem and prove it.

Let $f = f(z)$ be a non-constant meromorphic function of a complex variable z in $|z| < +\infty$ and let D be a non-empty simply connected domain, where f is regular and univalent. Let $A_1A_2A_3A_4$ be an arbitrary rectangle which is contained entirely in D and whose sides are parallel to the real and imaginary axes on the z-plane. We denote by S the set of all domains D satisfying the above conditions. Let a curvilinear rectangle formed by the images of the four sides of $A_1A_2A_3A_4$ under the mapping function $f = f(z)$, i.e., $f(A_1A_2)\cup f(A_2A_3)\cup f(A_3A_4)\cup f(A_4A_1)$ be $P_1P_2P_3P_4$. Here we put $P_1 = f(A_1A_2)\cap f(A_4A_1)$, $P_2 = f(A_1A_2)\cap f(A_2A_3)$, $P_3 = f(A_2A_3)\cap$

$\cap f(A_3A_4)$, $P_4 = f(A_3A_4) \cap f(A_4A_1)$. Furthermore, let $V = f(M)$, where M is the centre of the rectangle $A_1A_2A_3A_4$.

Remark. Let M_1, M_2, M_3, M_4 be the midpoints of the four sides A_1A_2, A_2A_3, A_3A_4, A_4A_1, respectively, and let the line segments M_2M_4, M_1M_3 be L_1, L_2, respectively. Then V is, by the univalence of $f = f(z)$ in D, the only point of intersection of the two arcs $f(L_1)$, $f(L_2)$ which lies inside the curvilinear rectangle $P_1P_2P_3P_4$ on the w-plane. Here, by the definition of generalized inversion (see [2], p. 87) we see that the opposite sides P_1P_2, P_4P_3 and the opposite sides P_1P_4, P_2P_3 are inverse with respect to L_1, L_2, respectively, on the w-plane.

The purpose of the present note is, as stated in Section 1, to prove the following theorem:

Theorem. *Let D be an arbitrary domain belonging to S. Suppose that* $w = f(z)$ ($\not\equiv$ const) *is meromorphic in* $|z| < +\infty$. *Then*

$$\frac{1}{\overline{P_1V}^2} + \frac{1}{\overline{P_3V}^2} = \frac{1}{\overline{P_2V}^2} + \frac{1}{\overline{P_4V}^2}$$

holds for an arbitrary rectangle which is contained entirely in D and whose sides are parallel to the real and imaginary axes on the z-plane if and only if

$$f(z) = \frac{az+b}{cz+d} \quad or \quad f(z) = \frac{a\exp(kz)+b}{c\exp(kz)+d},$$

where a, b, c, d are arbitrary complex constants and k is an arbitrary real or purely imaginary constant with $(ad-bc)k \neq 0$.

Remark. By the following two facts we see that the property in Theorem A characterizes orthogonal pencils of coaxal circles.

(i) The horizontal and vertical lines $\operatorname{Im}(z) =$ const and $\operatorname{Re}(z) =$ const on the z-plane are transformed by the function $w = f(z) = \dfrac{a\exp(kz)+b}{c\exp(kz)+d}$ ($(ad-bc)k \neq 0$, k real or purely imaginary) into orthogonal elliptic and hyperbolic pencils of coaxal circles on the w-plane, including the degenerate case, i.e., orthogonal pencils of concentric circles (not straight lines) and concurrent straight lines passing through the common centre.

(ii) The horizontal and vertical lines $\operatorname{Im}(z) =$ const and $\operatorname{Re}(z) =$ const on the z-plane are transformed by the function $w = f(z) = \dfrac{az+b}{cz+d}$ ($ad-bc \neq 0$) into orthogonal parabolic pencils of coaxal circles on the w-plane, including the degenerate case, i.e., orthogonal pencils of parallel straight lines.

Proof of the Theorem. By the above remark and by Theorem A we have only to prove the "only if" part of the theorem.

Let $A_1A_2A_3A_4$ be an arbitrary rectangle which is contained entirely in D and whose sides are parallel to the real and imaginary axes on the z-plane and let A_1, A_2, A_3, A_4 represent the complex numbers $x+y$, $x-\bar{y}$, $x-y$, $x+\bar{y}$, respectively, x denoting the centre of the rectangle $A_1A_2A_3A_4$. By hypothesis we have

$$\text{(1)} \qquad \frac{1}{|f(x+y)-f(x)|^2}+\frac{1}{|f(x-y)-f(x)|^2} = \frac{1}{|f(x+\bar{y})-f(x)|^2}+\frac{1}{|f(x-\bar{y})-f(x)|^2},$$

where $x+y, x-\bar{y}, x-y, x+\bar{y} \in D$ with $y \neq 0$.

Clearing the denominators of both sides of (1) yields

$$\text{(2)} \qquad |f(x+y)-f(x)|^2|f(x+\bar{y})-f(x)|^2|f(x-\bar{y})-f(x)|^2+|f(x-y)-f(x)|^2|f(x+\bar{y})-f(x)|^2|f(x-\bar{y})-f(x)|^2 = |f(x+\bar{y})-f(x)|^2|f(x+y)-f(x)|^2|f(x-y)-f(x)|^2+|f(x-\bar{y})-f(x)|^2|f(x+y)-f(x)|^2|f(x-y)-f(x)|^2.$$

Rewriting (2) by repeated application of the formula $|A-B|^2 = (A-B)(\bar{A}-\bar{B})$ (A, B complex), putting $y = t\exp(i\varphi)$, where t, φ are real and φ is arbitrarily fixed, differentiating both sides of the resulting equation eight times with respect to t and putting $t = 0$ yields

$$\begin{aligned}
\text{(3)} \qquad & 6720\Big(\exp(2i\varphi)\big(2f'''(x)\overline{f'(x)}|f'(x)|^4+4\overline{f'''(x)}f'(x)|f'(x)|^4- \\
& -3\overline{f''(x)^2}f'(x)^2|f'(x)|^2\big)+\exp(-2i\varphi)\big(4f'''(x)\overline{f'(x)}|f'(x)|^4+ \\
& +2\overline{f'''(x)}f'(x)|f'(x)|^4-3f''(x)^2\overline{f'(x)^2}|f'(x)|^2\big)+3|f'(x)|^4|f''(x)|^2\Big) \\
& = 6720\Big(\exp(2i\varphi)\big(4f'''(x)\overline{f'(x)}|f'(x)|^4+2\overline{f'''(x)}f'(x)|f'(x)|^4- \\
& -3f''(x)^2\overline{f'(x)}^2|f'(x)|^2\big)+\exp(-2i\varphi)\big(2f'''(x)\overline{f'(x)}|f'(x)|^4+ \\
& +4\overline{f'''(x)}f'(x)|f'(x)|^4-3\overline{f''(x)}^2f'(x)^2|f'(x)|^2\big)+3|f'(x)|^4|f''(x)|^2\Big).
\end{aligned}$$

Since the representation of a trigonometric polynomial is unique, we can equate the coefficients of $\exp(2i\varphi)$ of both sides of (3). Hence we have in D

$$\begin{aligned}
& 2f'''(x)\overline{f'(x)}|f'(x)|^4+4\overline{f'''(x)}f'(x)|f'(x)|^4-3\overline{f''(x)}^2f'(x)^2|f'(x)|^2 \\
& \quad = 4f'''(x)\overline{f'(x)}|f'(x)|^4+2\overline{f'''(x)}f'(x)|f'(x)|^4-3f''(x)^2\overline{f'(x)}^2|f'(x)|^2
\end{aligned}$$

or

$$\text{(4)} \qquad f'''(x)\overline{f'(x)}|f'(x)|^4-\tfrac{3}{2}f''(x)^2\overline{f'(x)^2}|f'(x)|^2 = \overline{f'''(x)}f'(x)|f'(x)|^4-\tfrac{3}{2}\overline{f''(x)^2}f'(x)^2|f'(x)|^2.$$

Since $f = f(z)$ is regular and univalent in D, we have in D

(5) $$|f'(x)|^6 \neq 0.$$

By (4), (5) we have in D

(6) $$\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = \overline{\left(\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2\right)}.$$

By (6) we infer that the regular function

$$\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \quad \text{in } D$$

is real in D. Hence, by a well-known theorem in analytic function theory we have in D

(7) $$\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = K,$$

where K is a real constant.

By the Identity Theorem, (7) holds at every point x, where $f = f(z)$ is regular with $f'(x) \neq 0$.

The left-hand side of (7) is the Schwarzian derivative of $f = f(z)$ (see [3]). Solving (7), we have, according as $K = 0$ or $K < 0$ or $K > 0$, $f(z) = \frac{az+b}{cz+d}$ or $f(z) = \frac{a\exp(kz)+b}{c\exp(kz)+d}$ (k is a real constant), or $f(z) = \frac{a\exp(kz)+b}{c\exp(kz)+d}$ (k is a purely imaginary constant), where a, b, c, d are complex constants with $(ad-bc)k \neq 0$. Thus the theorem is proved.

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