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A dual method for some class of systems of variational inequalities

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Abstract. The purpose of this paper is to give some necessary and sufficient conditions for the existence of solutions for a class of systems of variational inequalities. Dual methods are the main analytical tool of the paper.

1. Introduction. In the paper we establish some class of systems of variational inequalities. This problem arises from theory of solid mechanics with constraints where restrictions are imposed independently on deformations and stresses, [14]–[17]. The dual problem for this system is proposed and a relationship between solutions of these two problems is examined. The results are then applied to obtain some existence theorems and to formulate necessary conditions for the existence of solutions of the system under consideration. Some known theorems for monotone-type multivalued mappings are employed, [1]–[3], [5]–[6].

2. Notations. Throughout the paper the following conventions are used:

(A.1) V and Y are reflexive Banach spaces with dual V^* and Y^*, respectively. The norms on V and Y, the bilinear canonical pairings over $V^* \times V$ and $Y^* \times Y$ will be denoted by $\|\cdot\|_V$, $\|\cdot\|_Y$, $\langle\cdot,\cdot\rangle_V$, $\langle\cdot,\cdot\rangle_Y$, respectively;

(A.2) $L \in (V \to Y)$ is a linear continuous operator from V into Y with domain $D(L) = V$. The adjoint operator is denoted by $L^*\colon Y^* \to V^*$. Moreover, we suppose that there exists positive constant $c > 0$ such that $\|Lu\|_Y \geq c\|u\|_V$, $\forall u \in V$;

(A.3) $A\colon Y^* \to 2^Y$ is multivalued mapping from Y^* into 2^Y. The set of all $\eta \in Y^*$ such that $A\eta \neq \emptyset$ will be denoted by $D(A)$;

(A.4) $\varphi\colon V \to (-\infty, \infty]$ and $\psi\colon Y^* \to (-\infty, \infty]$ are convex lower semicontinuous functions with domains $D(\varphi) = \{v \in V\colon \varphi(v) < \infty\} \neq \emptyset$ and $D(\psi) = \{\eta \in Y^*\colon \psi(\eta) < \infty\} \neq \emptyset$. By $\varphi^*\colon V^* \to (-\infty, \infty]$ we denote the conjugate function of φ, $\partial\varphi\colon V \to 2^{V^*}$ and $\partial\psi\colon Y^* \to 2^Y$ are subdifferentials of φ and ψ with effective domains

$$D(\partial\varphi) = \{v \in V\colon \partial\varphi(v) \neq \emptyset\} \quad \text{and} \quad D(\partial\psi) = \{\eta \in Y^*\colon \partial\psi(\eta) \neq \emptyset\},$$

respectively ($\partial\varphi(v)$ and $\partial\psi(\eta)$ stand for the subgradients of φ at v and of ψ at η, respectively);

(A.5) $f \in V^*$ is any fixed element of V^*.

3. The primal problem and the dual problem. We are concerned with the following system of variational inequalities:

$$
(3.1)\qquad
\begin{aligned}
&(u, \sigma) \in V \times Y^*,\\
&\langle L^*\sigma - f, v-u\rangle_V + \varphi(v) - \varphi(u) \geq 0, \qquad \forall v \in V,\\
&\langle \eta-\sigma, A\sigma - Lu\rangle_Y + \psi(\eta) - \psi(\sigma) \geq 0, \qquad \forall \eta \in Y^*.
\end{aligned}
$$

DEFINITION 3.1. An element $(u, \sigma) \in V \times Y^*$ is called a *solution of system* (3.1) if the following three conditions are satisfied:

$$(3.2)\qquad u \in D(\partial\varphi), \qquad \sigma \in D(A) \cap D(\partial\psi),$$

$$(3.3)\qquad \langle L^*\sigma - f, v-u\rangle_V + \varphi(v) - \varphi(u) \geq 0 \quad \text{for every } v \in V, \text{ i.e.}$$

$$0 \in L^*\sigma - f + \partial\varphi(u) \equiv \{L^*\sigma - f + u^*:\ u^* \in \partial\varphi(u)\},$$

(3.4) there exists $y \in A\sigma$ such that

$$\langle \eta-\sigma, y - Lu\rangle_Y + \psi(\eta) - \psi(\sigma) \geq 0 \quad \text{for every } \eta \in Y^*, \text{ i.e.}$$

$$0 \in A\sigma - Lu + \partial\psi(\sigma) \equiv \{w - Lu + z:\ w \in A\sigma,\ z \in \partial\psi(\sigma)\}.$$

The primal problem (P) consists in finding solutions of system (3.1), meant in the sense of Definition 3.1.

LEMMA 3.2. *An element $(u, \sigma) \in V \times Y^*$ is a solution of (P) if and only if the following conditions*:

$$(3.5)\qquad -L^*\sigma + f \in D(\partial\varphi^*), \qquad \sigma \in D(A) \cap D(\partial\psi),$$

$$(3.6)\qquad u \in \partial\varphi^*(-L^*\sigma + f),$$

$$(3.7)\qquad Lu \in \partial\psi(\sigma) + A(\sigma)$$

are satisfied.

Proof. Under assumptions (A.4) the equivalence between (3.2)–(3.4) and (3.5)–(3.7) follows immediately from the fact that $\partial\varphi^*$ is the inverse mapping of $\partial\varphi$, [5], [10]. ■

Now, let us consider the following variational inequality:

$$
(3.8)\qquad
\begin{aligned}
&\sigma \in Y^*,\\
&\langle \eta-\sigma, \partial\psi(\sigma) + A\sigma\rangle_Y + \varphi^*(-L^*\eta + f) - \varphi^*(-L^*\sigma + f) \geq 0, \quad \forall \eta \in Y^*.
\end{aligned}
$$

DEFINITION 3.3. An element $\sigma \in Y^*$ will be called a *solution of the variational inequality* (3.8) if the following two conditions are satisfied:

$$(3.9)\qquad -L^*\sigma + f \in D(\partial\varphi^*), \qquad \sigma \in D(A) \cap D(\partial\psi),$$

(3.10) there exists $y \in \partial\psi(\sigma) + A\sigma$ such that

$$\langle \eta - \sigma, y\rangle_Y + \varphi^*(-L^*\eta + f) - \varphi^*(-L^*\sigma + f) \geqslant 0,$$
$$\forall \eta \in Y^*, \quad \text{i.e.} \quad 0 \in \partial\psi(\sigma) + A\sigma + \partial\alpha(\sigma),$$

where $\alpha\colon Y^* \to (-\infty, \infty]$ is the function defined by

$$\alpha(\eta) \equiv \varphi^*(-L^*\eta + f), \qquad \eta \in Y^*, \tag{3.11}$$

with domain $D(\alpha) = \{\eta \in Y^*\colon -L^*\eta + f \in D(\varphi^*)\}$ (under assumptions (A.2) and (A.4) it is convex lower semicontinuous and proper function).

The dual problem (P^*) of the primal problem (P) consists in finding solutions of the variational inequality (3.8), meant in the sense of Definition 3.3.

The following theorem establishes a relation between solutions of problem (P) and those of the dual problem (P^*).

THEOREM 3.4. *Let us assume that hypotheses* (A.1)–(A.4) *are satisfied and let σ be an element of Y^*. Then the following two conditions are equivalent to each other*:

(i) *There exists $u \in V$ such that $(u, \sigma) \in V \times Y^*$ is a solution of problem* (P);

(ii) *σ is a solution of problem* (P^*).

Proof. If $\sigma \in Y^*$ is a solution of (P^*), then (3.5) is satisfied immediately. From (3.10) it follows that there exists $y \in \partial\psi(\sigma) + A(\sigma)$ such that

$$\langle \eta - \sigma, y\rangle_Y + \varphi^*(-L^*\eta + f) - \varphi^*(-L^*\sigma + f) \geqslant 0, \qquad \forall \eta \in Y^*. \tag{3.12}$$

Let us define $\operatorname{Ker} L^* = \{\eta \in Y^*\colon L^*\eta = 0\}$, $\operatorname{Im} L = \{y \in Y\colon y = Lv$ for some $v \in V\}$. Setting in (3.12), $\eta = \sigma + \theta$, where $\theta \in \operatorname{Ker} L^*$, we obtain

$$\langle \theta, y\rangle_Y \geqslant 0, \qquad \forall \theta \in \operatorname{Ker} L^*.$$

Hence $y \in (\operatorname{Ker} L^*)^\perp$, where $(\operatorname{Ker} L^*)^\perp$ is an annihilator of $\operatorname{Ker} L^*$. Under assumption (A.2), $\operatorname{Im} L$ is a closed subspace of Y and in this case $(\operatorname{Ker} L^*)^\perp = \operatorname{Im} L$, [10]. It follows that there exists $u \in V$ such that $y = Lu$. Hence $Lu \in \partial\psi(\sigma) + A(\sigma)$ and it implies that (3.7) holds. Now, we shall prove that $u \in \partial\varphi^*(-L^*\sigma + f)$. To this aid, let us consider inequality (3.12) by using the fact that $y = Lu$:

$$\langle \eta - \sigma, Lu\rangle_Y + \varphi^*(-L^*\eta + f) - \varphi^*(-L^*\sigma + f) \geqslant 0, \qquad \forall \eta \in Y^*.$$

Hence

$$\langle L^*\eta - L^*\sigma, u\rangle_V + \varphi^*(-L^*\eta + f) - \varphi^*(-L^*\sigma + f) \geqslant 0, \qquad \forall \eta \in Y^*,$$

$$\langle -L^*\eta - f - (-L^*\sigma + f), -u\rangle_V + \varphi^*(-L^*\eta + f) -$$
$$- \varphi^*(-L^*\sigma + f) \geqslant 0, \qquad \forall \eta \in Y^*.$$

Under assumption (A.2), L^* is surjective, [6]. It implies that

$$\langle v^* - (-L^*\sigma + f), -u\rangle_V + \varphi^*(v^*) - \varphi^*(-L^*\sigma + f) \geqslant 0, \qquad \forall v^* \in V^*.$$

Hence $u \in \partial\varphi^*(-L^*\sigma+f)$, i.e. (3.6) holds. Finally, by virtue of Lemma 3.2 it follows that $(u, \sigma) \in V \times Y^*$ is a solution of (P). Conversely, if (u, σ) is a solution of (P) then, due to Lemma 3.2, conditions (3.5)–(3.7) hold and (3.9) is satisfied immediately. From (3.6) we have

$$\langle v^* - (-L^*\sigma+f), -u\rangle_V + \varphi^*(v^*) - \varphi^*(-L^*\sigma+f) \geqslant 0, \quad \forall v^* \in V^*.$$

Setting $v^* = -L^*\eta+f$, $\eta \in Y^*$ and taking into account that L^* is surjective, we obtain:

$$\langle L^*\eta - L^*\sigma, u\rangle_V + \varphi^*(-L^*\sigma+f) - \varphi^*(-L^*\sigma+f) \geqslant 0, \quad \forall \eta \in Y^*.$$

Hence

$$\langle \eta - \sigma, Lu\rangle_Y + \varphi^*(-L^*\eta+f) - \varphi^*(-L^*\sigma+f) \geqslant 0, \quad \forall \eta \in Y^*.$$

It is equivalent to $-Lu \in \partial\alpha(\sigma)$. Using (3.7) we conclude that (3.10) holds. This proves the assertion. ■

4. Potential case. In this section, in addition to hypotheses (A.1)–(A.4), we shall assume that A is a singlevalued monotone operator with the potential $F\colon Y^* \to (-\infty, \infty)$.

Let us define a function $\Phi\colon V \times Y^* \to [-\infty, \infty]$ by

$$\Phi(v, \eta) \equiv -F(\eta) - \psi(\eta) + \langle \eta, Lv\rangle_Y + \varphi(v) - \langle f, v\rangle_V, \quad v \in V,\ \eta \in Y^*. \tag{4.1}$$

We recall that $(u, \sigma) \in V \times Y^*$ is said to be a *saddle point* of a function Φ if the following conditions are satisfied [5]:

(k) $\Phi(u, \sigma)$ is finite (in our case it is equivalent to $u \in D(\varphi)$, $\sigma \in D(\psi)$),
(kk) $\Phi(u, \eta) \leqslant \Phi(u, \sigma) \leqslant \Phi(v, \sigma)$, $\forall v \in V$, $\forall \eta \in Y^*$.

PROPOSITION 4.1. *Let $A\colon Y^* \to Y$ be a potential monotone operator and let Φ be defined by* (4.1). *Then the following conditions are equivalent*:

(i) (u, σ) *is a solution of problem* (P);

(ii) (u, σ) *is a saddle point of* Φ.

Proof. Let $(u, \sigma) \in V \times Y^*$ be a solution (P); then due to (3.2) we have $u \in D(\partial\varphi) \subset D(\varphi)$, $\sigma \in D(\partial\psi) \subset D(\psi)$, [10]. From (3.3) we obtain

$$\langle L^*\sigma - f, v\rangle_V + \varphi(v) \geqslant \langle L^*\sigma - f, u\rangle_V + \varphi(u), \quad \forall v \in V. \tag{4.2}$$

Under the assumption that A is a potential monotone operator from (3.4) it follows that σ is a solution of the minimization problem [5]:

$$\inf_{\eta \in Y^*} \{F(\eta) + \psi(\eta) - \langle \eta, Lu\rangle_Y\}.$$

Hence

$$F(\eta) + \psi(\eta) - \langle \eta, Lu\rangle_Y \geqslant F(\sigma) + \psi(\sigma) - \langle \sigma, Lu\rangle_Y, \quad \forall \eta \in Y^*. \tag{4.3}$$

Using (4.1) and taking into account inequalities (4.2) and (4.3), we arrive to inequalities (kk).

Conversely, if $(u, \sigma) \in V \times Y^*$ is a saddle point of Φ, then from (kk) it is easy to see that inequalities (4.2) and (4.3) hold (see (4.1)). From (4.3) it follows that $\sigma \in Y^*$ is a solution of (3.4), [5], i.e. $-A\sigma + Lu \in \partial\psi(\sigma)$. According to (4.2) we have $-L^*\sigma + f \in \partial\varphi(u) \Leftrightarrow u \in \partial\varphi^*(-L^*\sigma + f)$, i.e. $-L^*\sigma + f \in D(\partial\varphi^*)$. Taking into account the above relations we arrive at (3.9) and (3.10). Thus σ is a solution of (P*). Due to Theorem 3.4, (u, σ) is a solution of (P). ■

Remark 4.2. Under the assumptions of this section problem (P) can be written as follows:

$$\inf_{v \in V} \sup_{\eta \in Y^*} \Phi(v, \eta). \tag{4.4}$$

Now, let us consider the dual problem of (4.4) in the sense of minimax problems [5]:

$$\sup_{\eta \in Y^*} \inf_{v \in V} \Phi(v, \eta). \tag{4.5}$$

Using (4.1), we get the following form of (4.5):

$$-\inf_{\eta \in Y^*} \{F(\eta) + \psi(\eta) + \alpha(\eta)\}, \tag{4.6}$$

where α is defined by (3.11).

PROPOSITION 4.3. *Let A be a potential monotone operator and let hypotheses* (A.1)–(A.4) *be satisfied. Then every solution of* (4.6) *is also a solution of problem* (P*). *If, in addition,*

$$\partial(\psi + \alpha) = \partial\psi + \partial\alpha,$$

then σ is a solution of (P*) *if and only if σ is a solution of* (4.6).

The simple proof of Proposition 4.3 is omitted.

5. Existence theorems. Using Theorem 3.4, we can formulate the following existence theorem for the system of variational inequalities (3.1)

THEOREM 5.1. *Let hypotheses* (A.1)–(A.4) *be satisfied. Then the system of variational inequalities* (3.1) *has at least one solution if and only if* $0 \in \mathrm{Range}(A + \partial\psi + \partial\alpha)$.

Theorem 5.2 below is related to the problem whether $0 \in \mathrm{Range}(A + \partial\psi + \partial\alpha)$. This theorem we use in the next section. Its proof follows directly from our Theorem 5.1 and Browder–Hess, Theorem 7, [3]. For further discussion concerning the problem $0 \in \mathrm{Range}(T)$, where T is a multivalued mapping, we refer the reader to [1]–[4], [7]–[9], [12] (below we use terminology of [4]).

THEOREM 5.2. *Let hypotheses* (A.1)–(A.4) *be satisfied. Let $\partial\psi + \partial\alpha$ be a maximal monotone mapping from Y^* into 2^Y with $0 \in D(\partial\psi + \partial\alpha)$. Let, moreover, A be a generalized pseudo-monotone mapping from Y^* into 2^Y which* *is coercive and has the property that for each bounded maximal monotone mapping T from Y^* into 2^Y with $D(T) = Y^*$,* Range$(A+T) = Y$. *Then the system of variational inequalities* (3.1) *has at least one solution.*

6. An example. In this section we shall formulate some necessary and sufficient conditions for the existence of solutions of (3.1) under the assumptions:

(A.5) $\varphi = \text{ind}_K$, where ind_K: $V \to (-\infty, \infty]$ is an indicator function of a closed linear subspace K of V;

(A.6) $\psi = \text{ind}_\Sigma$, where ind_Σ: $Y^* \to (-\infty, \infty]$ is an indicator function of a closed linear subspace Σ of Y^*;

(A.7) A: $Y^* \to Y$ is a singlevalued mapping with $D(A) = Y^*$.

Under assumptions (A.5)–(A.6), $\varphi^* = \text{ind}_{K^\perp}$: $V^* \to (-\infty, \infty]$, where $K^\perp = \{v^* \in V^*: \langle v^*, v\rangle_V = 0 \text{ for all } v \in K\}$. It is easy to see that the function α: $Y^* \to (-\infty, \infty]$ defined by (3.11) is the indicator function of

$$L_f^{*-1}(K^\perp) \equiv \{\eta \in Y^*: \ -L^*\eta + f \in K^\perp\},$$

i.e. $\alpha = \text{ind}_{L_f^{*-1}(K^\perp)}$.

Let us define the sets

$$L^{*-1}(K^\perp) \equiv \{\eta \in Y^*: \ L^*\eta \in K^\perp\}, \quad \Sigma^\perp = \{y \in Y: \ \langle \eta, y\rangle_Y = 0, \ \forall \eta \in \Sigma\}.$$

The subdifferentials $\partial\psi$: $Y^* \to 2^Y$ and $\partial\alpha$: $Y^* \to 2^Y$ are multivalued mappings with effective domains Σ and $L_f^{*-1}(K^\perp)$, respectively, defined by

$$\partial\psi: \ \Sigma \ni \eta \to \Sigma^\perp, \qquad \partial\alpha: \ L_f^{*-1}(K^\perp) \ni \eta \to \big(L^{*-1}(K^\perp)\big)^\perp,$$

where $\big(L^{*-1}(K^\perp)\big)^\perp \equiv \{y \in Y: \ \langle \eta, y\rangle_Y = 0, \ \forall \eta \in L^{*-1}(K^\perp)\}$.

Using the above relations, the primal problem (P) can be written as

$$(u, \sigma) \in K \times \Sigma, \quad -L^*\sigma + f \in K^\perp, \quad -A\sigma + Lu \in \Sigma^\perp. \tag{6.1}$$

and its dual problem (P*) takes the form

$$\sigma \in L_f^{*-1}(K^\perp) \cap \Sigma, \quad 0 \in A\sigma + \Sigma^\perp + \big(L^{*-1}(K^\perp)\big)^\perp. \tag{6.2}$$

From (6.1) immeditely follows the following propositions:

PROPOSITION 6.1. *A necessary condition for the existence of solutions of* (6.1) *has the form*

$$f \in L^*\Sigma + K^\perp. \tag{6.3}$$

PROPOSITION 6.2. *Let $f \in L^*\sigma$. If $(u, \sigma) \in V \times Y^*$ is a solution of* (6.1), *then*

$$\langle \sigma - g, A\sigma\rangle_Y = 0 \tag{6.4}$$

holds for any $g \in \Sigma$ such that $L^*g = f$.

Proof. Let $(u, \sigma) \in V \times Y^*$ be a solution of (6.1). By Theorem 3.4 it follows that (6.2) holds. Thus there exist $y \in \Sigma^\perp$ and $z \in (L^{*-1}(K^\perp))^\perp$ such that $A\sigma + y + z = 0$. From (6.1) it follows that for any $g \in \Sigma$ such that $L^* g = f$ we have $\sigma - g \in L^{*-1}(K^\perp) \cap \Sigma$. Hence

$$\langle \sigma - g, A\sigma \rangle_Y = \langle \sigma - g, -y - z \rangle_Y = 0. \quad \blacksquare$$

PROPOSITION 6.3. *Let $f \in K^\perp$. If $(u, \sigma) \in V \times Y^*$ is a solution of* (6.1), *then the following condition is satisfied*:

$$\langle \sigma, A\sigma \rangle_Y = 0. \tag{6.5}$$

Proof. From Theorem 3.4 it follows that (6.2) is satisfied and we have $\sigma \in \Sigma$, $-L^* \sigma + f \in K^\perp$, $A\sigma + y + z = 0$ for some $y \in \Sigma^\perp$ and $z \in (L^{*-1}(K^\perp))^\perp$. By the assumption $f \in K^\perp$ and due to the above conditions we obtain $-L^* \sigma \in K^\perp - f \subset K^\perp$. Hence $\sigma \in L^{*-1}(K^\perp) \cap \Sigma$. Thus

$$\langle \sigma, A\sigma \rangle_Y = \langle \sigma, -y - z \rangle_Y = 0. \quad \blacksquare$$

The case where $f \in L^* \Sigma + K^\perp$, i.e. $f = f_1 + f_2$, where $f_1 \in L^* \Sigma$, $f_2 = K^\perp$ leads to the following system:

$$(u, \sigma) \in K \times \Sigma, \qquad -L^* \sigma + f_1 \in K^\perp, \qquad -A\sigma + Lu \in \Sigma^\perp.$$

It is the case considered in Proposition 6.2.

Finally, using Theorem 5.2 we shall formulate some sufficient conditions for the existence of solutions of (6.1).

THEOREM 6.4. *Let hypotheses* (A.1)–(A.7) *be satisfied. Let, moreover, A be a generalized pseudo-monotone operator from Y^* into Y which is coercive and has property that, for each bounded maximal monotone mapping T from Y^* into 2^Y with $D(T) = Y^*$*, Range$(A + T) = Y$. *Suppose that $\Sigma^\perp + (L^{*-1}(K^\perp))^\perp$ is a closed subspace of Y and $f \in L^* \Sigma$. Then system* (6.1) *has at least one solution.*

Proof. By Theorem 5.2 it suffices to prove that the multivalued mapping $\partial\psi + \partial\alpha$ is maximal monotone. For closed subspaces Σ and $L^{*-1}(K^\perp)$ of Y^* we have

$$\left(\Sigma^\perp + (L^{*-1}(K^\perp))^\perp\right)^\perp = \Sigma \cap L^{*-1}(K^\perp).$$

By assumption, $\Sigma^\perp + (L^{*-1}(k^\perp))^\perp$ is a closed subspace of Y and therefore

$$\Sigma^\perp + (L^{*-1}(K^\perp))^\perp = (\Sigma \cap L^{*-1}(K^\perp))^\perp.$$

The mapping $\partial\psi + \partial\alpha \colon Y^* \to 2^Y$ takes the form

$$\partial\psi + \partial\alpha \colon \Sigma \cap L_f^{*-1}(K^\perp) \ni \eta \to (\Sigma \cap L^{*-1}(K^\perp))^\perp.$$

Now, let us observe that the above mapping is the subdifferential of the indicator function

$$\mathrm{ind}_{\Sigma \cap L_f^{*-1}(K^\perp)} \colon Y^* \to (-\infty, \infty]$$

which is a convex lower semicontinuous proper function. Then, due to the known theorem [13], its subdifferential is a maximal monotone mapping. ■

Remark 6.5. By the assumption that one of the subspaces $\Sigma^{\perp}$ or $(L^{*-1}(K^{\perp}))^{\perp}$ is finite-dimensional $\Sigma^{\perp}+(L^{*-1}(K^{\perp}))^{\perp}$ is a closed subspace of Y. Therefore in this case the assertion of Theorem 6.4 follows.

References

[1] H. Brezis, M.G. Crandall, A. Pazy, *Perturbation of nonlinear maximal monotone sets in Banach space*, Comm. Pure Appl. Math. 23 (1970), 123–144.

[2] F. E. Browder, *Nonlinear maximal monotone operators in Banach space*, Math. Ann. 75 (1968), 89–113.

[3] –, *Nonlinear variational inequalities and maximal monotone mappings in Banach spaces*, Math. Ann. 183 (1969), 213–231.

[4] –, P. Hess, *Nonlinear mappings of monotone type in Banach spaces*, J. Funct. Analysis 11 (1972), 251–294.

[5] I. Ekeland, R. Temam, *Convex analysis and variational problems*, North-Holland Publ. Comp., 1976.

[6] H. Gajewski, K. Groger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Akademie Verlag, Berlin 1974.

[7] P. Hess, *On nonlinear mapping of monotone type homotopic to odd operators*, J. Funct. Analysis 11 (1972), 138–167.

[8] –, *On nonlinear mappings of monotone type with respect to two Banach spaces*, J. Math. Pures Appl. 52 (1973), 13–26.

[9] –, *On a unilateral problem associated with elliptic operators*, Proc. Amer. Math. Soc. 39 (1973), 94–100.

[10] R. Kluge, *Nichtlineare Variationsungleichungen und Extremalaufgaben*, Veb Deutscher Verlag der Wissenschaften, Berlin 1976.

[11] R. T. Rockafellar, *Convex functions, monotone operators and variational inequalities*; In: *Theory and applications of monotone operator*, 35–65, Gubbio 1969.

[12] –, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. 149 (1970), 75–93.

[13] –, *On the maximal monotonicity of subdifferential mappings*, J. Pacific J. Math. 33 (1970), 209–216.

[14] C. Woźniak, *Large deformations of elastic and non-elastic plates, shells and rods*, Mitteilungen aus dem Institut für Mechanik No. 20, Marz L 1980, Ruhr-Universität Bochum.

[15] –, *On the non-classical boundary value problems in structural and solid mechanics* (Lecture Notes), Instytut Inżynierii Lądowej Politechniki Wrocławskiej, Raport serii PRE nr 3/81, Wrocław 1981.

[16] C. Woźniak, *Constrained continuous media* (Parts I, II, III), Bull. Acad. Polon. Sci., Ser. Techn. 21, 109–116; 2, 167–173; 3, 175–182 (1973).

[17] –, *On the formation of simplified theories of solid mechanics*, Arch. Mech. 31, 6 (1979).

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