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On affine geometric product objects

by JÓZEF JOACHIM TELEGA (Warszawa)

Abstract. In this paper the problem of determination of a general form of affine geometric product objects is dealt with. For the purpose we make use of the fact that an affine representation of a degree n can be considered as a linear representation of degree $n+1$. The determination of the general form of an affine geometric product object is then based upon a linear representation of the direct sum of groups.

1. Introduction. A satisfactory and general definition of product objects has been given by Kucharzewski [1]. Further development of the theory of product geometric objects was done by us in papers [5]–[7], but in these contributions only linear product objects of first class have been dealt with.

Kucharzewski [1] (see also [5]) has given a general method for the determination of linear product objects of first class.

In this paper we are concerned with a more general problem: how to determine affine product objects of arbitrary class. For the purpose we shall make use of the fact that an affine representation of a degree n can be considered as a linear representation of degree $(n+1)$ [8]. We shall also make use of the notion of a linear representation of the direct sum of groups [4].

2. Some basic relations. Let two manifolds be given: m-dimensional M_1, and $\overline{m}$-dimensional M_2. Let M_1 be of class C^r and M_2 of class C^t. An object defined on the Cartesian product $M_1 \times M_2$ is called a *product object* [1], [5]. With the manifold M_1 (M_2) one can associate a Lie group; namely the differential group L_r^m ($L_t^{\overline{m}}$) [2]. On the Cartesian product $M_1 \times M_2$ there acts the direct sum $L_r^m \oplus L_t^{\overline{m}}$ of these groups.

The transformation law of affine geometric product objects has the form

$$T' = H(L, \bar{L})T + h(L, \bar{L}), \tag{2.1}$$

where $L \in L_r^m$, $\bar{L} \in L_t^{\overline{m}}$, or $L \oplus \bar{L} \in L_r^m \oplus L_t^{\overline{m}}$.

We are concerned with the problem of determination of the matrices H, h.

3. Affine representations of the direct sum of groups. Let a group G be the direct sum of groups G_1, G_2, i.e., $G = G_1 \oplus G_2$. Further, let V_1, V_2 be affine spaces of dimensions n, $\bar{n}$ respectively. We denote by $\{0, e_i\}_{1\leqslant i\leqslant n}$, $\{\bar{0}, \bar{e}_a\}_{1\leqslant a\leqslant \bar{n}}$ bases in spaces V_1, V_2, respectively. Firstly we consider affine representations of groups G_1, G_2.

To an element $s \in G_1$ there corresponds the affine transformation [8]

$$x' = F(s)x + g(s), \qquad x, x' \in V_1, \tag{3.1}$$

where $F(s)$, $g(s)$ are $n\times n$, $n\times 1$ matrices, respectively, $\det F(s) \neq 0$. These matrices define an affine representation of G_1 if and only if the following two equations are satisfied:

$$F(st) = F(s)F(t), \qquad s, t \in G_1, \tag{3.2}$$

$$g(st) = F(s)g(t) + g(s) \tag{3.3}$$

with the initial condition $F(1) = E$, where E is the unit matrix.

Analogously, to an element $\bar{s} \in G_2$ there corresponds the affine transformation

$$\bar{x}' = \bar{F}(\bar{s})\bar{x} + \bar{g}(\bar{s}), \qquad \bar{x}, \bar{x}' \in V_2, \tag{3.4}$$

where $\bar{F}(\bar{s})$, $\bar{g}(\bar{s})$ denote $\bar{n}\times\bar{n}$, $\bar{n}\times 1$ matrices, respectively; $\det \bar{F}(\bar{s}) \neq 0$. The matrix functions $\bar{F}(\bar{s})$, $\bar{g}(\bar{s})$ define an affine representation of G_2 iff they satisfy the equations

$$\bar{F}(\bar{s}\bar{t}) = \bar{F}(\bar{s})\bar{F}(\bar{t}), \qquad \bar{s}, \bar{t} \in G_2, \tag{3.5}$$

$$\bar{g}(\bar{s}\bar{t}) = \bar{F}(\bar{s})\bar{g}(\bar{t}) + \bar{g}(\bar{s}) \tag{3.6}$$

with the initial condition $\bar{F}(\bar{1}) = \bar{E}$ ($\bar{E}$ – the unit matrix). The dimension n ($\bar{n}$) of V_1 (V_2) is called the *degree of the representation*. By the homomorphism

$$s \to \tilde{F}_1(s) = \begin{bmatrix} F(s) & g(s) \\ 0 & 1 \end{bmatrix}, \qquad s \in G, \tag{3.7}$$

any affine representation of degree n can be considered as a linear representation of degree $n+1$.

In an analogous way we obtain a linear representation of degree $(\bar{n}+1)$

$$\bar{s} \to \tilde{F}_2(\bar{s}) = \begin{bmatrix} \bar{F}(\bar{s}) & \bar{g}(\bar{s}) \\ 0 & 1 \end{bmatrix}, \qquad \bar{s} \in G_2. \tag{3.8}$$

Every linear representation of the group $G_1 \oplus G_2$ is given by the Kronecker product of matrices $\tilde{F}_1$, $\tilde{F}_2$ [4] (the definition of the Kronecker product of matrices is given in [3]). In our case we have

$$\tilde{F}_1 \otimes \tilde{F}_2 = \begin{bmatrix} F(s) & g(s) \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \bar{F}(\bar{s}) & \bar{g}(\bar{s}) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} K(s,\bar{s}) & h(s,\bar{s}) \\ 0 & 1 \end{bmatrix}, \tag{3.9}$$

where

$$K(s,\bar{s}) = \begin{bmatrix} \overbrace{F\otimes\bar{F}}^{n\cdot\bar{n}} & \overbrace{F\otimes\bar{g}}^{n} & \overbrace{g\otimes\bar{F}}^{\bar{n}} \\ 0 & F & 0 \\ \multicolumn{2}{c}{0} & \bar{F} \end{bmatrix} \begin{matrix} \}n\cdot\bar{n} \\ \}n \\ \}\bar{n} \end{matrix}, \tag{3.10}$$

$$h(s,\bar{s}) = \begin{bmatrix} g\otimes\bar{g} \\ g \\ \bar{g} \end{bmatrix} \begin{matrix} \}n\cdot\bar{n} \\ \}n \\ \}\bar{n} \end{matrix}. \tag{3.11}$$

From the results obtained so far we conclude that to an element $s\oplus\bar{s} \in G_1\oplus\oplus G_2$ there corresponds the following affine transformation

$$y' = K(s,\bar{s})y + h(s,\bar{s}), \tag{3.12}$$

where y, $y' \in (V_1 \otimes V_2)\times V_1 \times V_2$; the symbols "$\otimes$", "$\times$" denote the tensor and Cartesian product, respectively.

4. Determination of affine geometric product objects. Now we pass to the determination of the general form of affine geometric product objects (2.1).

Let manifolds M_1, M_2 be given (see Section 2 above). Write $G = L_r^m \oplus \oplus L_{\bar{r}}^{\bar{m}}$.

At any specified point of M_1 the tangent space can be treated as an affine space (S. Kobayashi, K. Nomizu, *Fundations of differential geometry*, New York 1963, vol. I). So now V_1 is simply this affine tangent space or a space obtained by taking the operations of tensor product, direct sum etc. In an analogous way the space V_2 can be obtained, but of course with regard to the manifold M_2. Elements T, $T' \in (V_1 \otimes V)_2 \times \times V_1 \times V_2$ can be written down in the form $T = (T^{a\alpha}, T^a, T^\alpha)$, $T' = (T^{a'\alpha'}, T^{a'}, T^{\alpha'})$, where $a = 1, \ldots, n$, $\alpha = 1, \ldots, \bar{n}$. We get from (3.9)–(3.11)

$$\begin{aligned} T^{a'\alpha'} &= F_a^{a'} \bar{F}_\alpha^{\alpha'} T^{a\alpha} + F_a^{a'} \bar{g}^{\alpha'} T^a + g^{a'} \bar{F}_\alpha^{\alpha'} T^\alpha + g^{a'} \bar{g}^{\alpha'}, \\ T^{a'} &= F_a^{a'} T^a + g^{a'}, \\ T^{\alpha'} &= \bar{F}_\alpha^{\alpha'} T^\alpha + \bar{g}^{\alpha'}, \end{aligned} \tag{4.1}$$

where $F_a^{a'}$, $\bar{F}_\alpha^{\alpha'}$, g^a, $\bar{g}^{\alpha'}$ denote the elements of matrices $F(L)$, $\bar{F}(\bar{L})$, $g(L)$, $\bar{g}(\bar{L})$, respectively. The matrices H, h which appear in formula (2.1) have the form (3.10), (3.11) respectively, but we must put $s = L$, $\bar{s} = \bar{L}$. Thus we conclude that if the matrices F, g, $\bar{F}$, $\bar{g}$ fulfilling equations (3.2), (3.3), (3.5), (3.6) accordingly are given, then the transformation law of affine product object has the form (4.1). T, T' denote the components of this object in two allowable coordinate systems.

Remark 4.1. The above considerations are also valid for affine double and connecting objects (see [1], [5]–[7]).

Remark 4.2. The method suggested by Kucharzewski [1] for linear product objects of first class is a special case of our method. To prove this it suffices to put $g = \bar{g} = T^{a} = T^{a} = 0$; $L_r^m = \mathrm{GL}(m, R)$, $L_l^{\overline{m}} = \mathrm{GL}(\overline{m}, R)$, where GL denotes the linear group. Thus we get

$$T^{a'a'} = F_a^{a'}(A)\bar{F}_a^{a'}(B)T^{aa}, \quad A \in \mathrm{GL}(m, R), \; B \in \mathrm{GL}(\overline{m}, R). \tag{4.2}$$

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