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MINIMAL ALGEBRAS IN SOME CLASS OF ALGEBRAS

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We write $\mathfrak{A} = (A; \varphi_1, \ldots, \varphi_n)$ to indicate that $\mathfrak{A}$ is an algebra, A the set of its elements, and $\varphi_1, \varphi_2, \ldots, \varphi_n$ its fundamental operations. An algebra $\mathfrak{A}_0$ is called *minimal* in a class $\mathscr{K}$ if, for every algebra $\mathfrak{B} \in \mathscr{K}$ and every n, $\omega_n(\mathfrak{B}) \geqslant \omega_n(\mathfrak{A}_0)$, where $\omega_n(\mathfrak{A})$ is the number of operations essentially depending on n variables (in other words, of essentially n-ary operations).

The notion of minimal algebra was introduced by Dudek [1] who has studied the existence of minimal algebras in some classes of algebras with two binary operations. Here we shall be concerned with a class $\mathscr{K}$ of algebras with two essentially binary algebraic operations, one of which, denoted by $+$, is commutative and idempotent, whereas the other, denoted by dot (which may be omitted), is *diagonal* (see [4]), i.e. idempotent and satisfying the identity $x(yz) = (xy)z = xz$. We prove that in $\mathscr{K}$ minimal algebras actually exist and, moreover, that any such algebra is the product of two non-trivial semilattices with an additional diagonal operation suitably defined therein. These results were announced in [2], unfortunately with a misprint: in the definition of a minimal algebra, the sign $=$ has to be replaced by $\geqslant$.

Following the paper [3] we define simple iterations of the binary operation $+$ by

$$f_1(x_1) = x_1, \quad f_2(x_1, x_2) = x_1 + x_2,$$
$$f_{n+1}(x_1, x_2, \ldots, x_{n+1}) = f_2\big(f_n(x_1, \ldots, x_n), x_{n+1}\big) \quad (n > 1),$$

and its complete iterations by

$$f_1^* = f_2 = x_1 + x_2,$$

$$f_n^*(x_1, \ldots, x_{2^n}) = f_n\big(f_2(x_1, x_2), f_1^*(x_3, x_4), f_2^*(x_5, \ldots, x_8), \ldots, f_{n-1}^*(x_{2^{n-1}+1}, \ldots, x_{2^n})\big).$$

LEMMA 1. *The operation*

$$x_1 y_1 + x_2 y_2 \tag{$*$}$$

is essentially 4-*ary*.

Proof. If an operation is invariant with respect to some permutation of variables interchanging, say, x_1 and x_2, then either it depends actually on both x_1 and x_2 or on none of them. Since the operation $+$ is commutative, we have $x_1y_1+x_2y_2 = x_2y_2+x_1y_1$, and so $(*)$ depends on both x_1 and x_2 or on none of them. The same is true for y_1 and y_2. Suppose that $(*)$ does not depend on x_1. Then the operation x_1y_1 which arises from $(*)$ by putting $x_1 = x_2$ and $y_1 = y_2$ is also independent of x_1, contrary to the assumption that the operation $\cdot$ is essentially binary.

LEMMA 2. *The operation*

$$(**) \qquad f(x_1, \ldots, x_n) = f_k(x_{i_1}, \ldots, x_{i_k})\cdot f_l(x_{j_1}, \ldots, x_{j_l}),$$

where $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} = [1, n]$ *and the arguments in each bracket on the right-hand side are all different, depends essentially on all* x_j*'s.*

Proof. Suppose that f does not depend on one of the variables $x_{i_1}, \ldots, x_{i_k}$, e.g. on x_{i_1}. The right-side multiplication of $(**)$ by a variable u and the diagonality of the operation $\cdot$ imply

$$f(x_1, \ldots, x_n)\cdot u = f_k(x_{i_1}, \ldots, x_{i_k})\cdot u,$$

where the right-hand term does not depend on x_{i_1}. We now extend the simple iteration f_k to the complete iteration f_k^*, thus replacing x_{i_1} by a block of new variables on which f_k does not depend. Since in a complete iteration any two variables may be interchanged, f_k^* does not depend on any variable. We identify in the term $f_k^*\cdot u$ all variables appearing in f_k^* and write x for each of them. Since $+$ is idempotent, we get $x\cdot u$. Since $f_k^*\cdot u$ does not depend on any of the variables just replaced by x, the operation $x\cdot u$ does not depend on x, contrary to the assumption that $\cdot$ is essentially binary.

LEMMA 3. *If*

$$f = f_k(x_{i_1}, \ldots, x_{i_k})\cdot f_l(x_{j_1}, \ldots, x_{j_l}),$$
$$g = f_p(x_{s_1}, \ldots, x_{s_p})\cdot f_q(x_{t_1}, \ldots, x_{t_q}),$$

where $\{i_1, \ldots, i_k, j_1, \ldots, j_l\} = \{s_1, \ldots, s_p, t_1, \ldots, t_q\} = [1, n]$ *and the variables in each bracket are all different, and if*

$$\{x_{i_1}, \ldots, x_{i_k}\} \neq \{x_{s_1}, \ldots, x_{s_p}\} \quad \text{or} \quad \{x_{j_1}, \ldots, x_{j_l}\} \neq \{x_{t_1}, \ldots, x_{t_q}\},$$

then $f \neq g$.

Proof. Suppose that

$$\{x_{i_1}, \ldots, x_{i_k}\} \neq \{x_{s_1}, \ldots, x_{s_p}\} \quad \text{and} \quad x_{i_1} \notin \{x_{s_1}, \ldots, x_{s_p}\}.$$

If we had $f = g$, then the right-hand side multiplication by $f_p(x_{s_1}, \ldots, x_{s_p})$ would lead to

$$f_k(x_{i_1}, \ldots, x_{i_k})\cdot f_p(x_{s_1}, \ldots, x_{s_p}) = f_p(x_{s_1}, \ldots, x_{s_p}).$$

Since x_{i_1} appears here on the left-hand side without appearing on the right-hand side, the product $f_k \cdot f_p$ does not depend on x_{i_1}, but this contradicts Lemma 2.

LEMMA 4. *Let $\tau_n(\mathfrak{A})$ denote the number of those operations in $\mathfrak{A}$ that obey scheme* (**), *where the arguments in each bracket are all different. Then $\tau_n(\mathfrak{A}) \geqslant 3^n - 2$. If $+$ is (commutative and) associative, then $\tau_n(\mathfrak{A}) = 3^n - 2$.*

Proof. By Lemma 2 every operation of type (**) depends on all variables and, in view of Lemma 3, any two such operations are different, provided the sets of their arguments do not coincide in each pair of corresponding brackets. Hence the number of operations (**) is not less than the number of representations of $X = \{x_1, x_2, \ldots, x_n\}$ as the union of two non-empty sets X_1 and X_2. If $X_1 = \{x_{i_1}, \ldots, x_{i_k}\}$, then X_2 consists of all remaining variables and, possibly, of some elements of X_1. Consequently, if X_1 is fixed and $k < n$, there are

$$\binom{k}{0} + \binom{k}{1} + \ldots + \binom{k}{k} = 2^k$$

possible choices of X_2. For $k = n$ there are only $2^n - 1$ such possibilities, since the term $\binom{n}{0}$ would correspond to $X_1 = \{x_1, x_2, \ldots, x_n\}$ and $X_2 = \emptyset$. There are $\binom{n}{k}$ ways for X_1 to contain exactly k elements. So, for a fixed $k < n$, there are $\binom{n}{k} 2^k$ representations of X as $X_1 \cup X_2$ and, for $k = n$, their number is $2^n - 1$. So we have finally

$$\begin{aligned} \tau_n(\mathfrak{A}) &\geqslant \binom{n}{1} 2^1 + \binom{n}{2} 2^2 + \ldots + \binom{n}{n} 2^n - 1 \\ &= \binom{n}{0} 2^0 + \binom{n}{1} 2^1 + \ldots + \binom{n}{n} 2^n - 2 = 3^n - 2. \end{aligned}$$

If $+$ is associative (and commutative as is throughout supposed), then

$$(***) \qquad f_n(x_1, \ldots, x_n) = (x_{i_1} + \ldots + x_{i_k}) \cdot (x_{j_1} + \ldots + x_{j_l}),$$

the ordering within the brackets being arbitrary. Hence, every representation of type $X = X_1 \cup X_2$ determines only one operation (**), and so the second assertion of the lemma follows.

Lemma 4 implies immediately

COROLLARY. *For $\mathfrak{A} \in \mathscr{K}$ we have $\omega_n(\mathfrak{A}) \geqslant 3^n - 2$.*

Remark 1. The estimation of Lemma 4 cannot be improved by taking into account operations of type $f_n(x_1, \ldots, x_n)$.

In fact, since the operation $\cdot$ is idempotent, we have $f_n \cdot f_n = f_n$, and so f_n is actually of type (**) with $k = l = n$.

Remark 2. It is obvious that the equation $\tau_n(\mathfrak{A}) = 3^n - 2$ holds if and only if every representation $X = X_1 \cup X_2$ determines only one operation.

LEMMA 5. *Suppose that in an algebra $\mathfrak{A} = (A;\ +, \cdot)$ with $+$ associative and idempotent and with $\cdot$ diagonal the following identity holds:*

$$(x_1 + x_2)(x_3 + x_4) = x_1 \cdot x_3 + x_2 \cdot x_4. \tag{1}$$

Then we have identically

$$(x_1 + x_2 + \ldots + x_n)(y_1 + y_2 + \ldots + y_n) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.$$

The proof is by induction.

LEMMA 6. *If the algebra $\mathfrak{A} = (A;\ +, \cdot)$ satisfies the assumption of Lemma 5, then every operation in $\mathfrak{A}$ is of type (***), where the variables in any bracket are all different.*

Proof. Assuming $x_1 = x_2$ ($x_3 = x_4$) in (1), we conclude that $\cdot$ is left-side (right-side) distributive with respect to $+$. Distributivity and diagonality of $\cdot$ imply that every operation in $\mathfrak{A}$ is representable as the sum of terms of type $x_i \cdot x_j$ or x_i. Consequently, since $x_i = x_i \cdot x_i$, every operation can be written as $x_{i_1} \cdot x_{j_1} + \ldots + x_{i_m} \cdot x_{j_m}$ and, by Lemma 5, this is equal to

$$(x_{i_1} + \ldots + x_{i_m}) \cdot (x_{j_1} + \ldots + x_{j_m}).$$

Since $+$ is idempotent, every summand in each bracket may be written only once, and thus scheme (***) actually appears.

THEOREM 1. *In the class $\mathscr{K}$ there exists a minimal algebra $\mathfrak{A}_0$ such that $\omega_n(\mathfrak{A}_0) = 3^n - 2$ for $n = 1, 2, \ldots$*

Proof. Let $\mathfrak{A}_1 = (X_1;\ +)$, $\mathfrak{A}_2 = (X_2;\ +)$, where $|X_1| > 1$, $|X_2| > 1$ and $+$ is essentially binary, associative, commutative, and idempotent. We introduce into the product $\mathfrak{A}_1 \times \mathfrak{A}_2 = (X_1 \times X_2;\ +)$ an additional operation $\cdot$ defined by

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1, b_2). \tag{2}$$

It is easily proved that the resulting algebra $\mathfrak{A}_0 = (X_1 \times X_2;\ +, \cdot)$ belongs to $\mathscr{K}$ and satisfies the assumption of Lemma 5; hence, in view of Lemma 6, every operation in $\mathfrak{A}_0$ is of type (***). Consequently, $\omega_n(\mathfrak{A}_0) = \tau_n(\mathfrak{A}_0)$ and so, by commutativity of $+$, we deduce from Lemma 4 that there are exactly $3^n - 2$ n-ary operations in $\mathfrak{A}_0$; the proof is completed.

The following theorem shows that the algebra $\mathfrak{A}_0$ we have just used is not a mere example.

THEOREM 2. *Every algebra that is minimal in $\mathscr{K}$ is isomorphic to an algebra of type $\mathfrak{A}_0$, i.e. to the product of two algebras $\mathfrak{A}_1 = (X_1;\ +)$ and $\mathfrak{A}_2 = (X_2;\ +)$, where $+$ is essentially binary, associative, commutative and idempotent, and $\cdot$ is defined by* (2).

The proof will be based on the following theorem.

THEOREM 3. *Suppose that in $\mathfrak{A} = (A;\ +, \cdot)$ both operations $+$ and $\cdot$ are essentially binary, with the first being associative, commutative and idempotent, and the second being diagonal. If* (1) *is fulfilled, then there exist algebras $\mathfrak{A}_1 = (X_1;\ +)$ and $\mathfrak{A}_2 = (X_2;\ +)$ with the operation $+$ essentially binary, associative, commutative and idempotent and such that $\mathfrak{A} \simeq \mathfrak{A}_0 = (X_1 \times X_2;\ +, \cdot)$, where $+$ operates coordinatewise, and $\cdot$ is defined by* (2).

Proof. We introduce in A two relations R_1 and R_2 defined by

$$a_1 R_1 a_2 \Leftrightarrow a_1 a_2 = a_2, \qquad a_1 R_2 a_2 \Leftrightarrow a_1 a_2 = a_1.$$

It is easy to check that these relations are of equivalence type and that they preserve $+$ and $\cdot$. Moreover,

$$(\alpha) \qquad a R_1 \cap R_2 b \Leftrightarrow a = b,$$

$$(\beta) \qquad \underset{a,b}{\forall} a R_1 R_2 b.$$

In fact,

$$a R_1 \cap R_2 b \Leftrightarrow (a R_1 b \text{ and } a R_2 b) \Leftrightarrow (ab = b \text{ and } ab = a) \Leftrightarrow a = b,$$

$$a R_1 R_2 b \Leftrightarrow \underset{c}{\exists} (a R_1 c \text{ and } c R_2 b).$$

Actually, $c = a \cdot b$ holds, since $ac = a(ab) = ab = c$ or else $aR_1 c$ and $cb = (ab)b = ab = c$, whence $cR_2 b$. For $a \in A$, let $[a]_{R_1}$ and $[a]_{R_2}$ be the cosets of R_1 and R_2, respectively, to which a belongs. Let $\mathfrak{A}_1 = (A/R_1;\ +)$, and $\mathfrak{A}_2 = (A/R_2;\ +)$. Each of the algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ contains at least two elements, since otherwise one of the relations R_1 and R_2 would hold identically, contrary to the assumption that $\cdot$ is essentially binary. It is obvious that $+$ is associative, commutative and idempotent in both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ (as was in $\mathfrak{A}$). Hence and from $|\mathfrak{A}_1| > 1$, $|\mathfrak{A}_2| > 1$ we easily deduce that $+$ is essentially binary. Further, (α) and (β) imply that the canonical mapping $\varphi(a) = ([a]_{R_1}, [a]_{R_2})$ of A into $A/R_1 \times A/R_2$ is bi-univalent and onto. In $A/R_1 \times A/R_2$ we define the operation $\cdot$ by formula (2).

It can be easily checked that $\cdot$ is diagonal. We now show that φ preserves $\cdot$. Indeed, we have

$$\varphi(a) \cdot \varphi(b) = ([a]_{R_1}, [a]_{R_2}) \cdot ([b]_{R_1}, [b]_{R_2}) = ([a]_{R_1}, [b]_{R_2}).$$

Since $ab = a$ or else $abR_1 a$, so $[ab]_{R_1} = [a]_{R_1}$ and, similarly, $[ab]_{R_2} = [b]_{R_2}$. Hence

$$\varphi(a) \cdot \varphi(b) = ([ab]_{R_1}, [ab]_{R_2}) = \varphi(ab).$$

Thus φ is shown to be an isomorphism of $\mathfrak{A}$ onto

$$\mathfrak{A}_0 = (A/R_1 \times A/R_2;\ +, \cdot).$$

Proof of Theorem 2. If an algebra $\mathfrak{A}$ is minimal in $\mathscr{K}$, then, for every n, it contains 3^n-2 essentially n-ary operations. By Lemma 4, in $\mathfrak{A}$ there are at least 3^n-2 n-ary operations of type (**), so $\mathfrak{A}$ cannot contain any other operation. This implies

(i) $\omega_n(\mathfrak{A}) = \tau_n(\mathfrak{A}) = 3^n-2$,

(ii) every operation in $\mathfrak{A}$ can be represented by means of $+$ and $\cdot$.

On account of (ii) we may admit $+$ and $\cdot$ as the only fundamental operations in $\mathfrak{A}$. It is enough to prove that

$$(x_1+x_2)+x_3 = x_1+(x_2+x_3) \quad \text{and} \quad (x_1+x_2)(x_3+x_4) = x_1x_3+x_2x_4,$$

and to use Theorem 3. Since $x_1+(x_2+x_3) = (x_2+x_3)+x_1$, the first identity will be proved by showing that $(x_1+x_2)+x_3 = (x_2+x_3)+x_1$.

In the opposite case, $\mathfrak{A}$ would admit two operations, with three variables, of type

$$f_n(x_1, \ldots, x_n) = f_n(x_1, \ldots, x_n)\cdot f_n(x_1, \ldots, x_n).$$

But, in view of (i), this would contradict Remark 2 after putting in it $X = X_1 = X_2 = \{x_1, x_2, x_3\}$.

To prove the second identity let us observe that, by associativity of $+$, the (essentially 4-ary) operation $x_1x_3+x_2x_4$ must coincide with some operation of type (***), i.e. it must be of the form

$$(x_{i_1}+ \ldots +x_{i_k})\cdot(x_{j_1}+ \ldots +x_{j_l}),$$

where every argument belongs to the set $\{x_1, x_2, x_3, x_4\}$. Hence

$$x_1x_3+x_2x_4 = (x_{i_1}+ \ldots +x_{i_k})\cdot(x_{j_1}+ \ldots +x_{j_l}).$$

Suppose that x_1 or x_2 appears in the second bracket. Then, if we identify x_1 with x_2 and x_3 with x_4, the above identity will imply one of the equations

$$x_1x_3 = x_1(x_1+x_3), \qquad x_1x_3 = x_3x_1,$$
$$x_1x_3 = (x_1+x_3)x_1, \qquad x_1x_3 = x_3(x_1+x_3),$$
$$x_1x_3 = (x_1+x_3)(x_1+x_3),$$

each of which contradicts Lemma 3. If x_3 or x_4 appears in the first bracket, a similar contradiction can be obtained. So only one possibility remains, namely in the first bracket x_1 and x_2 appear, whereas in the second x_3 and x_4 do. So it must be

$$x_1\cdot x_3+x_2\cdot x_4 = (x_1+x_2)\cdot(x_3+x_4)$$

which completes the proof.

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