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Index of the Dirac operator in R^n

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Abstract. In this paper a proof of the formula for the index of the Dirac operator in R^n is given following Bott and Seeley [1] but without using K-theory.

C. Callias [2] has calculated the index of the Dirac operator in R^n in connection with some problems of the Yang–Mills theory. An interesting implication of this paper is the fact that this index can be non-zero for odd n.

R. Bott and R. Seeley [1] have noticed that Callias' result can be derived from Fedosov's formula [3], [4] (see also [6]) in purely algebraic way. Modulo a constant depending only upon n the index formula may be obtained very simply. The calculation of the constant presented a certain problem, which was overcome by using K-theory.

The purpose of this paper is to compute the above-mentioned constant without the use of K-theory.

This will give a full purely algebraic derivation of Callias' formula for the index of the Dirac operator in R^n from Fedosov's formula for the index of an elliptic operator in R^n. We shall omit all those details that can be found in the paper by Bott and Seeley cited above. Our notation is also taken from that paper.

Let V' and V'' be finite-dimensional complex vector spaces and let $V = V' \otimes V''$.

Let us consider a differential operator D acting in $C^\infty(R^n, V)$ with the (full) symbol

$$\sigma(x, \xi) = \delta(\xi) \otimes I + iI \otimes U(x) \tag{1}$$

satisfying the conditions

$$\delta(\xi) = \sum_{j=1}^{n} \delta^j \xi_j, \qquad \delta(\xi)^2 = |\xi|^2 = \xi_1^2 + \ldots + \xi_n^2, \tag{2}$$

$$\begin{aligned} U(x)^2 &= I && \text{for } |x| \geqslant 1, \\ U(x) &= U(x/|x|) && \text{for } |x| \geqslant 1. \end{aligned} \tag{3}$$

Such an operator will be called the *Dirac operator* in R^n.

It is known that D has finite index given by Fedosov's formula

$$\text{index } D = -\left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} \int\limits_{S^{2n-1}_{x,\xi}} \operatorname{Tr}(\sigma^{-1}\, d\sigma)^{2n-1}, \tag{4}$$

where $S^{2n-1}_{x,\xi} = \{(x, \xi)\colon |x|^2+|\xi|^2 = 1\}$, $d\sigma$ is the exterior derivative of the matrix σ, and $\operatorname{Tr}(\sigma^{-1}\, d\sigma)^{2n-1}$ denotes the trace of the $(2n-1)$th exterior power of the matrix 1-form $\sigma^{-1}\, d\sigma$.

Bott and Seeley observed that formula (4) can be written as

$$\text{index } D = -\left(\frac{i}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} \int\limits_{Z_{\xi,x,t}} \operatorname{Tr}(z^{-1}\, dz)^{2n-1}, \tag{5}$$

where $z(\xi, x, t) = \delta(\xi)\otimes\cos t+i \sin t\otimes U(x)$, $Z_{\xi,x,t} = S^{n-1}_\xi \times S^{n-1}_x \times I_t$, S^{n-1}_ξ, S^{n-1}_x are unit spheres in $\boldsymbol{R}^n_\xi$ and $\boldsymbol{R}^n_x$, respectively, and $I_t = [0, \frac{1}{2}\pi]$. From (2) and (3) we easily derive that

$$z^{-1}(\xi, x, t) = \delta(\xi)\otimes\cos t-i \sin t\otimes U(x),$$

$$z^{-1}\, dz = \cos t(\delta\otimes\cos t-i \sin t\otimes U)\, d\delta\otimes I + i \sin t(\delta\otimes\cos t-i \sin t\otimes U) I\otimes dU+i\delta\otimes U dt. \tag{6}$$

From now on for brevity we shall omit the tensor and exterior multiplication signs and also instead of $\delta\otimes I$ and $I\otimes U$ we shall write δ and U.

With such convention the matrices δ and U satisfy the relations

$$U\delta = \delta U, \quad U^2 = I, \quad \delta^2 = I \quad (\text{for } |\xi| = 1), \tag{7}$$

$$dU\delta = \delta dU, \quad U d\delta = d\delta U. \tag{8}$$

By exterior differentiation of the above formulae we get

$$dUU = -UdU, \quad d\delta\delta = -\delta d\delta, \quad -dU d\delta = d\delta dU. \tag{9}$$

Formula (6) may be written as

$$z^{-1}\, dz = Ad\delta+BdU+Cdt, \tag{10}$$

where $A = \cos t\cdot M$, $M = \cos t\delta-i \sin t\cdot U$,

$$B = i \sin t\cdot M, \quad C = i\delta U.$$

From (7)–(9) it easily follows that

$$M^{-1} = \cos t\delta+i \sin tU, \quad d\delta M = -M^{-1}\, d\delta, \quad dUM = M^{-1}\, dU,$$

$$Ad\delta BdU = BdUAd\delta = i \sin t \cos t\, d\delta\, dU,$$

$$(Ad\delta)^2 = -\cos^2 t(d\delta)^2, \quad (BdU)^2 = -\sin^2 t(dU)^2.$$

By commutation of $Ad\delta$ and BdU as well as anticommutation of $d\delta$ and dU we obtain

$$(11)\qquad \begin{aligned}(Ad\delta)^k(BdU)^k &= (Ad\delta BdU)^k = (-i\sin t\cos t)^k(d\delta\, dU)^k\\ &= (-1)^{k(k+1)/2}\, i^k \sin^k t\cos^k t\,(d\delta)^k(dU)^k\\ &= i^{k^2+2k}\sin^k t\cos^k t\,(d\delta)^k(dU)^k.\end{aligned}$$

We now start to compute $\operatorname{Tr}(z^{-1}dz)^{2n-1}$. We have

$$\begin{aligned}(z^{-1}dz)^{2n-1} &= (Ad\delta+BdU+Cdt)^{2n-1}\\ &= \sum_{j=0}^{2n-2}(Ad\delta+BdU)^j\,Cdt\,(Ad\delta+BdU)^{2n-2-j}+(Ad\delta+BdU)^{2n-1}\end{aligned}$$

and thus

$$(12)\qquad \begin{aligned}&\operatorname{Tr}(z^{-1}dz)^{2n-1}\\ &\qquad = (2n-1)\operatorname{Tr}(Ad\delta+BdU)^{2n-2}\,Cdt+\operatorname{Tr}(Ad\delta+BdU)^{2n-1}.\end{aligned}$$

The second term of (12) vanishes on $S_\xi^{n-1}\times S_x^{n-1}$. Computing the first term by using the commutation of $Ad\delta$ and BdU, we get

$$(Ad\delta+BdU)^{2n-2} = \sum_{k=0}^{2n-2}\binom{2n-2}{k}(Ad\delta)^k(BdU)^{2n-2-k}.$$

Now, (5) may be written as follows:

$$\begin{aligned}\operatorname{index} D &= -\left(\frac{i}{2\pi}\right)^n\frac{(n-1)!}{(2n-2)!}\binom{2n-2}{n-1}\int_{Z_{\xi,x,t}}\operatorname{Tr}(Ad\delta)^{n-1}(BdU)^{n-1}\,Cdt\\ &= \frac{i^{n^2+n+2}}{(2\pi)^n(n-1)!}\int_{S_\xi^{n-1}}\operatorname{Tr}\delta(d\delta)^{n-1}\int_{S_x^{n-1}}\operatorname{Tr}U(dU)^{n-1}\cdot\int_0^{\frac{\pi}{2}}\sin^{n-1}t\cos^{n-1}t\,dt.\end{aligned}$$

If we use the well-known formula

$$\int_0^{\frac{\pi}{2}}\sin^{n-1}t\cos^{n-1}t\,dt = \frac{\Gamma\left(\frac{n}{2}\right)^n}{2\Gamma(n)} = \frac{\Gamma\left(\frac{n}{2}\right)^2}{2(n-1)!}$$

we get

$$(13)\qquad \operatorname{index} D = \frac{i^{n^2+n+2}}{2(2\pi)^n}\left(\frac{\Gamma\left(\frac{n}{2}\right)}{(n-1)!}\right)^2\int_{S_\xi^{n-1}}\operatorname{Tr}\delta(d\delta)^{n-1}\int_{S_x^{n-1}}\operatorname{Tr}U(dU)^{n-1}.$$

Next we have to compute $\int_{S_\xi^{n-1}} \operatorname{Tr} \delta(d\delta)^{n-1}$. In view of (2), for $|\xi| = 1$ we have

$$\delta(\xi) = \sum_{j=1}^{n} \delta^j \xi_j, \qquad \delta^2(\xi) = I,$$

and thus

(14) $$\delta^j\delta^k + \delta^k\delta^j = 0, \quad i \neq j, \quad \delta^j\delta^j = I.$$

We see that the matrices δ^j, $j = 1, \ldots, n$, generate the complex Clifford algebra C_n^c (cf. [5]), and so V' is a C_n^c-module.

We have $d\delta = \sum_{j=1}^{n} \delta^j d\xi_j$ and thus, for $k \leqslant n$,

$$(d\delta)^k = \sum_{j_1,\ldots,j_k} \delta^{j_1}\delta^{j_2}\ldots\delta^{j_k} d\xi_{j_1}\ldots d\xi_{j_k} = k! \sum_{j_1<j_2\ldots<j_k} \delta^{j_1}\delta^{j_2}\ldots\delta^{j_k} d\xi_{j_1}\ldots d\xi_{j_k}.$$

It follows that

$$\operatorname{Tr} \delta(d\delta)^{n-1} = (n-1)! \sum_{j=1}^{n} \sum_{1\leqslant j_1<\ldots<j_{n-1}\leqslant n} \operatorname{Tr} \delta^j\delta^{j_1}\ldots\delta^{j_{n-1}} \xi_j d\xi_{j_1}\ldots d\xi_{j_{n-1}}.$$

Now, let us observe that if n is even

$$\operatorname{Tr} \delta^j\delta^{j_1}\ldots\delta^{j_{n-1}} = 0.$$

In fact, if $j_1, \ldots, j_k$ are different and k is even, then in view of the anticommutativity of δ^{j_p}, $p = 1, \ldots, k$, and the basic property of trace we have

$$\operatorname{Tr} \delta^{j_1}\delta^{j_2}\ldots\delta^{j_k} = \operatorname{Tr}(-1)^{k-1}\delta^{j_2}\ldots\delta^{j_k}\delta^{j_1} = -\operatorname{Tr} \delta^{j_2}\ldots\delta^{j_k}\delta^{j_1}$$
$$= -\operatorname{Tr} \delta^{j_1}\delta^{j_2}\ldots\delta^{j_k}$$

and consequently $\operatorname{Tr} \delta^{j_1}\delta^{j_2}\ldots\delta^{j_k} = 0$. Therefore, if n is even, $\operatorname{Tr} \delta(d\delta)^{n-1} = 0$ and index $D = 0$.

Now, let us consider the case $n = 2m+1$. The algebra C_{2m+1}^c has only two inequivalent non-reducible representations (called *spin-representations*) and any representation of C_{2m+1}^c is a sum of spin-representations [5]. It suffices to consider the case where the matrices $\delta^1, \ldots, \delta^n$ generate a spin-representation. Such a representation may easily be given by using the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We may assume that the representation space is $C^{2^m} = C^2 \otimes \ldots \otimes C^2$ (m times) and δ^j are given by ([7], § 115)

$$\delta^{2k-1} = \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{k-1} \otimes \sigma_1 \otimes \underbrace{I_2 \otimes \ldots \otimes I_2}_{m-k}, \quad k = 1, \ldots, m,$$

$$\delta^{2k} = \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{k-1} \otimes \sigma_2 \otimes \underbrace{I_2 \otimes \ldots \otimes I_2}_{m-k}, \quad k = 1, \ldots, m,$$

$$\delta^{2m+1} = \pm \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{m},$$

where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The signs $\pm$ in δ^{2m+1} give two different (i.e., inequivalent) spin-representations. Observe that

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I_2, \quad \sigma_1 \sigma_2 = -i\sigma_3 = -\sigma_2 \sigma_1$$

and

$$\mathrm{Tr}\, \sigma_1 = \mathrm{Tr}\, \sigma_2 = \mathrm{Tr}\, \sigma_3 = 0.$$

From this it easily follows that

$$\mathrm{Tr}\, \delta^{j_1} \delta^{j_2} \ldots \delta^{j_k} = 0 \quad \text{for } j_1 < \ldots < j_k,\ k < n.$$

Moreover, we have $\delta^1 \delta^2 \ldots \delta^n = \pm(-i)^m I_2 \otimes \ldots \otimes I_2$, thus $\mathrm{Tr}\, \delta^1 \delta^2 \ldots \delta^n = \pm(-2i)^m = \pm i^{3m} 2^m$ and

$$\mathrm{Tr}\, \delta (d\delta)^{n-1} = \pm(n-1)!\, i^{3m} 2^m \sum_{j=1}^{n} (-1)^{j-1} \xi_j d\xi_1 \ldots \widehat{d\xi_j} \ldots d\xi_n$$

(as usual, the dot means that the factor should be omitted). Hence we have

$$\begin{aligned} \int_{S_\xi^{n-1}} \mathrm{Tr}\, \delta (d\delta)^{n-1} &= \pm(2m)!\, i^{3m} 2^m \sum_{j=0}^{n} (-1)^{j-1} \int_{S_\xi^{n-1}} \xi_j d\xi_1 \ldots \widehat{d\xi_j} \ldots d\xi_n \\ &= \pm(2m+1)!\, i^{3m} 2^m \int_{|\xi| \leq 1} d\xi_1 \ldots d\xi_n \\ &= \pm(2m+1)!\, i^{3m} 2^m \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} = \pm(2m)!\, i^{3m} 2^{m+1} \frac{\pi^{m+1/2}}{\Gamma(m+\frac{1}{2})}. \end{aligned}$$

Using the above formula and the equality

$$\Gamma(m+\tfrac{1}{2}) = \frac{(2m)!\, \pi^{1/2}}{2^{2m} m!},$$

we get from (13) the Callias formula

$$\text{index } D = \pm\left(\frac{i}{8\pi}\right)^m \frac{1}{2m!} \int\limits_{S_x^{2m}} \operatorname{Tr} U(dU)^{2m}. \tag{15}$$

This is the result we aimed at.

It is interesting to note that by (15) the index D depends on the full symbol of D.

References

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