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On the existence of solutions of differential equations of retarded type in a Banach space

by V. Lakshmikantham, A. R. Mitchell and R. W. Mitchell (Arlington, Texas)

Abstract. The existence and uniqueness of solutions of differential equations of retarded type in a Banach space are considered under a monotonicity type condition. The tools that are available for the study of equations without delay pose problems in this case since the domain and the range come from different Banach spaces. To overcome this difficulty, subsets of the domain have to be chosen carefully and weaker forms of differential inequalities have to be employed.

Introduction. The study of the Cauchy problem for differential equations in a Banach space has attracted a lot of attention in recent years. (See [1] for references.) The two main directions that are followed are to find compactness type or monotonicity type conditions. The corresponding theory for differential equations of retarded type in a Banach space is lacking a similar development. One reason seems to be the difficulty in imposing the assumptions since, in this case, the domain and the range of the function involved in differential equations are not in the same Banach space.

In this paper we attempt to overcome this difficulty by imposing conditions over a subset of the domain in a suitable way and employing the weaker forms of the theory of differential inequalities. We prove existence and uniqueness results which extend similar results in [4]. We believe that this approach would be fruitful in other situations.

1. Statement of main results. Let $\tau > 0$ be a given number and let E be a Banach space with $\|\cdot\|$. Let $E_0 = C\big[[-\tau, 0], E\big]$ denote the Banach space of continuous functions with the norm given by $\|\varphi\|_0 = \max\limits_{-\tau\leqslant s\leqslant 0} \|\varphi(s)\|$. If $t_0 \in \boldsymbol{R}^+$ and $x \in C\big[[t_0-\tau, \infty), E\big]$, then for any $t \in [t_0, \infty)$, we let $x_t \in E_0$ be defined by

$$x_t(s) = x(t+s), \qquad -\tau \leqslant s \leqslant 0.$$

Let $f \in C[\boldsymbol{R}^+ \times E_0, E]$. We consider the delay differential equation

$$x'(t) = f(t, x_t) \tag{1.1}$$

with the given initial function $\varphi_0 \in E_0$ at $t = t_0$, that is,

$$x_{t_0} = \varphi_0. \tag{1.2}$$

For notation and some discussion concerning such equations see [2], p. 185.

We shall be employing the following hypotheses:

$$f \in C[\boldsymbol{R}^+ \times E_0, E]. \tag{H$_1$}$$

$$\lim_{h \to 0-} \frac{1}{h}[\|\varphi(0) - \psi(0) + h\{f(t, \varphi) - f(t, \psi)\}\| - \|\varphi(0) - \varphi(0)\|] \leqslant g\big(t, \|\varphi(0) - \psi(0)\|\big), \tag{H$_2$}$$

whenever $\varphi, \psi \in \Omega$, where Ω is given by

$$\Omega = [\varphi, \psi \in E_0\colon \|\varphi(s) - \psi(s)\| \leqslant \|\varphi(0) - \psi(0)\|, \ -\tau \leqslant s \leqslant 0].$$

(H$_3$) $g \in C[\boldsymbol{R}^+ \times \boldsymbol{R}^+, \boldsymbol{R}^+]$, $g(t, 0) \equiv 0$ and $u \equiv 0$ is the only solution of the scalar differential equation

$$u' = g(t, u), \ u(t_0) = 0,$$

on $[t_0, \infty)$.

Let $S_p(\varphi_0) = [\varphi \in E_0\colon \|\varphi - \varphi_0\|_0 \leqslant p]$. Then (H$_1$) implies there exist positive numbers a, b, M such that

$$\|f(t, \varphi)\| \leqslant M, \qquad (t, \varphi) \in [t_0, t_0 + a] \times S_b(\varphi_0),$$

and $M \leqslant b/a$. This observation we shall be using without further mention.

(H$_4$) for each $T > t_0$, and $\varepsilon > 0$ there is a number $\delta(T, \varepsilon) > 0$ such that

$$\|f(t, \varphi) - f(s, \varphi)\| < \varepsilon \quad \text{whenever } (t, \varphi), (s, \varphi) \in [t_0, T] \times S_b(\varphi_0) \text{ and } |t - s| < \delta.$$

Our aim is to prove the following results which generalize the results of [4] to delay differential equations.

THEOREM 1. *Let hypotheses* (H$_1$), (H$_2$) *and* (H$_3$) *hold. Then for each* $\varphi_0 \in E_0$, *there is a unique solution for the initial value problem* (1.1) *and* (1.2) *existing on* $[t_0, t_0 + a]$.

THEOREM 2. *Let hypotheses* (H$_1$), (H$_2$), (H$_3$) *and* (H$_4$) *hold. Then for each* $\varphi_0 \in E_0$ *there is a unique solution for the initial value problem* (1.1) *and* (1.2) *existing on* $[t_0, \infty)$.

2. Auxiliary results. We need the following known results.

LEMMA 1. *Assume that*

(a) $g \in C[\boldsymbol{R}^+ \times \boldsymbol{R}^+, \boldsymbol{R}^+]$ *and* $[t_0, \infty)$ *is the largest interval of existence* *of the maximal solution* $r(t, t_0, u_0)$ *of*

$$u' = g(t, u), \quad u(t_0) = u_0 \geqslant 0;$$

(b) $m \in C[[t_0, -\tau, \infty), \mathbf{R}^+]$, S *a countable subset of* $[t_0, t_0+a]$, *and for every* $t_1 \geqslant t_0$, $t_1 \notin S$ *for which*

$$m_{t_1}(s) \leqslant m(t_1), \quad -\tau \leqslant s \leqslant 0,$$

the differential inequality

$$D_- m(t_1) \leqslant g(t_1, m(t_1))$$

holds.

Then, if $m_{t_0}(s) \leqslant u_0$, $-\tau \leqslant s \leqslant 0$, *we have*

$$m(t) \leqslant r(t, t_0, u_0), \quad t \in [t_0, \infty).$$

For a proof see [3], Vol. II, p. 8.

LEMMA 2. *Let the assumption of Lemma* 1 *hold. Suppose that* $[t_0, t_1] \subset [t_0, \infty)$. *Then there exists a* $\varepsilon_0 > 0$ *such that* $0 < \varepsilon < \varepsilon_0$, *the maximal solution* $r(t, t_0, u_0, \varepsilon)$ *of*

$$u' = g(t, u)+\varepsilon, \quad u(t_0) = u_0+\varepsilon,$$

exists on $[t_0, t_1]$ *and* $\lim_{\varepsilon \to 0} r(t, t_0, u_0, \varepsilon) = r(t, t_0, u_0)$ *uniformly on* $[t_0, t_1]$.

For a proof see [3], Vol. I, p. 12.

Our next result is concerned with the construction of ε-approximate solutions to problem (1.1) and (1.2) which is an extension of a similar important result in [4].

LEMMA 3. *Let* (H_1) *hold and let* a, b, $M > 0$ *be chosen such that* $\|f(t, \varphi)\| \leqslant M$ *on* $[t_0, t_0+a] \times S_b(\varphi_0)$, *where* $\varphi_0 \in E_0$ *is a given initial function. Then for each positive integer* n, *there is a positive integer* $N = N(n)$, *a partition* $\{t_i^n\}_{i=0}^N$ *of* $[t_0, t_0+a]$, *and a function* x^n *from* $[t_0-\tau, t_0+a]$ *into* E *such that*

(i) $x_{t_0}^n = \varphi_0$;

(ii) $|t_{i+1}^n - t_i^n| \leqslant 1/n$ *for each* $1 \leqslant i \leqslant N$;

(iii) $\|x_t^n - x_s^n\|_0 \leqslant M|t-s|$, $t, s \in [t_0, t_0+a]$;

(iv) *if* $t \in (t_{i-1}^n, t_i^n)$, *then the derivative* $(x^n)'(t)$ *exists and equals* $f(t_{i-1}^n, x_{t_{i-1}^n}^n)$ *for each* $1 \leqslant i \leqslant N$;

(v) *if* $\|\varphi - x_{t_{i-1}^n}^n\|_0 \leqslant M(t_i^n - t_{i-1}^n)$ *and* $t \in [t_{i-1}, t_i]$, *then* $\|f(t, \varphi) - f(t_{i-1}^n, x_{t_{i-1}}^n)\| \leqslant 1/n$.

Proof. Let $t_0^n = t_0$ and define $x^n(t) = \varphi_0(t-t_0)$ for $t \in [t_0-\tau, t_0]$. Assume we have defined $t_0^n, \ldots, t_i^n$ and $x^n(t)$ on $[t_0-\tau, t_i^n]$ so that (1)-(5) are satisfied. If $t_i^n = t_0+a$ we are finished. If $t_i^n < t_0^n+a$ choose δ_i subject to the following restrictions:

(i) $t_i^n + \delta_i \leqslant t_0 + a$;

(ii) $\delta_i \leqslant 1/n$;

(iii) using the continuity of f at $(t_i^n, x_{t_i}^n)$ we choose δ_i sufficiently small so that if $t \in [t_i^n, t_i^n + \delta_i]$ and $\|\psi - x_{t_i}^n\|_0 \leqslant M\delta_i$, then $\|f(t, \psi) - f(t_i^n, x_{t_i}^n)\| \leqslant 1/n$;

and precisely one of conditions (iv) and (v) below hold.

(iv) $\delta_i + t_i^n = t_0 + a$ or $\delta_i = 1/n$;

(v) for each $\beta > 0$ there exist t_β, ψ_β with $t_\beta \in [t_i^n, t_i^n + \delta_i + \beta]$ and $\|\psi_\beta - x_{t_i}^n\|_0 \leqslant M\delta_i + \beta$ and yet

$$\|f(t_\beta, \psi_\beta) - f(t_i^n, x_{t_i}^n)\| > 1/n.$$

Define $t_{i+1}^n = t_i^n + \delta_i$ and for $t \in [t_i^n, t_{i+1}^n]$ define

$$x^n(t) = x^n(t_i^n) + (t - t_i^n) f(t_i^n, x_{t_i}^n).$$

If we can show there exists N_n with $t_{N_n}^n = t_0 + a$ we are finished. If not $t_i^n \to s \leqslant t_0 + a$ and hence $\{x_{t_i^n}^n\}_{i=0}^\infty$ is Cauchy since $\|x_{t_i^n}^n - x_{t_j^n}^n\|_0 \leqslant M|t_i^n - t_j^n| \to 0$ as $i, j \to \infty$ and so $x_{t_i^n}^n \to \psi_0$. However, since f is continuous at (s, ψ_0) there exists $a < 1/n$ such that $\|\varphi - \psi_0\|_0 < a$ and $|t - s| < a$ then

$$\|f(s, \psi_0) - f(t, \varphi)\| < 1/3n.$$

For sufficiently large i,

$$|t_i^n - s| < \frac{a}{3(M+1)} \quad \text{and} \quad \|\psi_0 - x_{t_i}^n\|_0 < \tfrac{1}{3}a.$$

But

$$\delta_i = t_{i+1}^n - t_i^n < s - t_i^n < \frac{a}{3(M+1)} < \frac{1}{n} \quad \text{and} \quad t_{i+1}^n \neq t_0 + a$$

so (iv) does not hold forcing (v) to hold. Consider (v) for $\beta = \frac{1}{3}a$ and there exists $\varphi_{\frac{1}{3}a}, t_{\frac{1}{3}a}$ with

$\|\varphi_{\frac{1}{3}a} - x_{t_i^n}^n\|_0 \leqslant M\delta_i + \frac{1}{3}a = M(t_{i+1}^n - t_i^n) + \frac{1}{3}a < \frac{2}{3}a$ and $t_{\frac{1}{3}a} \in [t_i^n, t_i^n + \delta_i + \frac{1}{3}a]$ (this means $|t_{\frac{1}{3}a} - s| < a$) and yet $\|f(t_{\frac{1}{3}a}, \varphi_{\frac{1}{3}a}) - f(t_i^n, x_{t_i^n}^n)\| > 1/n$. Notice

$$\|\varphi_{\frac{1}{3}a} - \psi_0\|_0 \leqslant \|\varphi_{\frac{1}{3}a} - x_{t_i^n}^n\|_0 + \|x_{t_i^n}^n - \psi_0\|_0 < \tfrac{2}{3}a + \tfrac{1}{3}a = a < 1/n$$

and since $|t_{\frac{1}{3}a} - s| < a$ we have

$$\frac{1}{n} \leqslant \|f(t_{\frac{1}{3}a}, \varphi_{\frac{1}{3}a}) - f(t_i^n, x_{t_i^n}^n)\| \leqslant \|f(t_{\frac{1}{3}a}, \varphi_{\frac{1}{3}a}) - f(s, \psi_0)\| + \|f(s, \psi_0) - f(t_i^n, x_{t_i^n}^n)\|$$
$$\leqslant \frac{1}{3n} + \frac{1}{3n} < \frac{1}{n}.$$

This contradiction forces the existence of N_n such that $t_{N_n}^n = t_0 + a$. ∎

LEMMA 4. *Assume the hypotheses of Lemma 3 and for each positive integer n let $x^n(t)$ and $\{t_i^n\}_{i=1}^{N_n}$ be the function and partition assured by Lemma 3. If for each $t \in [t_0-\tau, t_0+a]$ we have $x^n(t)$ converges pointwise to a function $x(t)$, then $x(t)$ is a solution of $x'(t) = f(t, x_t)$, $x_{t_0} = \varphi_0$.*

Proof. Since $\{x^n\}$ is an equicontinuous family by (iii) of Lemma 3 we have $x^n \to x$ uniformly on $[t_0-\tau, t_0+a]$ and so $x(t)$ is continuous. Moreover, since $x^n \to x$ uniformly on $[t_0-\tau, t_0+a]$ we have $(x^n)_t \to x_t$ uniformly on $[-\tau, 0]$ for each $t \in [t_0, t_0+a]$. Furthermore by 3.1 of Lemma 3 $\|x_t^n - x_{t_0}^n\|_0 = \|x_t^n - \varphi_0\|_0 \leqslant |t-t_0|M \leqslant aM \leqslant b$ thus each $x_t^n \in S_b(\varphi_0)$ and so $x_t \in S_b(\varphi_0)$.

Let $S = \bigcup_{n=1}^{\infty} \{t_i^n | i = 1, \ldots, N_n\}$ and note S is countable. Let $t \in (t_{i_n}^n, t_{i_n+1}^n)$ for each n; then

$$\lim_{n\to\infty} \|x_t - x_{t_{i_n}^n}^n\|_0 \leqslant \lim_{n\to\infty} [\|x_t - (x^n)_t\|_0 + \|(x^n)_t - (x^n)_{t_{i_n}^n}\|_0] \leqslant \lim_{n\to\infty} [\|x_t - (x^n)_t\|_0 + |t - t_{i_n}^n| M] = 0. \tag{4.1}$$

Recall from part (iv) of Lemma 3 $(x^n)'(t) = f(t_{i_n}^n, (x^n)_{t_{i_n}^n})$ for $t \in (t_{i_n}^n, t_{i_n+1}^n)$. Note $K = \{(t, x(t)) | t \in [t_0, t_0+a]\}$ is compact, thus using (4.1), and (iv) of Lemma 3 we have $\lim_{n\to\infty} (x^n)' = \lim_{n\to\infty} f(t_{i_n}^n, (x^n)_{t_{i_n}^n}) = f(t, x_t)$ uniformly in $[t_0, t_0+a] - S$. Consequently for each $t \in [t_0, t_0+a]$

$$x(t) = \lim_{n\to\infty} x^n(t) = \lim_{n\to\infty} \Big\{\varphi_0(t_0) + \int_{t_0}^{t} (x^n)'(s)\,ds\Big\} = \varphi_0(t_0) + \int_{t_0}^{t} f(s, x_s)\,ds$$

and thus $x(t)$ is the desired solution.

LEMMA 5. *Assume the hypotheses of Theorem 1 and let $\{x^n\}_{n=1}^{\infty}$ be the approximation assured by Lemma 3. Then $\{x^n\}$ converges uniformly to a function $x(t)$ from $[t_0-\tau, t_0+a]$ into E such that $\varphi_0 = x_{t_0}$ and $x_t \in S_b(\varphi_0)$ for each $t \in [t_0, t_0+a]$.*

Proof. Let m, n be positive integers and define $w_{m,n}(t) = \|x^n(t) - x^m(t)\|$. To avoid notational difficulties we will write $w(t)$ instead of $w_{m,n}(t)$. Notice $w_t = \|x_t^n - x_t^m\|$. Let $\{t_i^\varkappa\}_{i=1}^{N_\varkappa}$ be the partition corresponding to $\varkappa$ and let $S = \bigcup_{\varkappa=1}^{\infty} \{t_i^\varkappa\}_{i=1}^{N_\varkappa}$ and observe S is countable. Suppose that for $t_1 \in (t_0, t_0+a] - S$ we have $w_{t_1}(s) \leqslant w(t_1)$ for all $s \in [-\tau, 0]$. Setting $\varphi = x_{t_1}^n$ and $\psi = x_{t_1}^m$ we see $\|\varphi(s) - \psi(s)\| \leqslant \|\varphi(0) - \psi(0)\|$ for all $s \in [-\tau, 0]$ and so $\varphi, \psi \in \Omega$. Let i, j be the integers such that $t \in (t_{i-1}^n, t_i^n) \cap (t_{j-1}^m, t_j^m)$.

For h sufficiently close to zero, $h < 0$ we have

$$w(t_1+h)-w(t_1) = \|x^n(t_1+h)-x^m(t_1+h)\| - \|x^n(t_1)-x^m(t_1)\|$$
$$= \|x^n(t_1)-x^m(t_1)+h\{f(t_1, x^n_{t_1})-f(t_1, x^m_{t_1})\}\| - \|x^n(t_1)-x^m(t_1)\| +$$
$$+\|x^n(t_1+h)-x^m(t_1+h)\| - \|x^n(t_1)-x^m(t_1)+h\{f(t_1, x^n_{t_1})-f(t_1, x^m_{t_1})\}\|$$
$$\geqslant \|x^n(t_1)-x^m(t_1)+h\{f(t_1, x^n_{t_1})-f(t_1, x^m_{t_1})\}\| - \|x^n(t_1)-x^m(t_1)\| -$$
$$-\|[x^n(t_1+h)-x^n(t_1)-hf(t_1, x^n_{t_1})]-[x^m(t_1+h)-x^m(t_1)-hf(t_1, x^m_{t_1})]\|$$
$$\geqslant \|x^n(t_1)-x^m(t_1)+h\{f(t_1, x^n_{t_1})-f(t, x^m_{t_1})\}\| - \|x^n(t_1)-x^m(t_1)\| -$$
$$-\|x^n(t_1+h)-x^n(t_1)-hf(t_1, x^n_{t_1})\| - \|x^m(t_1+h)-x^m(t_1)-hf(t_1, x^m_{t_1})\|.$$

Thus

$$\frac{w(t_1+h)-w(t_1)}{h} \leqslant \frac{1}{h}[\|x^n(t_1)-x^m(t_1)+h\{f(t_1, x^n_{t_1})-f(t_1, x^m_{t_1})\}\| -$$
$$-\|x^n(t_1)-x^m(t_1)\|]+$$
$$+\left\|\frac{x^n(t_1+h)-x^n(t_1)}{h}-f(t_1, x^n_{t_1})\right\| + \left\|\frac{x^m(t_1+h)-x^m(t_1)}{h}-f(t_1, x^m_{t_1})\right\|,$$

and so

$$\frac{w(t_1+h)-w(t_1)}{h} \leqslant \frac{1}{h}[\|x^n(t_1)-x^m(t_1)+h\{f(t_1, x^n_{t_1})-f(t_1, x^m_{t_1})\}\| -$$
$$-\|x^n(t_1)-x^m(t_1)\|]+$$
$$+\left\|\frac{x^n(t_1+h)-x^n(t_1)}{h}-(x^n)'(t_1)\right\| + \|(x^n)'(t_1)-f(t_1, x^n_{t_1})\| +$$
$$+\left\|\frac{x^m(t_1+h)-x^m(t_1)}{h}-(x^m)'(t_1)\right\| + \|(x^m)'(t_1)-f(t_1, x^m_{t_1})\|.$$

Consequently applying hypothesis (H_2) we obtain

$$D_-w(t_1) \leqslant g(t_1, w(t_1)) + \|(x^n)'(t_1)-f(t_1, x^n_{t_1})\| + \|(x^m)'(t_1)-f(t_1, x^m_{t_1})\|.$$

By part (iii) of Lemma 3,

$$\|x^n_{t_{i-1}}-x^n_{t_1}\|_0 \leqslant M|t_1-t_{i-1}| \leqslant M(t_i-t_{i-1})$$

and so using parts (iv) and (v) of Lemma 3

$$\|(x^n)'(t_1)-f(t_1, x^n_{t_1})\| \leqslant \frac{1}{n}$$

and similarly

$$\|(x^m)'(t_1)-f(t_1, x^m_{t_1})\| \leqslant \frac{1}{m}$$

and we obtain the inequality

$$D_- w(t_1) \leqslant g(t_1, w(t_1)) + \frac{1}{n} + \frac{1}{m}.$$

Applying Lemma 1 and noting $w_{t_0} \equiv 0$, we have

$$w(t) \leqslant r_{n,m}\left(t, t_0, \frac{1}{n} + \frac{1}{m}\right) \quad \text{for each } t \in [t_0, t_0 + a],$$

where $r_{n,m}\left(t, t_0, \frac{1}{n} + \frac{1}{m}\right)$ is the maximal solution of

$$u' = g(t, u) + \frac{1}{n} + \frac{1}{m}, \quad u(t_0) = \frac{1}{n} + \frac{1}{m}.$$

By Lemma 2, $\lim_{m,n\to\infty} r_{m,n}\left(t, t_0, \frac{1}{n} + \frac{1}{m}\right) = r(t, t_0, 0)$ uniformly on $[t_0, t_0 + a]$, where $r(t, t_0, 0)$ is the maximal solution of $u' = g(t, u)$, $u(t_0) = 0$ which is identically zero by (H_3). This implies $\{x^n(t)\}$ is Cauchy and since the sequence is equicontinuous we know $\lim_{n\to\infty} x_n(t) = x(t)$ uniformly on $[t_0 - \tau, t_0 + a]$.

3. Proof of the main results. We now have the tools necessary to prove Theorem 1.

Proof of Theorem 1. As a consequence of Lemmas 3, 4, and 5, it is clear that the initial value problem (1.1) and (1.2) has a local solution for each given initial function $\varphi_0 \in E_0$. To see uniqueness suppose $x(t, t_0, \varphi_0)$ and $y(t, t_0, \varphi_0)$ are both solutions of (1.1) and (1.2) existing on $[t_0, t_0 + a]$ and define $m(t) = \|x(t, t_0, \varphi_0) - y(t, t_0, \varphi_0)\|$. Notice $m_t = \|x_t(t_0, \varphi_0) - y_t(t_0, \varphi_0)\|$ and if $t_1 \in [t_0, t_0 + a]$ such that $m_{t_1}(s) \leqslant m(t_1)$ for all $s \in [-\tau, 0]$, then we can let $\varphi = x_{t_1}$ and $\psi = x_{t_1}$. Thus $\|\varphi(s) - \psi(s)\| \leqslant \|\varphi(0) - \psi(0)\|$ for all $s \in [-\tau, 0]$ so $\varphi, \psi \in \Omega$. Using the same technique as used in Lemma 5 we obtain

$$D_- m(t_1) \leqslant g(t_1, m(t_1)).$$

Applying Lemma 1 we get as before

$$m(t) \leqslant r(t, t_0, 0) \quad \text{for all } t \in [t_0 - \tau, t_0 + a],$$

where $r(t, t_0, 0)$ is a maximal solution of $u' = g(t, u)$, $u(t_0) = 0$. Thus by (H_3) we have $x(t, t_0, \varphi_0) = y(t, t_0, \varphi_0)$ for all $t \in [t_0 - \tau, t_0 + a]$.

Proof of Theorem 2. Theorem 1 assures for local existence and if global existence is shown the same proof assures global uniqueness. By local existence there is a $T > t_0$ such that $x(t, t_0, \varphi_0)$ exists on $[t_0 - \tau, T)$. Suppose $T < \infty$ and $\varepsilon > 0$ then by (H_4) there exists $\delta(T, \varepsilon) > 0$ such that

$\|f(t, \varphi)-f(s, \varphi)\| < \varepsilon$ whenever $(t, \varphi), (s, \varphi) \in [t_0, T]\times S_b(\varphi_0)$ and $|t-s| < \delta$. Choosing $0 < h < \delta(T, \varepsilon)$ and such that $t_0 < T-h$. Defining

$$m(t) = \|x(t+h, t_0, \varphi_0) - x(t, t_0, \varphi_0)\| \quad \text{for } t \in [t_0, T-h)$$

we see $m_t = \|x_{t+h}(t_0, \varphi_0) - x_t(t_0, \varphi_0)\|$. If for some $t_1 \in (t_0, T-h)$ we have $m_{t_1}(s) \leqslant m(t_1)$ for all $s \in [-\tau, 0]$, then setting $\varphi = x_{t_1+h}(t_0, \varphi_0)$, $\psi = x_{t_1}(t_0, \psi_0)$ we see $\varphi, \psi \in \Omega$. Proceeding as in Lemma 5 we arrive at

$$D_- m(t_1) \leqslant g(t_1, m(t_1)) + \|f(t_1, x_{t_1+h}(t_0, \varphi_0)) - f(t_1+h, x_{t_1+h}(t_0, \varphi_0))\|$$
$$\leqslant g_t'(t_1, m(t_1)) + \varepsilon, \quad \text{by } (\mathrm{H}_4).$$

Lemma 1 now yields

$$(3.1) \qquad \|x(t+h, t_0, \varphi_0) - x(t, t_0, \varphi_0)\| \leqslant r(t, t_0, \|x_{t_0+h}(t_0, \varphi_0) - \varphi_0\|_0 + \varepsilon)$$

for all $t \in [t_0, T-h)$, where $r(t, t_0, \varphi_0)$ is the maximal solution of $u' = g(t, u) + \varepsilon$, $u(t_0) = \|x_{t_0+h}(t_0, \varphi_0) - \varphi_0\|_0 + \varepsilon = y_0$.

As $t+h$ and t tend to T, h tends to zero and consequently by Lemma 2, (H_3) and (H_4), $\lim_{u_0\to 0} r(t, t_0, u_0) \equiv 0$ uniformly on every compact subinterval of $[t_0, \infty)$. Since $\lim_{\varepsilon > h\to 0} [\|x_{t_0+h}(t_0, \varphi_0) - \varphi_0\|_0 + \varepsilon] = 0$ it follows from (3.1) that $x(t, t_0, \varphi_0)$ tends to a limit as $t \to T$ and this is enough to define $x(t, t_0, \varphi_0)$ for all $t \in [t_0 - \tau, \infty)$.

References

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