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Mean growth and Taylor coefficients of some classes of functions

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Abstract. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a holomorphic function in the unit disk $\{|z| < 1\}$. We put

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \qquad 0 < p < \infty,$$

$$M_\infty(r, f) = \max_{|z|=r} |f(z)|.$$

A holomorphic function $f(z)$ is said to belong to the class N^+ if $\log^+|f|$ has a harmonic majorant represented by Poisson integral ([1], p. 25. Priwalow denotes the same class as D. See [6], p. 82.) Then we have $H^p \subset N^+ \subset N$, where N is the Nevanlinna class of functions of bounded characteristic.

For functions of H^p or N, growths of $M_q(r, f)$ as $r \to 1$ $(q > p)$ and of a_n as $n \to \infty$ are studied by several authors. We give here corresponding results for the class N^+.

Results obtained are:

1° $M_q(r, f) = O\left(\exp\left[\frac{o(1)}{1-r}\right]\right)$ as $r \to 1$ for $0 < q < \infty$.

2° $a_n = O(\exp[o(\sqrt{n})])$ as $n \to \infty$.

3° Let $\omega(r)$, $0 \leqslant r < 1$, be any continuous function such that $\omega(r) \downarrow 0$ as $r \to 1$. Then, there is a function $f(z) \in N^+$ such that

$$M_q(r, f) \neq O\left(\exp\left[\frac{\omega(r)}{1-r}\right]\right).$$

4° Let $\{\delta_n\}$ be any positive sequence such that $\delta_n \downarrow 0$ as $n \to \infty$. Then there is a function $f(z) \in N^+$ whose Taylor coefficients satisfy

$$a_n \neq O(\exp[\delta_n \sqrt{n}]).$$

3° and 4° show that the limitations in 1° and 2° are exact in a strong sense.

We follow, for proving 3° and 4°, to the saddle point method of W. K. Hayman, Acta Math. 112 (1964), p. 181-214.

Readers are recommended to consult Duren's book, p. 84 and 98 for results concerning H^p, and Priwalow's book, p. 106-108, concerning the class N.

1. Introduction. Let D be the unit disk $\{|z|<1\}$. For a holomorphic function $f(z)$ in D, we write

$$M_p(r,f)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p d\theta\right)^{1/p},\qquad 0<p<\infty,$$

$$M_\infty(r,f)=\max_{|z|=r}|f(z)|,$$

$$M_0(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta.$$

$M_0(r,f)$ is usually denoted as $T(r,f)$ and called the *Nevanlinna characteristic* of $f(z)$.

For $0<p\leqslant\infty$, a holomorphic function $f(z)$ is said to belong to the *Hardy class* H^p if $M_p(r,f)=O(1)$ as $r\to1$.

A holomorphic function $f(z)$ is said to belong to the *class* N of functions of bounded characteristic if $M_0(r,f)=O(1)$ as $r\to1$.

A function $f(z)\in N$ is said to belong to the *class* N^+ if $\log^+|f(z)|$ has a harmonic majorant represented by the Poisson integral. $f(z)\in N^+$ is factorized as follows [1], p. 25:

$$f(z)=B(z;f)S(z;f)\Phi(z;f), \tag{1.1}$$

where $B(z;f)$ is the Blaschke product relative to the zero points of $f(z)$, $S(z;f)$ is a *singular inner function*, i.e.,

$$S(z;f)=\exp\left[-\int\frac{e^{it}+z}{e^{it}-z}\,d\mu_f(t)\right].$$

with a positive singular measure $d\mu_f$, and $\Phi(z;f)$ is an *outer function for the class* N, i.e.,

$$\Phi(z;f)=\exp\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(\theta)|\,d\theta\right]$$

with a summable function $\log|f(\theta)|$, $|f(\theta)|=\lim_{r\to1}|f(re^{i\theta})|$ for almost every θ, $0\leqslant\theta<2\pi$.

Since for $0<p<q\leqslant\infty$

$$[p\times M_0(r,f)]^{1/p}\leqslant M_p(r,f)\leqslant M_q(r,f)\leqslant M_\infty(r,f), \tag{1.2}$$

we have

$$\bigcup H^p\subset N^+\subset N,$$

and these inclusion relations are proper [6], p. 82.

Hardy and Littlewood [4], [5] proved that $f(z) \in H^p$ implies

$$M_q(r,f) = o\left((1-r)^{1/q-1/p}\right), \quad 0 < p < q \leqslant \infty, \tag{1.3}$$

and they pointed out that the exponent $(1/q-1/p)$ is best possible. Duren and Taylor [2] showed that the estimate (1.3) is best possible in a stronger sense.

Hardy and Littlewood [5] proved also that if $f(z) = \sum a_n z^n \in H^p$, $0 < p \leqslant 1$, then

$$a_n = o(n^{1/p-1}) \tag{1.4}$$

and that the exponent $(1/p-1)$ in (1.4) is best possible. It was shown [2], [3], that the estimate (1.4) cannot be improved at all.

We consider here corresponding problems for $p = 0$, i.e., for functions of the class N or N^+.

It is well known that

$$\log M_\infty(r,f) = O\left(\frac{1}{1-r}\right) \quad \text{if } f(z) \in N. \tag{1.5}$$

The estimate (1.5) is best possible, as seen from the trivial example

$$f(z) = \exp\left[c\frac{1+z}{1-z}\right], \quad c > 0, \tag{1.6}$$

S. N. Mergelyan showed that if $f(z) = \sum a_n z^n \in N$, then

$$\log|a_n| = O(\sqrt{n}) \tag{1.7}$$

and that the estimate (1.7) is best possible, using example (1.6), [6], p. 106.

For functions of the class N^+, we shall prove in this note the

THEOREM 1. *Let $f(z) \in N^+$. Then*

$$\log M_p(r,f) = o\left(\frac{1}{1-r}\right), \quad 0 < p \leqslant \infty. \tag{1.8}$$

THEOREM 2. *Let $f(z) = \sum a_n z^n \in N^+$. Then*

$$\log|a_n| = o(\sqrt{n}). \tag{1.9}$$

THEOREM 3. *Let $0 < p \leqslant \infty$, and let $\omega(r)$ be an arbitrary positive, continuous, non-increasing function on $0 \leqslant r < 1$, with $\omega(r) \downarrow 0$ as $r \uparrow 1$. Then there exists a function $f(z) \in N^+$ such that*

$$\log M_p(r,f) \neq O\left(\frac{\omega(r)}{1-r}\right). \tag{1.10}$$

THEOREM 4. *Let $\{\delta_n\}$ be an arbitrary sequence of positive numbers tending monotonically to 0. Then there exists a function $f(z) = \sum a_n z^n \in N^+$ such that*

$$\log|a_n| \neq O(\delta_n\sqrt{n}). \tag{1.11}$$

2. Proofs of Theorems 1 and 2. Let $u(r, \theta)$ be a harmonic majorant of $\log^+|f(z)|$, represented by the Poisson integral of a boundary function $h(\varphi) \geqslant 0$. Then

$$\log M_\infty(r,f) \leqslant \max_{0\leqslant\theta<2\pi} u(r, \theta). \tag{2.1}$$

Take a number $\varepsilon > 0$. Let K be a sufficiently large positive number so that, for $h^K(\varphi) = \min(K, h(\varphi))$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} \big(h(\varphi) - h^K(\varphi)\big)d\varphi < \varepsilon. \tag{2.2}$$

Then

$$u(r, \theta) = \int_0^{2\pi} P(r, \theta; \varphi)h^K(\varphi)d\varphi + \int_0^{2\pi} P(r, \theta; \varphi)\big(h(\varphi) - h^K(\varphi)\big)d\varphi$$

$$= u_0(r, \theta) + u_1(r, \theta),$$

where

$$P(r, \theta; \varphi) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}.$$

Since $0 \leqslant h^K(\varphi) \leqslant K$, we have

$$0 \leqslant u_0(r, \theta) \leqslant K.$$

On the other hand, as easily seen,

$$u_1(r, \theta) \leqslant \frac{1}{2\pi}\frac{1+r}{1-r}\int_0^{2\pi} \big(h(\varphi) - h^K(\varphi)\big)d\varphi \leqslant \frac{2\varepsilon}{1-r}.$$

Thus

$$u(r, \theta) \leqslant K + \frac{2\varepsilon}{1-r}.$$

Hence

$$\overline{\lim_{r\to1}}(1-r)\big(\max_{0\leqslant\theta<2\pi} u(r, \theta)\big) \leqslant 2\varepsilon. \tag{2.3}$$

As $\varepsilon > 0$ is arbitrary, we have our Theorem 1, using inequalities (2.1) and (1.2).

Next we prove Theorem 2, using the method of Mergelyan [6], p. 106.

As is well known,

$$|a_n| \leqslant \inf_{0<r<1} \left(r^{-n} M_\infty(r, f)\right). \tag{2.4}$$

For any $\varepsilon > 0$, there is a number $r_0 = r_0(\varepsilon)$, $0 < r_0 < 1$, such that

$$M_\infty(r, f) < \exp\left[\frac{\varepsilon}{1-r}\right] \quad \text{for } r \geqslant r_0. \tag{2.5}$$

Put

$$g_n(r) = g_n(r; \varepsilon) = r^{-n}\exp\left[\frac{\varepsilon}{1-r}\right]. \tag{2.6}$$

Then there holds

$$|a_n| \leqslant g_n(r) \quad \text{for } r \geqslant r_0. \tag{2.7}$$

We wish to seek the minimum value of $g_n(r)$ for $r \geqslant r_0$.

Since $g_n'(r)/g_n(r) = \varepsilon(1-r)^{-2} - n/r$, we have for the root $r = r_n$ of the equation $g_n' = 0$,

$$r_n = 1 - \sqrt{\varepsilon/n}\left(1 + o(1)\right). \tag{2.8}$$

Thus $r_n \geqslant r_0$ if n is sufficiently large. Substituting (2.8) into (2.6), we obtain by an easy calculation

$$|a_n| \leqslant e^{2\sqrt{n\varepsilon}(1+o(1))},$$

which proves Theorem 2.

Now we turn to the proofs of Theorems 3 and 4, by means of constructions of examples.

3. Proof of Theorem 3.

3.1. Construction of the example. We can suppose $\frac{1}{2} \leqslant \omega(0) \leqslant 1$. Let ϱ_n, $0 < \varrho_n < \varrho_{n+1} < 1$, $n \geqslant 1$, be numbers such that

$$\frac{1}{\log 4}\log\frac{1}{\omega(\varrho_n)} = n. \tag{3.1.1}$$

We define a function $\Omega(s)$, $0 \leqslant s < \infty$, as follows:

(3.1.2′) $\quad \Omega(n) = 20^n/(1-\varrho_n)(1-\varrho_{n+1}) \quad \text{for } s = n,$

(3.1.2) $\quad \Omega(s) = \Omega(n) + \left(\Omega(n+1) - \Omega(n)\right)(s-n) \quad \text{for } n \leqslant s \leqslant n+1.$

Then we have that

(3.1.3) $\Omega(s)$ is positive, continuous, increasing, and

$$\Omega\left(\frac{1}{\log 4}\frac{1}{\omega(r)}\right)\Big/\left(\frac{1}{1-r}\right)\to\infty \quad \text{as } r\to 1.$$

In fact, there holds for $\varrho_n\leqslant r\leqslant\varrho_{n+1}$,

$$\begin{aligned}\Omega\left(\frac{1}{\log 4}\log\frac{1}{\omega(r)}\right)\Big/\left(\frac{1}{1-r}\right)&\geqslant\Omega\left(\frac{1}{\log 4}\log\frac{1}{\omega(\varrho_n)}\right)\Big/\left(\frac{1}{1-\varrho_{n+1}}\right)\\&=\frac{20^n}{(1-\varrho_n)(1-\varrho_{n+1})}(1-\varrho_{n+1})=\frac{20^n}{(1-\varrho_n)}\\&\geqslant 20^n\to\infty \quad \text{as } n\to\infty,\ r\to 1.\end{aligned}$$

Moreover, we have obviously

$$\frac{1}{20}>\frac{\Omega(n)}{\Omega(n+1)}. \tag{3.1.4}$$

Put

$$b_n=\Omega(n)/\Omega(n+1),\quad n\geqslant 1, \tag{3.1.5}$$

and

$$c_n=b_1b_2\dots b_n=\Omega(1)/\Omega(n+1)\downarrow 0 \quad \text{as } n\to\infty. \tag{3.1.6}$$

We will define sequences of intervals $\{I_{n,\nu}\}$, $\{I^*_{n,\nu}\}$, $n=0,1,\dots$, $\nu=1,2,\dots,2^n$, as follows:

(i) $I_{0,1}=[0,1]$,

(ii) $I_{1,1}=[0,c_1]$, $\quad I_{1,2}=[1-c_1,1]$;

$I^*_{1,1}=[3c_1, 3\frac{1}{2}\times c_1]$, $\quad I^*_{1,2}=[1-3\frac{1}{2}\times c_1, 1-3c_1]$.

(iii) Suppose $I_{n,\nu}$, $\nu=1,2,\dots,2^n$, be defined so that the length of $I_{n,\nu}$ for each ν equals to c_n. If, for a ν, $I_{n,\nu}=[S,T]$, $0<S<T<1$, $T-S=c_n$, we define

$$\begin{aligned}I_{n+1,2\nu-1}&=[S, S+c_{n+1}],\\I_{n+1,2\nu}&=[T-c_{n+1}, T];\\I^*_{n+1,2\nu-1}&=[S+3c_{n+1}, S+3\tfrac{1}{2}\times c_{n+1}],\\I^*_{n+1,2\nu}&=[T-3\tfrac{1}{2}c_{n+1}, T-3c_{n+1}].\end{aligned} \tag{3.1.7}$$

Thus the construction proceeds inductively.

Let $k(t)$ be a function defined on $[0,1]$ as follows:

(i) $k(t)=0$ for $t\notin\bigcup_{n=1}^{\infty}\bigcup_{\nu=1}^{2^n}I^*_{n,\nu}$,

(ii) $k(t) = k_n$ on each $I^*_{n,\nu}$, $\nu = 1, \ldots, 2^n$, where k_n is a constant independent of ν.

$$\int_{I^*_{n,\nu}} k(t)\,dt = 2^{-n} q_1 q_2 \ldots q_n (q_n^{-1} - 1), \quad n \geqslant 1, \tag{3.1.8}$$
$$= 2^{-n} q_1 q_2 \ldots q_{n-1} (1 - q_n), \quad n \geqslant 2,$$

where

$$q_n = \left(\frac{58+n}{59+n}\right)^\beta, \quad n \geqslant 1, \tag{3.1.9}$$

in which β is a number such that $0 < \beta < 1$.

Then

$$\tfrac{59}{60} < q_n \uparrow 1, \quad q_1 q_2 \ldots q_n \downarrow 0, \quad \text{as } n \uparrow \infty, \tag{3.1.10}$$

and

$$\int_{I_{n,\nu}} k(t)\,dt = 2^{-n} q_1 q_2 \ldots q_n. \tag{3.1.11}$$

We put

$$f(z) = \exp\left[\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} k(t)\,dt\right]. \tag{3.1.12}$$

$f(z)$ can be easily seen to be a function of the class N^+.

Now we will estimate, from below, the mean growth $M_p(r, F)$ of the integral $F(z) = \int_0^z f(z)\,dz$ of $f(z)$. If this done, we can obtain the estimate from below of $M_p(r, f)$, as follows:

Put

$$f_1(r, \theta) = \sup_{0 \leqslant t \leqslant r} |f(te^{i\theta})|.$$

Then we have

$$|F(re^{i\theta})| = \left|\int_0^r f(te^{i\theta}) e^{i\theta}\,dt\right| \leqslant f_1(r, \theta) \tag{3.1.13}$$

and, by the maximal theorem of Hardy–Littlewood [7], p. 186, Theorem IV. 40,

$$\int_0^{2\pi} f_1(r, \theta)^p\,d\theta \leqslant A \int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta, \quad p > 0, \tag{3.1.14}$$

for an absolute constant A. Thus, from (3.1.13) and (3.1.14), we obtain

$$M_p(r, F) \leqslant A^{1/p} M_p(r, f), \tag{3.1.15}$$

which gives the required result.

We proceed to estimate $M_p(r, F)$ by the saddle point method used in [8].

3.2. Proof of Theorem 3. Fix numbers N and ν. If $I_{N,\nu} = [S, T]$, write

(3.2.1)
$$\theta_0 = (S+T)/2, \qquad \Delta = \Delta(N) = c_N/2,$$
$$a_0 = \int_{I_{N,\nu}} k(t)\,dt = q_1 q_2 \ldots q_N/2^N.$$

We will define sequences of intervals $\{J_m\}$, $\{J'_m\}$, $\{J^*_m\}$, $\{J^{**}_m\}$, $m = 0, 1, \ldots$ $\ldots, N-1$, satisfying condition (3.2.2) below, as follows:

Put
$$J_0 = J_{N,\nu}.$$

If $\nu = 2\mu$, we set
$$J'_0 = I_{N,2\mu-1} = I_{N,\nu-1},$$
$$J_1 = I_{N-1,\mu} = I_{N-1,\nu/2},$$
$$J^*_0 = I^*_{N,2\mu} = I^*_{N,\nu},$$
$$J^{**}_0 = I^*_{N,2\mu-1} = I^*_{N,\nu-1}.$$

If $\nu = 2\mu-1$, we set
$$J'_0 = I_{N,2\mu}, = I_{N,\nu+1},$$
$$J_1 = I_{N-1,\mu} = I_{N-1,(\nu+1)/2},$$
$$J^*_0 = I^*_{N,2\mu-1} = I^*_{N,\nu},$$
$$J^{**}_0 = I^*_{N,2\mu} = I^*_{N,\nu+1}.$$

Suppose J_m be defined so as to satisfy the condition

(3.2.2) $\quad J_m = I_{N-m,\varkappa}$ for a suitable number $\varkappa$, $0 \leqslant \varkappa \leqslant 2^{N-m}$.

Then we set, if $\varkappa = 2\lambda$,
$$J'_m = I_{N-m,2\lambda-1} = I_{N-m,\varkappa-1},$$
$$J_{m+1} = I_{N-m-1,\lambda} = I_{N-m-1,\varkappa/2},$$
$$J^*_m = I^*_{N-m,2\lambda} = I^*_{N-m,\varkappa},$$
$$J^{**}_m = I^*_{N-m,2\lambda-1} = I^*_{N-m,\varkappa-1}.$$

If $\varkappa = 2\lambda - 1$, we set

$$J'_m = I_{N-m,2\lambda} = I_{N-m,\varkappa+1},$$
$$J_{m+1} = I_{N-m-1,\lambda} = I_{N-m-1,(\varkappa+1)/2},$$
$$J^*_m = I^*_{N-m,2\lambda-1} = I^*_{N-m,\varkappa},$$
$$J^{**}_m = I^*_{N-m,2\lambda} = I^*_{N-m,\varkappa+1}.$$

Thus J_m, J'_m, J^*_m, and J^{**}_m are defined inductively up to $m = N-1$.

If $t \in J'_m$ and $t' \in J_m$, we have from (3.1.6) and (3.1.5),

$$|t-t'| \geqslant c_{N-m-1} - 2c_{N-m} = b_1 \dots b_{N-m}(b^{-1}_{N-m} - 2) \tag{3.2.3}$$
$$= 2\varDelta(b_{N-m+1} \dots b_N)^{-1}(b^{-1}_{N-m} - 2)$$
$$\geqslant 2 \times 18 \times 20^m \varDelta = 36\varDelta \times 20^m,$$

since $b_k^{-1} \geqslant 20$, $k \geqslant 1$, as seen from (3.1.4) and (3.1.5).

If $t \in J^*_m$ and $t' \in J_m$,

$$|t-t'| \geqslant 2c_{N-m} = 2\varDelta \times (b_{N-m+1} \dots b_N)^{-1} \geqslant 4\varDelta \times 20^m, \tag{3.2.4}$$

and if $t \in J^{**}_m$ and $t' \in J_m$,

$$|t-t'| \geqslant c_{N-m-1} - 4.5c_{N-m} = b_1 \dots b_{N-m}(b^{-1}_{N-m} - 4.5) \tag{3.2.5}$$
$$\geqslant 2 \times 15.5 \times 20^m = 31\varDelta \times 20^m.$$

These inequalities (3.2.3)–(3.2.5) correspond to (4.7)–(4.9) of [8], hence the arguments in [8] can be applied here, and we get for a number $\delta, 0 < \delta < 0.2$,

$$|F(re^{i\theta})| > \exp[(1.01 - \delta)a_0/\varDelta] \tag{3.2.6}$$

if

$$r \geqslant 1 - \delta^2\varDelta/50, \quad |\theta - \theta_0| < 0.6(1-\delta)\varDelta, \tag{3.2.7}$$

as seen from (6.6) in [8]. Accordingly,

$$\int_{\theta_0-0.6(1-\delta)\varDelta}^{\theta_0+0.6(1-\delta)\varDelta} |F(re^{i\theta})|^p d\theta > \frac{6\varDelta(1-\delta)}{5} \exp[p(1.01-\delta)a_0/\varDelta]. \tag{3.2.8}$$

There are just $2^N = a_0^{-1}q_1 \dots q_N$ different values of θ_0 for a fixed N, and their total contributions are therefore at least

$$1.2(1-\delta)q_1 \dots q_N \frac{\varDelta}{a_0} \exp[p(1.01-\delta)a_0/\varDelta] \tag{3.2.9}$$
$$\geqslant 1.2(1-\delta)\exp[p(1.01-2\delta)a_0/\varDelta]$$

for $N \geqslant A(\delta)$, where $A(\delta)$ is a constant depending only on δ. Hence we have, if r satisfies (3.2.7),

$$M_p(r, F) \geqslant B\exp[(1.01-2\delta)a_0/\Delta] \tag{3.2.10}$$

for a constant $B = B(\delta)$.

If we take

$$1-r = \delta^2 \Delta/50, \quad \frac{1}{\Delta} = \frac{\delta^2}{50}\frac{1}{1-r}, \tag{3.2.11}$$

then, since $\Delta = c_N/2 = \Omega(1)/2\Omega(N+1)$, we have from (3.2.11),

$$\begin{aligned} \Omega(N+1) &= \Omega(1)/2\Delta = (\delta/10)^2\Omega(1)(1-r)^{-1}, \\ \Omega(N+1)\Big/\left(\frac{1}{1-r}\right) &= (\delta/10)^2\Omega(1). \end{aligned} \tag{3.2.12}$$

If N is sufficiently large such that corresponding r in (3.2.11) is near 1 so as to satisfy

$$\Omega\left(\frac{1}{\log 4}\log\frac{1}{\omega(r)}\right)\Big/\left(\frac{1}{1-r}\right) > \frac{\delta^2}{100} \times \Omega(1), \tag{3.2.13}$$

we obtain, by the monotonicity of $\Omega(s)$, from (3.2.12) and (3.2.13),

$$N < N+1 \leqslant \frac{1}{\log 4}\log\frac{1}{\omega(r)}, \quad \omega(r) < 1/4^N. \tag{3.2.14}$$

Now

$$\begin{aligned} a_0 = q_1 \cdots q_N/2^N &\geqslant (\tfrac{59}{60})^N/2^N > 1/3^N = (4/3)^N \times (1/4^N) \\ &\geqslant \Psi(r)\omega(r), \end{aligned} \tag{3.2.15}$$

where $\Psi(r)$ is a function defined as follows: If we write $r_N = 1-\delta^2\Delta/50$, $\Delta = \Delta(N) = c_N/2$,

$\Psi(r) = (4/3)^N \quad$ for $r = r_N$,

$\Psi(r) = \big(\Psi(r_{N+1})(r-r_N) + \Psi(r_N)(r_{N+1}-r)\big)/(r_{N+1}-r_N) \quad$ for $r_N \leqslant r \leqslant r_{N+1}$.

Then, obviously

$$\Psi(r) \to \infty \quad \text{as } r \to 1. \tag{3.2.16}$$

Hence we obtain, from (3.2.11) and (3.2.15),

$$a_0/\Delta \geqslant (\delta^2/50) \times \Psi(r) \times \frac{\omega(r)}{1-r}. \tag{3.2.17}$$

Thus we get, from (3.2.10), (3.2.17) and (3.2.16),

$$\overline{\lim_{r\to 1}} \log M_p(r, F) \Big/ \left(\frac{\omega(r)}{1-r}\right) = \infty,$$

which proves our Theorem 3, as stated in connection with (3.1.15).

4. Proof of Theorem 4. We can suppose $\delta_n \leqslant 1$, $n = 0, 1, \ldots$ Put

$$a_n' = \exp[\delta_n \sqrt{n}] \quad \text{and} \quad g(r) = \sum_{n=0}^{\infty} a_n' r^n. \tag{4.1}$$

For each r, $0 < r < 1$, let $\nu(r)$ be the least number such that

$$e^{1/\sqrt{n}} \times \sqrt{r} < 1 \quad \text{for } n \geqslant \nu(r)+1. \tag{4.2}$$

Then

$$\nu(r) \leqslant \left(\frac{2}{\log(1/r)}\right)^2 \leqslant \left(\frac{4}{1-r}\right)^2, \tag{4.3}$$

and

$$\begin{aligned} g(r) &\leqslant \sum_{n=0}^{\nu(r)} e^{\delta_n\sqrt{n}} + \sum_{\nu(r)+1}^{\infty} (e^{1/\sqrt{n}}\sqrt{r})^n(\sqrt{r})^n \\ &\leqslant \nu(r)\exp[\delta_{\nu(r)}\sqrt{\nu(r)}] + (1-\sqrt{r})^{-1} \\ &\leqslant \exp[2\delta_{\nu(r)}\sqrt{\nu(r)}] \leqslant \exp[8\delta_{\nu(r)}/(1-r)] \end{aligned} \tag{4.4}$$

for $r \geqslant r_0$, where r_0 is a suitable constant.

Let $r = \sigma_n$ be the least number such that $\nu(r) = n$. We define a function $\omega(r)$ as follows:

$$\begin{aligned} &\omega(r) = 8\delta_{n-1} && \text{for } r = \sigma_n, \\ &\omega(r) = \big(8\delta_{n-1}(\sigma_{n+1}-r) + 8\delta_n(r-\sigma_n)\big)/(\sigma_{n+1}-\sigma_n') && \\ & && \text{for } \sigma_n \leqslant r \leqslant \sigma_{n+1}. \end{aligned} \tag{4.5}$$

Then $\omega(r)$ is continuous, non-increasing, and $\omega(r)\downarrow 0$ as $r\uparrow 1$, and

$$g(r) \leqslant \exp\left[\frac{\omega(r)}{1-r}\right]. \tag{4.6}$$

By Theorem 3, there is a function $f(z) = \sum a_n z^n \in N^+$, constructed as in (3.1.12), such that

$$M_p(r, f) \neq O\left(\exp\left[\frac{\omega(r)}{1-r}\right]\right) \quad \text{for the } \omega(r) \text{ in (4.5).} \tag{4.7}$$

From (4.7) we have, obviously,

$$a_n \neq O(a_n'), \quad \text{i.e.,} \quad a_n \neq O(\exp[\delta_n\sqrt{n}]),$$

which is easily seen to be equivalent to Theorem 4.

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