ANNALES
POLONICI MATHEMATICI
XLIII (1983)

On approximate solutions of a system of functional equations

by KAROL BARON (Katowice)

Abstract. Assume that S is a set, (X, ϱ) is a metric space and $(Y_i, \| \|_i)$, $1 \leqslant i \leqslant m$, are normed spaces, where m is a positive integer. Given a function f mapping $S \times X$ into X and functions h_i, $1 \leqslant i \leqslant m$, mapping subsets of the set $X \times Y_1^S \times \ldots \times Y_m^S$ into Y_i, respectively, the problem is considered, under what conditions, for every $\varepsilon > 0$, there exist Lipschitz continuous functions φ_i from X into Y_i, $1 \leqslant i \leqslant m$, such that

$$\|\varphi_i(x) - h_i(x, \varphi_1 \circ f(\cdot, x), \ldots, \varphi_m \circ f(\cdot, x))\|_i \leqslant \varepsilon$$

for all $i \in \{1, \ldots, m\}$ and $x \in X$.

Here we shall continue a study, initiated by R. C. Buck, of approximate solutions of functional equations (cf. [1], Chapter VI of [2], [4] and [5]).

In the whole paper we shall denote the set of all functions mapping a set X into a set Y by $\mathscr{F}(X, Y)$ and by $\mathscr{C}(X, Y)$ we shall denote the set of all continuous functions mapping a topological space X into a topological space Y. The set of all positive integers will be denoted by N, whereas m will stand for the set $N \cap [1, m]$, where m is a fixed positive integer. Moreover, the components of vectors will be indicated by lower indices and we shall write Φ to denote the set $\Phi_1 \times \ldots \times \Phi_m$ whenever a suitable m-tuple of sets $(\Phi_1, \ldots, \Phi_m)$ will be given.

Assume that

(i) (X, ϱ) is a compact metric space, whereas $(Y_i, \| \|_i)$, $i \in \mathrm{m}$, are finite dimensional Banach spaces.

(ii) Φ_i, $i \in \mathrm{m}$, is a subset of $\mathscr{F}(X, Y_i)$ closed under uniform convergence, containing the zero function and such that $(0, 1) \cdot \Phi_i \subset \Phi_i$.

(iii) S is a non-void set and $f: S \times X \to X$ is a function fulfilling the condition

$$\bigwedge_{x \in X} \bigvee_{n \in N} \bigvee_{s_1, \ldots, s_n \in S} (f(s_1, \cdot) \circ \ldots \circ f(s_n, \cdot)(x) \in U)$$

for every neighbourhood U of a point $\xi \in X$ (¹).

(¹) It follows from Theorems 3.2 and 3.3 of [2] (cf. also [3]) that under hypotheses (i) we have (iii) whenever the family $\{f(s, \cdot): s \in S\}$ is locally equicontinuous and $\sup\{\varrho(f(s, x), \xi): s \in S\} < \varrho(x, \xi)$ holds for all $x \in X \setminus \{\xi\}$ with a $\xi \in X$.

(iv) There exists a function $M\colon S \to [0, +\infty)$ such that

$$\bigwedge_{x,\bar{x}\in X} [\varrho(f(\cdot, x), f(\cdot, \bar{x})) \leqslant M\varrho(x, \bar{x})] \tag{1}$$

and

$$\bigwedge_{s\in S} (f(s, \xi) = \xi). \tag{2}$$

(v) For every $i \in \mathrm{m}$, h_i is a function defined on a set containing the set

$$\Omega := \{(x, y_1, \ldots, y_m) \in X \times \mathscr{F}(S, Y_1) \times \ldots \times \mathscr{F}(S, Y_m)\colon \bigwedge_{i\in \mathrm{m}} \bigvee_{\varphi_i \in \Phi_i} (y_i = \varphi_i \circ f(\cdot, x))\} \tag{3}$$

and taking values in Y_i in such a manner that for every $\varphi \in \Phi$ the function ψ given by $\psi(x) = h_i(x, \varphi \circ f(\cdot, x))$, $x \in X$, belongs to Φ_i and

$$h_i(\xi, 0) = 0. \tag{4}$$

(vi) For every $i \in \mathrm{m}$ there exists an $L_i \in [0, +\infty)$ and an extended-real-valued function β_i defined on a subset B_i of $\mathscr{F}(S, [0, +\infty)^m)$ such that

$$\bigwedge_{(x,y),(\bar{x},\bar{y})\in\Omega} \big(\|h_i(x, y) - h_i(\bar{x}, \bar{y})\|_i \leqslant L_i\varrho(x, \bar{x}) + \beta_i(\|y_1 - \bar{y}_1\|_1, \ldots, \|y_m - \bar{y}_m\|_m)\big). \tag{5}$$

(vii) There exists a positive matrix $(a_{ij})_{(i,j)\in \mathrm{m}\times\mathrm{m}}$ such that for every $c \in [0, +\infty)^m$ and $u \in \mathscr{F}(S, [0, +\infty)^m)$, if $u(s) = c$ for all $s \in S$ or $u = (c_1 M, \ldots, c_m M)$ (cf. (1)), then $u \in B_i$ and $\beta_i(u) \leqslant \sum_{j=1}^{m} a_{ij} c_j$ for every $i \in \mathrm{m}$. Moreover, for every $i \in \mathrm{m}$, $\beta_i(u) \leqslant \beta_i(\bar{u})$ whenever $u, \bar{u} \in B_i$ and $u_j \leqslant \bar{u}_j$ for all $j \in \mathrm{m}$.

(viii) The characteristic roots of the matrix $(a_{ij})_{(i,j)\in \mathrm{m}\times\mathrm{m}}$ are less than or equal one in absolute value.

(ix) For every $i \in \mathrm{m}$ we have

$$\bigwedge_{u\in B_i} \bigwedge_{\vartheta\in(0,1)} (\vartheta u \in B_i \text{ implies } \vartheta\beta_i(u) \leqslant \beta_i(\vartheta u)) \tag{6}$$

and $\liminf_{n\to\infty} \beta_i(u_n) \leqslant \beta_i(u)$ for every uniformly convergent sequence $(u_n\colon n \in N)$ of mappings from B_i such that its limit u belongs to B_i and fulfilling the inequalities $u_{nj} \leqslant cM$, with a certain $c \in [0, +\infty)$, for all $n \in N$ and $j \in \mathrm{m}$.

Let us start with the following

THEOREM 1. *If hypotheses* (i)–(ix) *are satisfied, then for every* $\varepsilon > 0$ *there exists a* $\varphi \in \Phi$ *such that*

$$\varphi_i \text{ fulfils a Lipschitz condition,} \tag{7}$$

$$\varphi_i(\xi) = 0 \tag{8}$$

and

$$\bigwedge_{x \in X} \left[\| \varphi_i(x) - h_i(x, \varphi \circ f(\cdot, x)) \|_i \leqslant \varepsilon \right] \tag{9}$$

hold for every $i \in \mathrm{m}$.

We first prove two lemmas. The first one concerns the system of functional inequalities

$$\lambda_i(x) \leqslant \beta_i(\lambda \circ f(\cdot, x)), \quad i \in \mathrm{m}. \tag{10}$$

We assume additionally that

(x) For every $i \in \mathrm{m}$ the extended-real-valued function β_i is defined on a subset B_i of $\mathscr{F}(S, [0, +\infty)^m)$. Moreover, there exists a positive matrix $(a_{ij})_{(i,j) \in \mathrm{m} \times \mathrm{m}}$ fulfilling (viii) and such that for every $i \in \mathrm{m}$, $u \in B_i$ and $c \in (0, +\infty)^m$, if $u_j \leqslant c_j$ for all $j \in \mathrm{m}$, then $\beta_i(u) \leqslant \sum_{j=1}^{m} a_{ij} c_j$.

LEMMA 1. *Let* X *be a topological space and suppose that hypotheses* (iii) *and* (x) *are satisfied. If* $\lambda\colon X \to [0, +\infty)^m$ *is a solution of the system* (10) *continuous at the point* ξ *such that*

$$\bigwedge_{i \in \mathrm{m}} (\lambda_i(\xi) = 0), \tag{11}$$

then $\lambda = 0$.

Proof. Fix arbitrarily an $\varepsilon > 0$. By (viii) and by the Perron Theorem ([6], p. 354) there exists an $\eta \in (0, +\infty)^m$ such that

$$\bigwedge_{i \in \mathrm{m}} \Big(\sum_{j=1}^{m} a_{ij} \eta_j \leqslant \eta_i \Big). \tag{12}$$

We may assume that

$$\bigwedge_{i \in \mathrm{m}} (\eta_i \leqslant \varepsilon). \tag{13}$$

It follows from the continuity of λ at the point ξ and from (11) that there exists a neighbourhood U_0 of ξ such that

$$\bigwedge_{i \in \mathrm{m}} (\lambda_i|_{U_0} \leqslant \eta_i). \tag{14}$$

Put $U_{n+1} := \bigcap \{f(s, \cdot)^{-1}(U_n): s \in S\}$ for every $n \in N \cup \{0\}$. Hence and from (iii) we have

(15) $$\bigwedge_{n \in N \cup \{0\}} (f(S \times U_{n+1}) \subset U_n)$$

and

(16) $$X = \bigcup_{n=0}^{\infty} U_n.$$

We shall show that

(17) $$\bigwedge_{i \in \mathrm{m}} (\lambda_i|_{U_n} \leqslant \eta_i)$$

holds for every $n \in N \cup \{0\}$. In fact, assume (17) for an $n \in N \cup \{0\}$ and fix arbitrarily $i \in \mathrm{m}$ and $x \in U_{n+1}$. Making use of (15) and (17) we obtain $\lambda_j \circ f(\cdot, x) \leqslant \eta_j$ for all $j \in \mathrm{m}$. This together with (10), (x) and (12) gives

$$\lambda_i(x) \leqslant \sum_{j=1}^{m} a_{ij}\eta_j \leqslant \eta_i.$$

Hence and from (14) we have (17) for all $n \in N \cup \{0\}$, which jointly with (16) allows us to state that $\lambda_i \leqslant \eta_i$ for every $i \in \mathrm{m}$. Taking (13) and the unrestricted choice of a positive real number ε into account we end the proof of Lemma 1.

LEMMA 2. *Under hypotheses* (i)–(vii) ([2]), *if the characteristic roots of* $(a_{ij})_{(i,j) \in \mathrm{m} \times \mathrm{m}}$ *are less than one in absolute value, then the system of equations*

(18) $$\varphi_i(x) = h_i(x, \varphi \circ f(\cdot, x)), \quad i \in \mathrm{m},$$

has exactly one solution $\varphi \in \Phi$ *continuous at the point* ξ *such that* $\varphi_i(\xi) = 0$ *for every* $i \in \mathrm{m}$. *This solution fulfils a Lipschitz condition. More exactly, if E is the*

([2]) It is enough to assume, instead of (i), that (X, ϱ) is a compact metric space and (Y_i, σ_i), $i \in \mathrm{m}$, are complete metric spaces. As regards hypotheses (ii), it suffices to assume that for every $i \in \mathrm{m}$, Φ_i is a subset of $\mathscr{F}(X, Y_i)$ closed under uniform convergence and such that $0 \in \Phi_i$. Finally, instead of (vii) it is enough to assume:

(vii') There exist positive matrices $(a_{ij})_{(i,j) \in \mathrm{m} \times \mathrm{m}}$ with all characteristic roots less than one in absolute value and $(b_{ij})_{(i,j) \in \mathrm{m} \times \mathrm{m}}$ with all characteristic roots not greater that one such that for every $c \in [0, +\infty)^m$ and $u \in \mathscr{F}(S, [0, +\infty)^m)$:

if $u(s) = c$ for all $s \in S$, then $u \in B_i$ and $\beta_i(u) \leqslant \sum_{j=1}^{m} b_{ij}c_j$ for all $i \in \mathrm{m}$,

if $u = (c_1 M, \ldots, c_m M)$, then $u \in B_i$ and $\beta_i(u) \leqslant \sum_{j=1}^{m} a_{ij}c_j$ for all $i \in \mathrm{m}$.

Moreover, for every $i \in \mathrm{m}$, we have $\beta_i(u) \leqslant \beta_i(\bar{u})$ whenever $u, \bar{u} \in B_i$ and $u_j \leqslant \bar{u}_j$ for all $j \in \mathrm{m}$.

set of all characteristic roots of $(a_{ij})_{(i,j)\in \mathrm{m}\times \mathrm{m}}$, $\Theta := \max|E|$ *and if* $c\in(0, +\infty)^m$ *satisfies the condition*

$$\bigwedge_{i\in \mathrm{m}} \Big(\sum_{j=1}^{m} a_{ij}c_j = \Theta c_i \text{ and } L_i \leqslant c_i\Big), \tag{19}$$

then the function φ_i *fulfils the Lipschitz condition with the constant* $c_i/(1-\Theta)$, $i\in \mathrm{m}$.

Proof. Making use of Perron's Theorem once more, we fix a $c\in(0, +\infty)^m$ satisfying (19) and we define $\mathscr{L}_i$, $i\in \mathrm{m}$, to be the set of all functions from Φ_i which fulfil the Lipschitz condition with the constant $c_i/(1-\Theta)$ and have value zero at ξ. For every $i\in \mathrm{m}$ the set $\mathscr{L}_i$ is non-void, since $0\in\mathscr{L}_i$. Put $\mathscr{L} := \mathscr{L}_1\times\ldots\times\mathscr{L}_m$ and fix a

$$\varphi\in\mathscr{L}, \tag{20}$$

$x, \bar{x}\in X$ and $i\in \mathrm{m}$. Recalling (20), (5), (1), (vii) and (19) we get

$$\begin{aligned}
&\big\|h_i\big(x, \varphi\circ f(\cdot, x)\big)-h_i\big(\bar{x}, \varphi\circ f(\cdot, \bar{x})\big)\big\|_i\\
&\leqslant L_i\varrho(x, \bar{x})+\beta_i\big(\|\varphi_1\circ f(\cdot, x)-\varphi_1\circ f(\cdot, \bar{x})\|_1, \ldots, \|\varphi_m\circ f(\cdot, x)-\varphi_m\circ f(\cdot, \bar{x})\|_m\big)\\
&\leqslant L_i\varrho(x, \bar{x})+\beta_i\Big(\frac{c_1}{1-\Theta} M\varrho(x, \bar{x}), \ldots, \frac{c_m}{1-\Theta} M\varrho(x, \bar{x})\Big)\\
&\leqslant L_i\varrho(x, \bar{x})+\sum_{j=1}^{m} a_{ij}\frac{c_j}{1-\Theta}\varrho(x, \bar{x})\leqslant \Big(c_i+\frac{1}{1-\Theta}\sum_{j=1}^{m} a_{ij}c_j\Big)\varrho(x, \bar{x})\\
&=\Big(c_i+\frac{\Theta}{1-\Theta}c_i\Big)\varrho(x, \bar{x}) = \frac{1}{1-\Theta}c_i\varrho(x, \bar{x}).
\end{aligned}$$

Moreover, if we have (20), then in view of (2) and (4) we obtain $h_i\big(\xi, \varphi\circ f(\cdot, \xi)\big) = 0$ for every $i\in \mathrm{m}$. Hence and from (v) for every $i\in \mathrm{m}$ we may define a function $T_i\colon \mathscr{L}\to\mathscr{L}_i$ by

$$T_i(\varphi)(x) = h_i\big(x, \varphi\circ f(\cdot, x)\big), \quad \varphi\in\mathscr{L},\ x\in X. \tag{21}$$

Now, fix $\varphi, \bar{\varphi}\in\mathscr{L}$ and $i\in \mathrm{m}$. If $\bar{x}\in X$, then, referring to (21), (5), (i) and (vii), we get

$$\begin{aligned}
&\|T_i(\varphi)(\bar{x})-T_i(\bar{\varphi})(\bar{x})\|_i\\
&\leqslant \beta_i\big(\|\varphi_1\circ f(\cdot, \bar{x})-\bar{\varphi}_1\circ f(\cdot, \bar{x})\|_1, \ldots, \|\varphi_m\circ f(\cdot, \bar{x})-\bar{\varphi}_m\circ f(\cdot, \bar{x})\|_m\big)\\
&\leqslant \beta_i\big(\sup\{\|\varphi_1(x)-\bar{\varphi}_1(x)\|_1\colon x\in X\}, \ldots, \sup\{\|\varphi_m(x)-\bar{\varphi}_m(x)\|_m\colon x\in X\}\big)\\
&\leqslant \sum_{j=1}^{m} a_{ij}\sup\{\|\varphi_j(x)-\bar{\varphi}_j(x)\|_j\colon x\in X\}.
\end{aligned}$$

By this inequality, (i) and (ii), endowing $\mathscr{L}_i$ with the supremum metric d_i, we have $d_i(T_i(\varphi), T_i(\bar{\varphi})) \leqslant \sum_{j=1}^{m} a_{ij} d_j(\varphi_j, \bar{\varphi}_j)$ and $(\mathscr{L}_i, d_i)$ is a complete metric space. By Theorem 1.4 and Lemma 1.2 from [7] the function $(T_1, \ldots, T_m)$ has (exactly one) fixed point.

It remains to show that the system (18) has at most one solution $\varphi \in \Phi$ continuous at the point ξ and vanishing at ξ. Suppose that φ and $\bar{\varphi}$ are two such solutions and put $\lambda_i := \|\varphi_i - \bar{\varphi}_i\|_i$, $i \in \mathrm{m}$. Then $(\lambda_1, \ldots, \lambda_m)$ is a solution of the system (10) continuous at ξ and condition (11) is satisfied. Applying Lemma 1 we see that $\varphi = \bar{\varphi}$. Thus Lemma 2 is proved.

Proof of Theorem 1. By Lemma 2 and the Perron Theorem we may assume that 1 is a characteristic root of $(a_{ij})_{(i,j) \in \mathrm{m} \times \mathrm{m}}$.

Fix a $c \in (0, +\infty)^m$ such that

$$\bigwedge_{i \in \mathrm{m}} \Big(\sum_{j=1}^{m} a_{ij} c_j = c_i \text{ and } L_i \leqslant c_i \Big) \tag{22}$$

and put $h_i^*(x, y) := h_i(x, (1-\vartheta)y)$, $(x, y) \in \Omega$, for every $i \in \mathrm{m}$ and for an arbitrarily fixed $\vartheta \in (0, 1)$ (cf. (ii)). Applying Lemma 2 to the system

$$\varphi_i(x) = h_i^*(x, \varphi \circ f(\cdot, x)), \quad i \in \mathrm{m}, \tag{23}$$

and taking (22) into account, we get the existence of a solution $\varphi \in \Phi$ of (23) such that $\varphi_i(\xi) = 0$ and φ_i fulfils the Lipschitz condition with the constant $c_i/(1-(1-\vartheta))$ for every $i \in \mathrm{m}$. Therefore, fixing a sequence $(\vartheta_n: n \in N)$ of numbers from the interval (0, 1) such that

$$\lim_{n \to \infty} \vartheta_n = 0, \tag{24}$$

we have also a sequence $(\varphi_n: n \in N)$ of functions from Φ such that

$$\bigwedge_{x \in X} [\varphi_{ni}(x) = h_i(x, (1-\vartheta_n)\varphi_n \circ f(\cdot, x))], \tag{25}$$

$$\varphi_{ni}(\xi) = 0 \tag{26}$$

and

$$\bigwedge_{x, \bar{x} \in X} (\|\varphi_{ni}(x) - \varphi_{ni}(\bar{x})\|_i \leqslant (c_i/\vartheta_n)\varrho(x, \bar{x})) \tag{27}$$

for every $n \in N$ and $i \in \mathrm{m}$. Moreover, recalling the Arzela–Ascoli Theorem, we may assume that the sequences $(\vartheta_n \varphi_{ni}: n \in N)$, $i \in \mathrm{m}$, converge uniformly.

We shall show that

$$\bigwedge_{i \in \mathrm{m}} (\lim_{n \to \infty} \vartheta_n \varphi_{ni} = 0). \tag{28}$$

To this end observe that in view of (vi) we have

$$(29)\quad \bigwedge_{\varphi\in\Phi}\bigwedge_{x\in X}\Big[\big(\|\varphi_1\circ f(\cdot,x)-\varphi_1\circ f(\cdot,\xi)\|_1,\ldots,\|\varphi_m\circ f(\cdot,x)-\\ -\varphi_m\circ f(\cdot,\xi)\|_m\big)\in\bigcap_{i=1}^{m}B_i\Big],$$

put

$$(30)\qquad \bar\varphi_i:=\lim_{n\to\infty}\vartheta_n\varphi_{ni},\qquad i\in\mathfrak{m},$$

and fix $n\in N$, $i\in\mathfrak{m}$ and $x\in X$. In view of (25), (4), (ii) and (5) we get $\|\varphi_{ni}(x)\|_i\leqslant L_i\varrho(x,\xi)+\beta_i\big((1-\vartheta_n)\|\varphi_{n1}\circ f(\cdot,x)\|_1,\ldots,(1-\vartheta_n)\|\varphi_{nm}\circ f(\cdot,x)\|_m\big)$, which jointly with (ii), (26), (29) and (6) shows that

$$(31)\quad \|\vartheta_n\varphi_{ni}(x)\|_i\\ \leqslant\vartheta_nL_i\varrho(x,\xi)+\beta_i\big((1-\vartheta_n)\|\vartheta_n\varphi_{n1}\circ f(\cdot,x)\|_1,\ldots,(1-\vartheta_n)\|\vartheta_n\varphi_{nm}\circ f(\cdot,x)\|_m\big).$$

Moreover, it follows from (26), (2), (27) and (1) that

$$(1-\vartheta_n)\|\vartheta_n\varphi_{nj}\circ f(\cdot,x)\|_j\leqslant\|\vartheta_n\varphi_{nj}\circ f(\cdot,x)\|_j\\ =\vartheta_n\|\varphi_{nj}\circ f(\cdot,x)-\varphi_{nj}\circ f(\cdot,\xi)\|_j\leqslant c_jM\varrho(x,\xi)$$

for all $j\in\mathfrak{m}$. Hence and from (30), (24), (ii), (26), (29) and from the second condition in (ix), passing to the limit in (31) we obtain

$$\|\bar\varphi_i(x)\|_i\leqslant\beta_i\big(\|\bar\varphi_1\circ f(\cdot,x)\|_1,\ldots,\|\bar\varphi_m\circ f(\cdot,x)\|_m\big).$$

In other words, $(\|\bar\varphi_1\|_1,\ldots,\|\bar\varphi_m\|_m)$ is a solution of (10). In view of (26), (27) and (30) this solution is continuous at the point ξ and vanishes at ξ. Applying Lemma 1 we have (28).

Now, fix arbitrarily $\varepsilon>0$ and $\eta\in(0,+\infty)^m$ such that (12) and (13) are satisfied. It follows from (28) that there exists an $n\in N$ such that

$$(32)\qquad \bigwedge_{x\in X}\big(\vartheta_n\|\varphi_{ni}(x)\|_i\leqslant\eta_i\big)$$

holds for all $i\in\mathfrak{m}$. We put

$$(33)\qquad \varphi_i(x):=h_i(x,\varphi_n\circ f(\cdot,x)),\qquad x\in X,\ i\in\mathfrak{m}.$$

It follows from (v) that $\varphi:=(\varphi_1,\ldots,\varphi_m)$ belongs to Φ. Fix $i\in\mathfrak{m}$ and pass to the proof of (7)–(9). Making use of (33), (5), (27), (1) and (vii) we obtain $\|\varphi_i(x)-\varphi_i(\bar x)\|_i\leqslant\big(L_i+\sum_{j=1}^{m}a_{ij}(c_j/\vartheta_n)\big)\varrho(x,\bar x)$ for all $x,\bar x\in X$, whereas (33), (2), (26) and (4) gives (8). Therefore it remains to prove (9) only.

Fix an $x \in X$. Taking (33), (v) and (5) into account we see that

$$(34)\quad \|\varphi_i(x) - h_i(x, \varphi \circ f(\cdot, x))\|_i$$
$$\leqslant \beta_i(\|\varphi_{n1} \circ f(\cdot, x) - \varphi_1 \circ f(\cdot, x)\|_1, \ldots, \|\varphi_{nm} \circ f(\cdot, x) - \varphi_m \circ f(\cdot, x)\|_m).$$

Moreover, applying (25), (33), (5), (32), (vii) and (12) we get

$$\|\varphi_{ni}[f(s, x)] - \varphi_i[f(s, x)]\|_i$$
$$\leqslant \beta_i[\vartheta_n \|\varphi_{n1} \circ f(\cdot, f(s, x))\|_1, \ldots, \vartheta_n \|\varphi_{nm} \circ f(\cdot, f(s, x))\|_m]$$
$$\leqslant \beta_i(\eta_1, \ldots, \eta_m) \leqslant \sum_{j=1}^{m} a_{ij}\eta_j \leqslant \eta_i$$

for all $s \in S$. This together with (34), (vii) and (12) allows us to state that $\|\varphi_i(x) - h_i(x, \varphi \circ f(\cdot, x))\|_i \leqslant \eta_i$, which jointly with (13) ends the proof of (9) and of Theorem 1.

In the next theorem condition (5) will be weakened at the cost of (ii). Namely, we assume the following:

(xi) For every $i \in \mathfrak{m}$ there exists an $F_i \in \mathscr{C}(X, Y_i)$, and $L_i \in [0, +\infty)$ and an extended-real-valued function β_i, defined on a subset B_i of $\mathscr{F}(S, [0, +\infty)^m)$, such that

$$(35)\quad \|(h_i(x, y) + F_i(x)) - (h_i(\bar{x}, \bar{y}) + F_i(\bar{x}))\|_i$$
$$\leqslant L_i \varrho(x, \bar{x}) + \beta_i(\|y_1 - \bar{y}_1\|_1, \ldots, \|y_m - \bar{y}_m\|_m)$$

holds for all $(x, y), (\bar{x}, \bar{y}) \in \Omega$ and conditions (vii)–(ix) are satisfied.

THEOREM 2. *Under hypotheses* (i)–(v) *and* (xi), *if*

$$(36)\quad \bigwedge_{i \in \mathfrak{m}} (\Phi_i + \mathscr{C}(X, Y_i) \subset \Phi_i),$$

then for every $\varepsilon > 0$ *there exists a* $\varphi \in \Phi$ *such that* (7)–(9) *hold for all* $i \in \mathfrak{m}$.

Proof. Take an $\varepsilon > 0$ and, making use of Lemma 6.2 of [2], a function $G: X \to Y_1 \times \ldots \times Y_m$ such that

$$(37)\quad G_i \text{ fulfils a Lipschitz condition}$$

and

$$(38)\quad \|F_i - G_i\|_i \leqslant \varepsilon/4$$

for every $i \in \mathfrak{m}$. Put

$$(39)\quad h_i^*(x, y) := h_i(x, y) + (F_i(x) - F_i(\xi)) - (G_i(x) - G_i(\xi)), \quad (x, y) \in \Omega,$$

for each $i \in \mathfrak{m}$.

We shall show that (v) and (vi) with h replaced by h^* are fulfilled. To this end fix an $i \in \mathfrak{m}$. If $\varphi \in \Phi$ and $\psi^*(x) := h_i^*(x, \varphi \circ f(\cdot, x))$, $x \in X$, then by

(39) we have $\psi^* = \psi + (F_i - F_i(\xi)) - (G_i - G_i(\xi))$, where ψ is defined as in (v). Hence and from (v), (xi) and (37) we get $\psi^* \in \Phi_i + \mathscr{C}(X, Y_i)$, which jointly with (36) shows that $\psi^* \in \Phi_i$, as required. Moreover, it follows from (39) and (4) that $h_i^*(\xi, 0) = 0$.

In order to get (vi), take (37) into account and denote by C_i a Lipschitz constant for G_i. Then, in view of (39) and (35), we have

$$\begin{aligned}\|h_i^*(x, y) - h_i^*(\bar{x}, \bar{y})\|_i &\\ &\leqslant \|(h_i(x, y) + F_i(x)) - (h_i(\bar{x}, \bar{y}) + F_i(\bar{x}))\|_i + \|G_i(x) - G_i(\bar{x})\|_i \\ &\leqslant (L_i + C_i)\varrho(x, \bar{x}) + \beta_i(\|y_1 - \bar{y}_1\|_1, \ldots, \|y_m - \bar{y}_m\|_m)\end{aligned}$$

for all $(x, y), (\bar{x}, \bar{y}) \in \Omega$. Therefore, making use of Theorem 1, there exists a $\varphi \in \Phi$ such that (7), (8) and

$$\bigwedge_{x \in X} \left[\|\varphi_i(x) - h_i^*(x, \varphi \circ f(\cdot, x))\|_i \leqslant \tfrac{1}{2}\varepsilon\right] \tag{40}$$

holds for every $i \in \mathfrak{m}$. This ends the proof of Theorem 2, because (39), (40) and (38) give (9).

The above theorems in many instances afford approximate solutions of (18) of higher regularity, as Theorem 3 below shows. Assume that:

(xii) X is a compact subset of a k-dimensional Euclidean space, ξ is a fixed element of X and Y_i, $i \in \mathfrak{m}$, are Euclidean spaces.

(xiii) Φ_i, $i \in \mathfrak{m}$, is a subset of $\mathscr{F}(X, Y_i)$ containing the set of all functions from X into Y_i whose all components are restrictions to X of polynomials in k variables.

(xiv) The function f maps $S \times X$ into X, where S is a non-void set.

(xv) For every $i \in \mathfrak{m}$, h_i is a function defined on a set containing the set Ω given by (3) and taking values in Y_i in such a manner that

$$\bigwedge_{(x,y)(x,\bar{y}) \in \Omega} (\|h_i(x, y) - h_i(x, \bar{y})\|_i \leqslant \beta_i(\|y_1 - \bar{y}_1\|_1, \ldots, \|y_m - \bar{y}_m\|_m))$$

and (x) hold.

THEOREM 3. *Let hypotheses* (xii)–(xv) *be fulfilled. If for every* $\varepsilon > 0$ *there exists a continuous function* $\varphi \in \Phi$ *satisfying* (8) *and* (9) *for every* $i \in \mathfrak{m}$, *then for every* $\varepsilon > 0$ *there exists a* $\varphi \in \Phi$ *such that for every* $i \in \mathfrak{m}$ *we have* (8) *and* (9) *and all components of* φ_i *are restrictions to* X *of polynomials in* k *variables.*

Proof. Fix a positive real number ε, suppose that $\eta \in (0, +\infty)^m$ is chosen in such a manner that (12) and

$$\bigwedge_{i \in \mathfrak{m}} (\eta_i \leqslant \tfrac{1}{3}\varepsilon) \tag{41}$$

hold and let $\bar{\varphi} \in \Phi$ be a continuous function fulfilling (8) and

$$\bigwedge_{x \in X} [\|\bar{\varphi}_i(x) - h_i(x, \bar{\varphi} \circ f(\cdot, x))\|_i \leqslant \tfrac{1}{3}\varepsilon] \tag{42}$$

for every $i \in \mathrm{m}$. Finally, take a $\varphi \in \Phi$ such that for every $i \in \mathrm{m}$ all components of φ_i are restrictions to X of polynomials in k variables and fulfilling conditions (8) and

$$\|\bar{\varphi}_i - \varphi_i\|_i \leqslant \eta_i. \tag{43}$$

Then, taking (xiii)–(xv), (41)–(43) and (12) into account, we get

$$\begin{aligned}
&\|\varphi_i(x) - h_i(x, \varphi \circ f(\cdot, x))\|_i \\
&\leqslant \|\varphi_i(x) - \bar{\varphi}_i(x)\|_i + \\
&\quad + \|\bar{\varphi}_i(x) - h_i(x, \bar{\varphi} \circ f(\cdot, x))\|_i + \|h_i(x, \bar{\varphi} \circ f(\cdot, x)) - h_i(x, \varphi \circ f(\cdot, x))\|_i \\
&\leqslant \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \beta_i(\|\bar{\varphi}_1 \circ f(\cdot, x) - \varphi_1 \circ f(\cdot, x)\|_1, \ldots, \|\bar{\varphi}_m \circ f(\cdot, x) - \varphi_m \circ f(\cdot, x)\|_m) \\
&\leqslant \tfrac{2}{3}\varepsilon + \sum_{j=1}^{m} a_{ij}\eta_j \leqslant \tfrac{2}{3}\varepsilon + \eta_i \leqslant \varepsilon
\end{aligned}$$

for every $x \in X$, and that is all we had to prove.

The next two theorems show a type of application of the theorems proved above.

THEOREM 4. *Suppose that X is a topological space and let $(Y_i, \|\ \|_i)$, $i \in \mathrm{m}$, be normed spaces. Moreover, assume* (iii) *and* (xv), *where Φ_i, $i \in \mathrm{m}$, is a subet of $\mathscr{F}(X, Y_i)$. If ε is a positive real number and if $\varphi \in \Phi$ fulfils* (8), (9) *and is continuous at the point ξ, then for every $i \in \mathrm{m}$ there exists a function h_i^* mapping the domain of h_i into Y_i in such a manner that*

$$\|h_i - h_i^*\|_i \leqslant \varepsilon \tag{44}$$

and φ is the only solution of (23) *in the class of all functions in Φ which are continuous at ξ and vanish at ξ.*

Before passing to the proof observe that it follows from the theorems given above that in many instances of systems of equations which have no continuous solutions there exist systems close to the ones under consideration which possess solutions of higher regularity.

Proof of Theorem 4. Put

$$h_i^*(x, y) := h_i(x, y) + \varphi_i(x) - h_i(x, \varphi \circ f(\cdot, x))$$

for all (x, y) from the domain of h and $i \in \mathrm{m}$. Then, directly from (9) and from the definition of h_i, we have (44) for every $i \in \mathrm{m}$ and φ is a solution of (23). It follows from Lemma 1 that φ is the only solution of (23) in the class considered (cf. the end of the proof of Lemma 2).

Finally, we assume

(xvi) (X, ϱ) is a compact metric space and $(Y, \|\ \|)$ is a finite-dimensional Banach space.

(xvii) The function $g\colon X \to X$ fulfils the Lipschitz condition with the constant 1 and

$$\bigwedge_{x \in X \setminus \{\xi\}} [\varrho(g(x), \xi) < \varrho(x, \xi)]$$

holds for a certain $\xi \in X$.

THEOREM 5. *Assume* (xvi) *and* (xvii). *If A is a positive matrix with characteristic roots less than or equal one in absolute value, then for every continuous function $F\colon X \to Y^m$ vanishing at ξ and for every $\varepsilon > 0$ there exists a function $G\colon X \to Y^m$ such that*

(45) $\quad G_i$ *fulfils a Lipschitz condition,*

(46) $$\|F_i - G_i\| \leqslant \varepsilon$$

for every $i \in \mathrm{m}$, and the series

(47) $$\sum_{n=0}^{\infty} A^n G \circ g^n$$

converges uniformly to a function mapping X into Y^m and fulfilling a Lipschitz condition.

Proof. Fix an $\varepsilon > 0$ and a suitable function $F\colon X \to Y^m$. In order to apply Theorem 2 we put $Y_i := Y$ and $\Phi_i := \mathscr{F}(X, Y)$ for all $i \in \mathrm{m}$, $S := \{1\}$, $f(1, x) := g(x)$ for every $x \in X$, $M := 1$, $h_i(x, y) := \sum_{j=1}^{m} a_{ij} y_j(1) - F_i(x)$ for each $(x, y) \in X \times \mathscr{F}(\{1\}, Y^m)$ and $i \in \mathrm{m}$, where $(a_{ij})_{(i,j) \in \mathrm{m} \times \mathrm{m}} = A$, and $\beta_i(u) := \sum_{j=1}^{m} a_{ij} u_j(1)$ for every $u \in \mathscr{F}(\{1\}, [0, +\infty)^m)$ and $i \in \mathrm{m}$. After this specification it is easy to see that all the assumptions of Theorem 2 are fulfilled except, perhaps, conditions (iii) and (2). These, however, follow from Theorem 3.3 of [2] (cf. also [3]). In fact, it follows from this theorem that

(48) $\quad$ the sequence $(g^n\colon n \in N)$ tends to ξ uniformly.

Hence, making use of Theorem 2, we see that there exists a function $\varphi\colon X \to Y^m$ such that (7), (8) and

(49) $$\bigwedge_{x \in X} [\|\varphi_i(x) - (\sum_{j=1}^{m} a_{ij} \varphi_j[g(x)] - F_i(x))\| \leqslant \varepsilon]$$

hold for all $i \in \mathrm{m}$.

Put

(50) $$G := A\varphi \circ g - \varphi$$

and fix an $i \in \mathrm{m}$. It follows from (50) and (7) that G_i is a linear combination of functions which fulfil a Lipschitz condition, and so we have (45). Condition (46) follows from (50) and (49). We shall now show that the series (47) converges uniformly to the function $-\varphi$.

Fix an $\eta \in (0, +\infty)^m$ satisfying (12) and (13) and $\delta > 0$ such that

(51) $$\bigwedge_{x \in X} \left(\varrho(x, \xi) < \delta \text{ implies } \|\varphi_i(x)\| \leqslant \eta_i\right)$$

is true for all $i \in \mathrm{m}$ (cf. (7) and (8)). Making use of (48), we find an $N \in N$ such that $\varrho(g^n(x), \xi) \leqslant \delta$ for all $x \in X$ and $n \geqslant N$, and recalling (51) we have

(52) $$\bigwedge_{i \in \mathrm{m}} \left(\|\varphi_i \circ g^n\| \leqslant \eta_i\right)$$

for all $n \geqslant N$. Let $A^n = (a_{ij}^{(n)})_{(i,j) \in \mathrm{m} \times \mathrm{m}}$ for each $n \in N$. Then, taking (12) into account and using the induction principle, we obtain $\sum_{j=1}^{m} a_{ij}^{(n)} \eta_j \leqslant \eta_i$ for all $n \in N$ and $i \in \mathrm{m}$. Hence and from (52) and (13) we get

$$\sum_{j=1}^{m} a_{ij}^{(n)} \|\varphi_j \circ g^n\| \leqslant \varepsilon$$

for all $i \in \mathrm{m}$ and $n \geqslant N$, which shows that the sequence $(A^n \varphi \circ g^n \colon n \in N)$ converges uniformly to zero. But, in view of (50),

$$\sum_{n=0}^{N} A^n G \circ g^n = A^{N+1} \varphi \circ g^{N+1} - \varphi$$

for all $N \in N \cup \{0\}$, and so the series (47) converges uniformly to the function $-\varphi$. Thus the theorem is proved.

References

[1] K. Baron, *On approximate solutions of a functional equation*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 23 (1975), p. 1065–1068.

[2] —, *Functional equations of infinite order*, Uniw. Śląski w Katowicach, Prace Naukowe 265 (1978).

[3] —, *On a majorization of distances between the values of a family of functions and a fixed point*, General Inequalities 2, Edited by E. F. Beckenbach, ISNM 27, Birkhäuser Verlag, Basel 1980, p. 463–464.

[4] R. C. Buck, *On approximation theory and functional equations*, J. Approximation Theory 5 (1972), p. 228–237.

[5] —, *Approximation theory and functional equations* (II), ibidem 9 (1973), p. 121–125.

[6] F. R. Gantmacher, *Matrix theory*, Izd. „Nauka", Moskwa 1966 (Russian transl.).

[7] J. Matkowski, *Integrable solutions of functional equations*, Dissertationes Math. 127 (1975).

Reçu par la Rédaction le 18.01.1980