Functional equations of delay type in L^1 spaces

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Abstract. The functional equation $x(t) = F(x_t)$, $x_0 = \varphi$, where φ belongs to $L^1(-r, 0; X)$, X a Banach space, is related to the semigroups generated by the operator $A\varphi = -\varphi'$, $D_A = \{\varphi \in W^{1,1}(-r, 0; X), \varphi(0) = F(\varphi)\}$. From the properties of the semigroup, regularity and asymptotic results for the solutions are deduced.

1. Introduction. In this paper we consider the functional equation of delay type

(FE) $x(t) = F(x_t)$, $t \geqslant 0$, $x_0 = \varphi$,

with initial data φ in $L^1(-r, 0; X)$. Here X is a real Banach space, r, $0 < r < +\infty$, is the delay and $x_t \in L^1(-r, 0; X)$ is the history of $x(t)$ at time t, $x_t(\vartheta) = x(t + \vartheta)$ a.e. Equation (FE) is studied by associating with it the operator in $L^1(-r, 0; X)$

(1) $A\varphi = -\varphi'$, $D_A = \{\varphi \in W^{1,1}(-r, 0; X), \varphi(0) = F(\varphi)\}$.

We prove that if F is Lipschitz continuous, then A generates a semigroup $T(t)$. By relating $T(t)$ to the semigroup generated in $C(-r, 0; X)$ by solutions with continuous initial data, we prove that $T(t)\varphi$ gives the segments of solutions of (FE). From the properties of the semigroup we then deduce regularity and asymptotic results for the solutions.

The semigroup approach to (FE) in the case of continuous initial data has been discussed in [3] and [8]. In [6] the particular case $F(\varphi) = H(\varphi(-r))$, $H: X \to X$, was considered in L^p spaces. In [9] and [10] the nonautonomous case $F(t, \varphi) = \int_{-t}^{0} g(-\vartheta, \varphi(\vartheta))d\vartheta$ was studied as a semigroup of operators in the product space $L^p \times X$. Our results are also related to the semigroup approach to age dependent population problems ([11]).

We suppose that

(H) $F: L^1(-r, 0; X) \to X$ is Lipschitz continuous with constant $|F|$;

we associate with (FE) the operator A and prove the following theorem:

THEOREM 1. *Let F satisfy* (H). *Then $A + |F| I$ is m-accretive in*

$L^1(-r, 0; X)$ *and therefore generates a semigroup of operators,* $T(t)$. $T(t)\varphi$ *is continuous in* $]-t, 0]$ *and if we set*

$$x(t) = \begin{cases} \varphi(t), & -r < t < 0, \\ F(\varphi), & t = 0, \\ T(t)\varphi(0), & t > 0; \end{cases}$$

$x(t)$ *is the unique solution of* (FE). *Such a solution is continuous for* $t > 0$ *and continuous from the right at* $t = 0$.

To obtain information on the behaviour of solution we consider then the weighted spaces $L_\sigma^1 = L^1(-r, 0; e^{-\sigma\vartheta}; X)$ and prove that if $|F|r \leqslant e^{-1}$, then there exist $\omega \leqslant 0$ and $\sigma \in R$ such that $A + \omega I$ is m-accretive in L_σ^1; more precisely, we can choose $\omega = \dfrac{1 + \log|F|r}{r}$ and $\sigma = \omega - 1/r$. Note that from Theorem 1 we have that, if $t \geqslant r$, $T(t)\varphi$ is continuous and the behaviour of solutions is known for all $\varphi \in L^1$ if it is known for continuous initial data. We deduce, using results in [3], that if $|F|r < 1$ then there exists a unique "equilibrium" solution of (FE) and this solution is exponentially asymptotically stable.

Theorem 1 says, in particular, that the sets $M = \{\varphi \in C(-r, 0; X), \varphi(0) = F(\varphi)\}$ and $M_1 = \{\varphi: [-r, 0] \to X, \varphi$ is piecewise continuous$\}$ are flow-invariant, that is, $T(t)M \subset M$ and $T(t)M_1 \subset M_1$. The following sets which extend D_A are also flow-invariant

$$D_{A,\sigma} = \left\{\varphi \in L^1(-r, 0; X), \lim_{\lambda \to 0} \frac{\|\varphi - (I + \lambda A)^{-1}\varphi\|}{\lambda^\sigma} < +\infty\right\}.$$

We shall see that

$$D_{A,\sigma} = \left\{\varphi \in L^1(-r, 0; X): \int_{-r}^{-t} |\varphi(t+\vartheta) - \varphi(\vartheta)|d\vartheta + \int_{-t}^{0} |\varphi(\vartheta)|d\vartheta \leqslant t^\sigma K_\varphi\right\},$$

and hence the sets $D_{A,\sigma}$ do not depend on F. This is the crucial difference between the cases of initial data in $C(-r, 0; X)$ and $L^1(-r, 0; X)$; and so in the L^1 spaces it is possible to use the Crandall-Pazy theory on nonlinear semigroups to study the non-autonomous version of (FE). This will be done in a join paper with G. F. Webb.

2. The semigroup associated with (FE). In this section we study the equation

(FE) $x(t) = F(x_t), \quad t > 0, \quad x_0 = \varphi,$

where $F: L^1 = L^1(-r, 0; X) \to X$ and $\varphi \in L^1$, by relating the semigroup associated with (FE) in L^1 to the semigroup generated in $C(-r, 0; X)$ by solutions with continuous initial data.

We denote by $|\cdot|$ the norm in X and by $\|\cdot\|$ the usual norm in L^1. We consider the operator

$$A\varphi = -\varphi', \quad D_A = \{\varphi \in W^{1,1}(-r, 0; X); \ \varphi(0) = F(\varphi)\}$$

and prove that

PROPOSITION 1. *Let* F *satisfy* (H). *Then* $A + |F| I$ *is* m-*accretive in* L^1. *Moreover,* $\bar{D}_A = L^1$.

Proof. Let φ_1, $\varphi_2 \in D_A$ and set $\psi_i = (I + \lambda A)\varphi_i = \varphi_i - \lambda\varphi_i'$, $i = 1, 2$, $\varphi = \varphi_1 - \varphi_2$ and $\psi = \psi_1 - \psi_2$. Then

$$\|\varphi\| = \left\| e^{\vartheta/\lambda}\varphi(0) + \int_\vartheta^0 (e^{(\vartheta - s)/\lambda}/\lambda)\psi(s)\,ds \right\|$$

$$\leqslant |\varphi(0)|\, \|e^{\vartheta/\lambda}\| + \left\| \int_\vartheta^0 (e^{(\vartheta - s)/\lambda}/\lambda)\psi(s)\,ds \right\|$$

$$\leqslant |F(\varphi_1) - F(\varphi_2)|(1 - e^{-r/\lambda})\lambda + \|\psi\|(1 - e^{-r/\lambda})$$

$$\leqslant |F|\cdot\|\varphi\|\lambda + \|\psi\|;$$

hence, if $|F|\lambda < 1$,

$$\|\varphi\| \leqslant \frac{1}{1 - \lambda|F|}\,\|\psi\|$$

and $A + |F| I$ is accretive.

In order to prove that $A + |F| I$ is m-accretive, choose $\lambda > 0$ such that $\lambda(1 - e^{-r/\lambda})|F| < 1$ and define, for a given $\psi \in Y$,

$$H: X \to X, \quad H(x) = F\left(e^{\vartheta/\lambda}x + \int_\vartheta^0 (e^{(\vartheta - s)/\lambda}/\lambda)\psi(s)\,ds\right).$$

We have

$$|H(x_1) - H(x_2)| \leqslant |F|\, \|e^{\vartheta/\lambda}(x_1 - x_2)\| = |F|\,\lambda(1 - e^{-r/\lambda})|x_1 - x_2|;$$

hence H is a contraction, and if $\bar{x}$ is the unique fixed point of H, the function $\bar{\varphi}(x) = e^{\vartheta/\lambda}\bar{x} + \int_\vartheta^0 (e^{(\vartheta - s)/\lambda}/\lambda)\psi(s)\,ds$ belongs to D_A and is such that $(I + \lambda A)\bar{\varphi} = \psi$.

That $\bar{D}_A = L^1$ is consequence of the following lemma.

LEMMA 1. *For all* $\psi \in L^1$

$$\lim_{\lambda \to 0} (I + \lambda A)^{-1}\psi = \psi.$$

Proof. Let A_0 be the operator

$$A_0\varphi = -\varphi', \quad D_{A_0} = \{\varphi \in W^{1,1}(-r, 0; X), \ \varphi(0) = 0\}.$$

We have

$$\|\psi - (I + \lambda A)^{-1} \psi\| = \|\psi(\vartheta) - e^{\vartheta/\lambda}(I + \lambda A)^{-1} \psi(0) - \int_{\vartheta}^{0} (e^{(\vartheta - s)\lambda}/\lambda) \psi(s) \, ds\|$$

$$\leqslant \|e^{\vartheta/\lambda}(I + \lambda A)^{-1} \psi(0)\| + \|\psi - (I + \lambda A_0)^{-1} \psi\|$$

$$= |(I + \lambda A)^{-1} \psi(0)| \lambda(1 - e^{-r/\lambda}) + \|\psi - (I + \lambda A_0)^{-1} \psi\|$$

$$\leqslant (|F(\psi)| + |F| \|\psi - (I + \lambda A)^{-1} \psi\|) \lambda(1 - e^{-r/\lambda}) +$$

$$+ \|\psi - (I + \lambda A_0)^{-1} \psi\|.$$

Hence, for λ small,

$$\|\psi - (I + \lambda A)^{-1} \psi\| \leqslant \frac{1}{1 - \lambda(1 - e^{-r/\lambda})|F|} (\lambda(1 - e^{-r/\lambda})|F(\psi)| +$$

$$+ \|\psi - (I + \lambda A_0)^{-1} \psi\|)$$

and the right part tends to zero as $\lambda \to 0$.

It follows in particular that

Lemma 2. *For all $\psi \in L^1$, $|(I + \lambda A)^{-1} \psi(0)|$ is bounded as $\lambda \to 0$.*

Proof. We have

$$\lim_{\lambda \to 0} (I + \lambda A)^{-1} \psi(0) = \lim_{\lambda \to 0} F((I + \lambda A)^{-1} \psi) = F(\psi)$$

as F is continuous. And so $\lim_{\lambda \to 0} |(I + \lambda A)^{-1} \psi(0)| = |F(\psi)|$.

From Proposition 1 it follows that A generates a semigroup of operators, $T(t)$, in L^1.

Let C_ω denote the space of continuous functions in $[-r, 0]$ with values in X endowed with the norm $\|\|\varphi\|\|_\omega = \sup_{-r \leqslant \vartheta \leqslant 0} e^{-\omega\vartheta} |\varphi(\vartheta)|$. We prove that if we consider the restrictions of F and A to C and define the operator

$$\tilde{A}\varphi = -\varphi', \quad D_{\tilde{A}} = \{\varphi \in C^1(-r, 0; X), \varphi(0) = F(\varphi)\},$$

then $\tilde{A}$ generates a semigroup of operators, $\tilde{T}(t)$, in C_ω, for some $\omega \in R$, which can be identified with the restriction of $T(t)$ to the set $M = \{\varphi \in C, \varphi(0) = F(\varphi)\}$.

We first prove that

Proposition 2. *Let $F: L^1 \to X$ satisfy (H). Then there exist $\omega \in R$ such that $F_{|C}: C_\omega \to X$ is Lipschitz continuous with constant $\gamma_\omega < 1$.*

Proof. Let $0 < p < r$ such that $|F| \cdot p < 1$. Then if $\varphi, \psi \in C$,

$$|F(\varphi) - F(\psi)| \leqslant |F| \int_{-r}^{0} |\varphi(\vartheta) - \psi(\vartheta)| \, d\vartheta$$

$$= |F| \int_{-r}^{-p} |\varphi(\vartheta) - \psi(\vartheta)| \, d\vartheta + |F| \int_{-p}^{0} |\varphi(\vartheta) - \psi(\vartheta)| \, d\vartheta$$

$$\leqslant |F|(r - p) \sup_{-r \leqslant \vartheta \leqslant -p} |\varphi(\vartheta) - \psi(\vartheta)| + |F| \cdot p \|\|\varphi - \psi\|\|_0$$

and the result follows from Proposition 2 of [8]. Note that any

$$\omega > \max\left\{0, \frac{1}{p}\log\frac{|F|(r-p)}{1-|F|p}\right\} \text{ would do.}$$

It follows from Theorems 4 and 5 of [3] that $\tilde{A}+\omega I$ is m-accretive in C. Let $\tilde{T}(t)$ be the semigroup generated by $\tilde{A}$; then set

$$\tilde{x}(t) = \begin{cases} \varphi(t), & -r \leqslant t < 0, \\ \tilde{T}(t)\,\varphi(0), & t \geqslant 0, \end{cases}$$

we have that $\tilde{x}_t = \tilde{T}(t)\,\varphi$ and $\tilde{x}(t)$ is the unique solution of $x(t) = F(x_t)$, $x_0 = \varphi$, $\varphi \in C$, $\varphi(0) = F(\varphi)$. Hence if $T(t)$ is the semigroup generated by the solutions of (FE), it must be $T(t)\,\varphi = \tilde{T}(t)\,\varphi$ for all $\varphi \in C$ such that $\varphi(0) = F(\varphi)$. And, in fact, we prove that

PROPOSITION 3. *Let $T(t)$ and $\tilde{T}(t)$ be the semigroups generated respectively by A and $\tilde{A}$. Then, for all $\varphi \in \bar{D}_{\tilde{A}} = \{\varphi \in C, \varphi(0) = F(\varphi)\}$, $T(t)\,\varphi$ is continuous and $T(t)\,\varphi = \tilde{T}(t)\,\varphi$, $\forall t \geqslant 0$.*

Proof. For all $\varphi \in D_{\tilde{A}}$ we have $(I+\lambda A)^{-1}\varphi = (I+\lambda\tilde{A})^{-1}\varphi$, and hence

$$\tilde{T}(t)\,\varphi = (C)- \lim_{n\to\infty}\left(I+\frac{t}{n}\,\tilde{A}\right)^{-n}\varphi = (L^1)- \lim_{n\to\infty}\left(I+\frac{t}{n}\,A\right)^{-n}\varphi = T(t)\,\varphi,$$

as $\psi_n \xrightarrow{C} \psi$ implies $\psi_n \xrightarrow{L^1} \psi$.

Since $D_A \subset \bar{D}_{\tilde{A}}$ from Proposition 1, it follows that, given $\varphi \in L^1$, there exists $\varphi_n \in \bar{D}_{\tilde{A}}$ such that $\varphi_n \xrightarrow{L^1} \varphi$; hence $T(t)\,\varphi_n \xrightarrow{L^1} T(t)\,\varphi$ for all $t \geqslant 0$.

For all fixed $t \geqslant 0$ we can suppose that

$$(2) \qquad \begin{aligned} \varphi_n(\vartheta) &\to \varphi(\vartheta) & \text{a.e.} \quad -r < \vartheta < 0, \\ T(t)\,\varphi_n(\vartheta) &\to T(t)\,\varphi(\vartheta) & \text{a.e.} \quad t+\vartheta > 0. \end{aligned}$$

We know that

$$\tilde{T}(t)\,\varphi_n(\vartheta) = \begin{cases} \varphi_n(t+\vartheta), & t+\vartheta < 0, \\ F(\tilde{T}(t+\vartheta)\,\varphi_n), & t+\vartheta \geqslant 0; \end{cases}$$

hence from Proposition 3,

$$T(t)\,\varphi_n(\vartheta) = \begin{cases} \varphi_n(t+\vartheta), & t+\vartheta < 0, \\ F(T(t+\vartheta)\,\varphi_n), & t+\vartheta \geqslant 0, \end{cases}$$

and so

$$\lim_{n\to\infty} T(t)\,\varphi_n(\vartheta) = \begin{cases} \varphi(t+\vartheta) & \text{a.e.,} \quad t+\vartheta < 0, \\ F(T(t+\vartheta)\,\varphi), & t+\vartheta \geqslant 0, \end{cases}$$

and from (2)

(3) $$T(t)\,\varphi(\vartheta) = \begin{cases} \varphi(t+\vartheta) & \text{a.e.,} \quad t+\vartheta < 0, \\ F\big(T(t+\vartheta)\,\varphi\big) & \text{a.e.,} \quad t+\vartheta > 0. \end{cases}$$

Set

$$x(t) = \begin{cases} \varphi(t), & -r \leqslant t < 0, \\ F\big(T(t)\,\varphi\big), & t \geqslant 0; \end{cases}$$

then $x_{|[0,+\infty[}$ is continuous as $T(t)\,\varphi$ is continuous in t and F is continuous. Moreover, from (3) we have

$$x_t(\vartheta) = x(t+\vartheta) = T(t)\,\varphi(\vartheta) \qquad \text{a.e.,} \qquad -r < \vartheta < 0,$$

that is, $x_t = T(t)\,\varphi$. It follows that $T(t)\,\varphi$ is continuous if $t \geqslant r$, and continuous in $]-t, 0]$ if $t < r$ and that $x(t)$ can also be defined as

$$x(t) = \begin{cases} \varphi(t) & \text{a.e.,} \quad t \in [-r, 0[, \\ F(\varphi), & t = 0, \\ T(t)\,\varphi(0), & t > 0, \end{cases}$$

and is a solution of (FE)'.

Any solution $x(t)$ of (FE)' which is $L^1_{\text{loc}}(-r, \infty; X)$ is continuous for $t \geqslant 0$ if $F\colon L^1 \to X$ is continuous, as $t \to x_t$, $[0, \infty[\to L^1$ is continuous. Moreover, if F is Lipschitz continuous, then the solution is unique: let $x(t)$ and $\tilde{x}(t)$ be solutions of (FE). Then for $0 < t \leqslant r$

$$|x(t) - \tilde{x}(t)| \leqslant |F|\,\|x_t - \tilde{x}_t\| = |F| \int_0^t |x(s) - \tilde{x}(s)|\,ds$$

and from Gronwall's lemma, $x(t) = \tilde{x}(t)$ for $0 \leqslant t \leqslant r$. In the same way one can prove that $x(t) = \tilde{x}(t)$ for $r \leqslant t \leqslant 2r$, and so on. And Theorem 1 is proved.

3. Asymptotic results. The semigroup generated by A in L^1 is of type $|F|$, that is,

(4) $$\|T(t)\,\varphi - T(t)\,\psi\| \leqslant e^{t|F|}\|\varphi - \psi\|, \qquad \varphi, \psi \in L^1,$$

and as $|F| > 0$, the information on the behaviour of solutions given by (4) is not in general satisfactory. Hence we consider the weighted space $L^1_\sigma = L^1(-r, 0; e^{-\vartheta\sigma}; X)$, that is, the space $L^1(-r, 0; X)$ endowed with the norm $\|\varphi\|_\sigma = \int_{-r}^0 e^{-\sigma\vartheta}|\varphi(\vartheta)|\,d\vartheta$, and we prove that if $|F|$ is small enough, then there exist $\omega \leqslant 0$ and σ such that $A + \omega I$ is m-accretive in L^1_σ and therefore the semigroup generated by A in L^1_σ is of type $\omega \leqslant 0$. We prove first

PROPOSITION 4. *Let F satisfy condition* (H). *If $|F|\,r \leqslant e^{-1}$, then A is m-accretive in L^1_σ for all σ such that $\sigma e^{\sigma r} \leqslant -|F|$.*

Proof. We have

$$|F(\varphi_1) - F(\varphi_2)| \leqslant |F| \, \|\varphi_1 - \varphi_2\|$$

$$= |F| \int_{-r}^{0} e^{\sigma\vartheta} e^{-\sigma\vartheta} |\varphi_1(\vartheta) - \varphi_2(\vartheta)| \, d\vartheta$$

$$\leqslant |F| \max\{1, e^{-\sigma r}\} \, \|\varphi_1 - \varphi_2\|_{\sigma}.$$

Hence if $\sigma \leqslant 0$, then $|F|_\sigma \leqslant |F| e^{-\sigma r}$, where $|F|_\sigma$ is the Lipschitz constant of $F \colon L_\sigma^1 \to X$.

Let $\varphi_i \in D_A$, $\psi_i = \varphi_i - \lambda\varphi_i'$, $i = 1, 2$, $\varphi = \varphi_1 - \varphi_2$, $\psi = \psi_1 - \psi_2$. Then

$$|\varphi(\vartheta)| \, e^{-\sigma\vartheta} \leqslant |\varphi(0)| \, e^{(1-\sigma\lambda)\vartheta/\lambda} + \int_{\vartheta}^{0} e^{-\sigma s} |\psi(s)| \, (e^{((1-\sigma\lambda)/\lambda)(\vartheta-s)}/h) \, ds$$

and so

$$(5) \qquad \|\varphi\|_\sigma \leqslant |F|_\sigma \|\varphi\|_\sigma \int_{-r}^{0} e^{(1-\sigma\lambda)\vartheta/\lambda} \, d\vartheta +$$

$$+ \int_{-r}^{0} e^{-\sigma s} |\psi(s)| \left(\int_{-r}^{s} (e^{(1-\sigma\lambda)(\vartheta-s)/\lambda}/\lambda) \, d\vartheta \right) ds$$

$$\leqslant |F|_\sigma \|\varphi\|_\sigma \frac{\lambda}{1-\sigma\lambda} (1 - e^{(1-\sigma\lambda)(-r)/\lambda}) +$$

$$+ \|\psi\|_\sigma \frac{1}{1-\sigma\lambda} (1 - e^{(1-\sigma\lambda)(-r)/\lambda}).$$

We have

$$(6) \qquad 1 - \frac{\lambda |F|_\sigma}{1-\sigma\lambda} (1 - e^{(1-\sigma\lambda)(-r)/\lambda}) \geqslant \frac{1}{1-\sigma\lambda} (1 - e^{(1-\sigma\lambda)(-r)/\lambda})$$

provided $\sigma + |F|_\sigma \leqslant 0$. Hence if $\sigma + |F|_\sigma \leqslant 0$, that is, if $\sigma e^{\sigma r} < -|F|$, then $\|\varphi\|_\sigma \leqslant \|\psi\|_\sigma$ and A is accretive in L_σ^1. Note that the minimum of $\sigma e^{\sigma r}$ is $-(e \cdot r)^{-1}$; hence such a σ exists only if $-(e \cdot r)^{-1} \leqslant -|F|$, i.e. $|F| \cdot r \leqslant e^{-1}$.

If $|F| \cdot r = e^{-1}$ there is only one σ which satisfies Proposition 4, $\sigma = -1/r$. If $|F| \cdot r < e^{-1}$ we can sharpen the result of Proposition 4, that is, we can choose $\omega < 0$ such that $A + \omega I$ is m-accretive in L_σ^1.

PROPOSITION 5. *Let F satisfy* (H) *and let* $|F| \cdot r < e^{-1}$. *If* $\omega = \dfrac{1 + \log r|F|}{r}$

and $\sigma = \omega - 1/r$, *then* $A + \omega I$ *is m-accretive in* L_σ^1.

Proof. We have to prove that if φ_1, $\varphi_2 \in D_A$ and $\varphi = \varphi_1 - \varphi_2$, then

$$(7) \qquad \|\varphi\|_\sigma \leqslant \frac{1}{1-\lambda\omega} \|\varphi - \lambda\varphi'\|_\sigma \qquad \text{for } \lambda > 0 \text{ small.}$$

From (5) we have to verify, instead of (6), the following

$$1-\frac{\lambda|F|_\sigma}{1-\sigma\lambda}(1-e^{(1-\sigma\lambda)(-r)/\lambda}) \geqslant \frac{1-\omega\lambda}{1-\sigma\lambda}(1-e^{(1-\sigma\lambda)(-r)/\lambda}),$$

that is,

$$(\lambda|F|_\sigma+1-\omega\lambda)e^{(1-\sigma\lambda)(-r)/\lambda} \geqslant \lambda(\sigma-\omega+|F|_\sigma),$$

which is satisfied if $\sigma-\omega+|F|_\sigma \leqslant 0$. And

$$\sigma-\omega+|F|_\sigma \leqslant \sigma-\omega+|F|e^{-\sigma r} = \sigma-\omega+|F|e^{-\omega r}e^{-(\sigma-\omega)r} = 0$$

if $\sigma-\omega = -1/r$ and $|F|e^{-\omega r} = (r\cdot e)^{-1}$.

Note that for all φ, $\psi\in C$

$$|F(\varphi)-F(\psi)| \leqslant |F|\,\|\varphi-\psi\| \leqslant |F|\cdot r\,\|\|\varphi-\psi\|\|_0,$$

that is, $F_{|C}\colon C\to X$ is Lipschitz continuous with constant $|F|\cdot r$. Let $|F|\cdot r < 1$; then $F_{|X}$ is a contraction and therefore has a unique fixed point φ_0. And $\varphi_0(t) = \varphi_0$, $t \geqslant -r$, is the unique constant solution of (FE). Moreover, as it is proved in [3], we can choose $\omega < 0$ such that $\tilde{A}+\omega I$ is m-accretive in C_ω and from Proposition 3 it follows that, for $t \geqslant r$, $\varphi\in L^1$,

$$\|\|T(t)\varphi - T(t)\varphi_0\|\|_\omega = \|\|\tilde{T}(t-r)T(r)\varphi - \tilde{T}(t-r)\varphi_0\|\|_\omega$$

$$\leqslant e^{(t-r)\omega}\|\|T(r)\varphi-\varphi_0\|\|_\omega.$$

Hence if $x(t)$ is the solution of (FE) with initial data φ, we have

$$|x(t)-\varphi_0| \leqslant e^{\omega t}e^{-r\omega}\|\|T(r)\varphi-\varphi_0\|\|_\omega$$

and φ_0 is exponentially asymptotically stable.

4. Flow-invariant sets. Regularity results. In Section 2 it is implicitly proved that the sets $M = \{\varphi\in C, \varphi(0) = F(\varphi)\}$ and $M_1 = \{\varphi\colon [-r, 0] \to X$, is piecewise continuous$\}$ are flow-invariant in the sense that $T(t)M \subset M$ and $T(t)M_1 \subset M_1$ for all $t \geqslant 0$. The following sets which extend D_A are clearly flow-invariant

$$(8)\qquad D_{A,\sigma} = \left\{\varphi\in L^1\colon \overline{\lim_{t\to 0}}\,\frac{\|\varphi-T(t)\varphi\|}{.t^\sigma} < +\infty\right\}, \qquad 0 < \sigma \leqslant 1,$$

and so are the sets

$$(9)\qquad D_{\tilde{A},\sigma} = \left\{\varphi\in M, \overline{\lim_{t\to 0}}\,\frac{\|\|\varphi-T(t)\varphi\|\|_0}{t^\sigma} < +\infty\right\}, \qquad 0 < \sigma \leqslant 1,$$

as follows from the results of Section 2. It is easy to prove ([7]) that

$$D_{\tilde{A},\sigma} = \{\varphi\in C \text{ is Hölder continuous with exponent } \sigma, \varphi(0) = F(\varphi)\}$$

for $0 < \sigma < 1$ and that

$$D_{\tilde{A},1} = \{\varphi \in C \text{ is Lipschitz continuous}, \ \varphi(0) = F(\varphi)\},$$

and so if inital data are in M and are Lipschitz or Hölder continuous, then solutions are respectively Lipschitz or Hölder continuous on bounded sets. The following proposition characterizes the sets $D_{A,\sigma}$.

PROPOSITION 6. *For all* σ, $0 < \sigma \leqslant 1$, *we have*

$$D_{A,\sigma} = D_{A_0,\sigma} = \left\{ \varphi \in L^1, \ \int_{-r}^{-t} |\varphi(t+\vartheta) - \varphi(\vartheta)| \, d\vartheta + \int_{-t}^{0} |\varphi(\vartheta)| \, d\vartheta \leqslant K_\varphi t^\sigma \right\},$$

where K *depends on* φ *and* σ.

Proof. Let $\psi \in D_{A_0,\sigma}$, so that

$$\varlimsup_{\lambda \to 0} \frac{\|\psi - (I + \lambda A_0)^{-1}\psi\|}{\lambda^\sigma} = K < +\infty$$

([1], [5]). Set $\varphi = (I + \lambda A)^{-1}\psi$; we have

$$\frac{\|\psi - \varphi\|}{\lambda^\sigma} \leqslant \frac{1}{\lambda^\sigma} \|e^{\vartheta/\lambda} \varphi(0)\| + \frac{1}{\lambda^\sigma} \|\psi - (I + \lambda A_0)^{-1}\psi\|$$

$$= \lambda^{1-\sigma}(1 - e^{-r/\lambda}) |(I + \lambda A)^{-1}\psi(0)| + \frac{1}{\lambda^\sigma} \|\psi - (I + \lambda A_0)^{-1}\psi\|.$$

From Lemma 2 it follows that

$$\varlimsup_{\lambda \to 0} \frac{\|\psi - \varphi\|}{\lambda^\sigma} \leqslant K \quad \text{for } 0 < \sigma < 1$$

and

$$\varlimsup_{\lambda \to 0} \frac{\|\psi - \varphi\|}{\lambda} \leqslant K + |F(\psi)|,$$

and hence $\psi \in D_{A,\sigma}$.

Vice versa, let $\psi \in D_{A,\sigma}$; then

$$\|\psi - (I + \lambda A_0)^{-1}\psi\| \leqslant \|\psi - (I + \lambda A)^{-1}\psi\| + \|e^{\vartheta/\lambda} \varphi(0)\|$$

and, using again Lemma 2, we have that $\psi \in D_{A_0,\sigma}$.

We have to verify that $D_{A_0,\sigma}$ coincides with the set

$$E_\sigma = \left\{ \varphi \in L^1, \ \int_{-r}^{-t} |\varphi(t+\vartheta) - \varphi(\vartheta)| \, d\vartheta + \int_{-t}^{0} |\varphi(\vartheta)| \, d\vartheta < K_\varphi t^\sigma \right\}.$$

The semigroup, $T_0(t)$, generated by A_0 is such that

$$T_0(t)\varphi(\vartheta) = \begin{cases} \varphi(t+\vartheta) & \text{a.e.,} \quad t+\vartheta < 0, \\ 0, & t+\vartheta \geqslant 0, \end{cases}$$

and so if $0 < t < r$

$$\|T_0(t)\varphi - \varphi\| = \int\limits_{-r}^{0} |T(t)\varphi(\vartheta) - \varphi(\vartheta)| \, d\vartheta = \int\limits_{-r}^{-t} |\varphi(t+\vartheta) - \varphi(\vartheta)| \, d\vartheta + \int\limits_{-t}^{0} |\varphi(\vartheta)| \, d\vartheta;$$

hence $D_{A_0,\sigma} = E_\sigma$.

Note that $D_{\tilde{A},\sigma}$ depends on F. It follows that it is not possible to use the Crandall–Pazy theory on nonlinear semigroups to study the non-autonomous version of (FE) in spaces of continuous functions. This is possible in L^1 spaces as, from Proposition 6, it follows that the set $D_{A,\sigma}$ does not depend on F. We will discuss the non-autonomous equation $x(t) = F(t, x_t)$ in another paper.

5. An Example. We show now how the problem

(P)
$$\begin{aligned}
Du(t, a) &= g\big(u(t, \cdot)\big)(a), & 0 \leqslant t < r, \ a > 0, \\
u(t, 0) &= f\big(u(t, \cdot)\big), & 0 \leqslant t < r, \\
u(0, a) &= u_0(a), & a > 0,
\end{aligned}$$

is related to (FE).

Here

$$Du(t, a) = \lim_{h \to 0^+} \frac{u(t+h, a+h) - u(t, a)}{h}, \qquad g \colon L^1(0, r; X) \to L^1(0, r; X),$$

$$f \colon L^1(0, r; X) \to X \quad \text{and} \quad u_0 \in L^1(0, r; X).$$

If $X = R^n$ and we take

$$g(\varphi)(a) = -\mu\Big(a, \int\limits_0^\infty \varphi(b) \, db\Big)\varphi(a) \quad \text{and} \quad f(\varphi) = \int\limits_0^\infty \beta\Big(a, \varphi(a), \int\limits_0^\infty \varphi(b) \, db\Big) da,$$

then problem (P) is the Gurtin–Mac-Camy model of age-dependent population problems (see [4], [11]).

We consider first the case where $g = 0$, and so (P) becomes

(P)'
$$\begin{aligned}
Du(t, a) &= 0, & 0 \leqslant t < r, \ a > 0, \\
u(t, 0) &= f\big(u(t, \cdot)\big), & 0 \leqslant t < r, \\
u(0, a) &= u_0(a), & a > 0.
\end{aligned}$$

Suppose that

(H)' $f \colon L^1(0, r; X) \to X$ is Lipschitz continuous with constant $|f|$.

We set $h \colon L^1(-r, 0; X) \to L^1(0, r; X)$, $(h\varphi)(\vartheta) = \varphi(-\vartheta)$ and $F = f \circ h$; clearly, $F \colon L^1(-r, 0; X) \to X$ is Lipschitz continuous with constant $|F| = |f|$. Hence

$$A\varphi = -\varphi', \qquad D_A = \{\varphi \in W^{1,1}(-r, 0; X), \ \varphi(0) = (f \circ h)\varphi\},$$

generates a semigroup $T(t)$ in $L^1(-r, 0; X)$ which satisfies Theorem 1. It follows that, setting

$$(5) \qquad u(t, a) = \begin{cases} u_0(a-t) & \text{a.e.,} \quad t < a, \\ (h\,T(t)\,h^{-1}u_0)(a), & t \geqslant a, \end{cases}$$

$u(t, a)$ is the unique solution of (P)'; moreover, $u_{|\Delta}$, $\Delta = \{(t, a),\ 0 \leqslant a \leqslant r,\ t \geqslant a\}$, is continuous. In fact, from (3) we have

$$(6) \qquad T(t)(h^{-1}u_0)(-a) = \begin{cases} u_0(-t+a) & \text{a.e.,} \quad t-a < 0, \\ T(t-a)(h^{-1}u_0)(0), & t-a \geqslant 0, \end{cases}$$

and

$$(7) \qquad T(t)(h^{-1}u_0)(-a) = F\big(T(t-a)\,h^{-1}u_0\big) = f\big(hT(t-a)\,h^{-1}u_0\big), \quad t \geqslant a.$$

And so from (5) and (6) we have

$$(8) \qquad u(t+h, a+h) - u(t, a) = 0 \quad \text{a.e.,} \quad h > 0$$

and from (7)

$$u(t, 0) = \big(hT(t)\,h^{-1}u_0\big)(0) = \big(T(t)\,h^{-1}u_0\big)(0) = f\big(hT(t)\,h^{-1}u_0\big) = f\big(u(t, \cdot)\big).$$

Hence $u(t, a)$ is a solution of (P)'.

Vice versa, if $u(t, a)$ is a solution of (P)', then set

$$(9) \qquad x(t) = \begin{cases} u_0(-t), & t < 0, \\ u(t, 0), & t \geqslant 0; \end{cases}$$

$x(t)$ is a solution of (FE). In fact, from (8) and (9) it follows that, if $t \geqslant 0$, $u(t, \cdot) = hx_t$ and so $x(t) = u(t, 0) = f\big(u(t, \cdot)\big) = F(x_t)$. If $f(\varphi) = \int_0^r \beta\big(a, \varphi(a), \int_0^r \varphi(b)\,db\big)\,da$, then problem (P)' is equivalent to the integral equation

$$x(t) = \int_0^t \beta\Big(a, x(t-a), \int_0^t x(t-b)\,db + \int_t^r u_0(b-t)\,db\Big)\,da +$$

$$+ \int_t^r \beta\Big(a, u_0(a-t), \int_0^t x(t-b)\,db + \int_t^r u_0(b-t)\,db\Big)\,da.$$

Note that $hT(t)\,h^{-1}$ is the semigroup generated in $L^1(0, r; X)$ by the m-accretive operator $B\varphi = \varphi'$, $D_B = \{\varphi \in W^{1,1}(0, r; X),\ \varphi(0) = f(\varphi)\}$. If g is sufficiently smooth, for example if is Lipschitz continuous, then $B - g$ is m-accretive in $L^1(0, r; X)$ and generates a semigroup which gives the solutions of (P).

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