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Properties of a class of functions with bounded boundary rotation

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Abstract. Let $P_k(\varrho)$ denote the class of regular functions $p_\varrho(z)$ in $E = \{z\colon |z| < 1\}$, satisfying $p_\varrho(0) = 1$ and $\int\limits_0^{2\pi} \left| \frac{\operatorname{Re} p_\varrho(z) - \varrho}{1-\varrho} \right| d\theta \leqslant k\pi$ for $k \geqslant 2$ and $0 \leqslant \varrho < 1$. Let $V_k(\varrho)$ denote the class of regular functions $f_\varrho(z)$ in E with normalizations $f_\varrho(0) = 0$ and $f'_\varrho(0) = 1$, also satisfying the condition $1 + z\frac{f''_\varrho(z)}{f'_\varrho(z)} \epsilon P_k(\varrho)$, $0 \leqslant \varrho < 1$. This class generalizes the class of convex functions of the order ϱ in the same way as the class V_k of functions of bounded boundary rotation generalizes the class of convex functions. Let $U_k(\varrho)$ denote the class of regular functions $f_\varrho(z)$ in E with $f_\varrho(0) = 0$, $f'_\varrho(0) = 1$ and satisfying $z\frac{f'_\varrho(z)}{f_\varrho(z)} \epsilon P_k(\varrho)$. This class generalizes the class of starlike functions of the order ϱ. In this paper we investigate certain properties of the above-mentioned classes and determine the radius of convexity for both the classes $V_k(\varrho)$ and $U_k(\varrho)$.

1. Introduction. Let $P_k(\varrho)$ denote the class of regular functions $p_\varrho(z)$ in $E = \{z\colon |z| < 1\}$, satisfying the properties $p_\varrho(0) = 1$ and

$$\int\limits_0^{2\pi} \left| \frac{\operatorname{Re} p_\varrho(z) - \varrho}{1-\varrho} \right| d\theta \leqslant k\pi \quad \text{for } k \geqslant 2 \text{ and } 0 \leqslant \varrho < 1. \tag{1.1}$$

When $\varrho = 0$ we get the class P_k defined by Pinchuk [4]. Let $V_k(\varrho)$ denote the class of regular functions $f_\varrho(x)$ in E with normalizations $f_\varrho(0) = 0$ and $f'_\varrho(0) = 1$, also satisfying the condition

$$1 + \frac{zf''_\varrho(z)}{f'_\varrho(z)} \epsilon P_k(\varrho), \quad 0 \leqslant \varrho < 1. \tag{1.2}$$

When $\varrho = 0$ we get the class V_k of functions of bounded boundary rotation studied by Paatero [3]. This class $V_k(\varrho)$ generalizes the class $K(\varrho)$ of convex functions of the order ϱ introduced by Robertson [7]. Let $U_k(\varrho)$ denote the class of regular functions $f_\varrho(z)$ in E with $f_\varrho(0) = 0$, $f'_\varrho(0) = 1$ and satisfying

$$\frac{zf'_\varrho(z)}{f_\varrho(z)} \epsilon P_k(\varrho). \tag{1.3}$$

This class generalizes the class $S^*(\varrho)$ of starlike functions of the order ϱ, also investigated by Robertson [7].

In this paper we investigate certain properties of the above-mentioned classes and determine the radius of convexity for the class $V_k(\varrho)$ and also the radius of convexity for the class $V_k(\varrho)$.

2. A representation theorem for the class $V_k(\varrho)$.

LEMMA 1. *If $p_\varrho(z) \in P_k(\varrho)$, then*

$$p_\varrho(z) = \frac{1}{2}\int_0^{2\pi} \frac{1+(1-2\varrho)ze^{-it}}{1-ze^{-it}}\, dm(t), \tag{2.1}$$

where $m(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} dm(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |dm(t)| \leqslant k. \tag{2.2}$$

Proof. Setting $f(z) = \dfrac{p_\varrho(z)-\varrho}{1-\varrho} = u(z)+iv(z)$ we get

$$u(z) = \mathrm{Re} f(z) = \mathrm{Re}\left\{\frac{p_\varrho(z)-\varrho}{1-\varrho}\right\},$$

$u(0) = 1$ and $v(0) = 0$. Since $p_\varrho(z) \in P_k(\varrho)$,

$$\int_0^{2\pi} \left|\frac{\mathrm{Re}\, p_\varrho(z)-\varrho}{1-\varrho}\right| d\theta \leqslant k\pi, \qquad k \geqslant 2,\ 0 \leqslant |z| = r < 1.$$

For $\varrho < 1$, $f(z)$ is regular in E, hence by Paatero's theorem [3] there exists a function $m(t)$ of bounded variation in $[0, 2\pi]$, satisfying $\int_0^{2\pi} dm(t) = 2$ and $\int_0^{2\pi} |dm(t)| \leqslant k$ such that

$$f(z) = \frac{p_\varrho(z)-\varrho}{1-\varrho} = \frac{1}{2}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\, dm(t), \qquad |z| < 1,$$

which yields

$$p_\varrho(z) = \frac{1}{2}\int_0^{2\pi} \frac{1+(1-2\varrho)ze^{-it}}{1-ze^{-it}}\, dm(t).$$

LEMMA 2. *$f_\varrho(z) \in V_k(\varrho)$ if and only if there exists an $f(z) \in V_k$ such that*

$$f_\varrho'(z) = \{f'(z)\}^{(1-\varrho)}.$$

Proof. Paatero [3] proved that $f(z) \in V_k$ if and only if there exists a function $m(t)$ of bounded variation on $[0, 2\pi]$ such that

$$f'(z) = \exp\left\{-\int_0^{2\pi} \log(1-ze^{-it})\,dm(t)\right\}; \tag{2.3}$$

$$\int_0^{2\pi} dm(t) = 2; \qquad \int_0^{2\pi} |dm(t)| \leqslant k.$$

Hence

$$1+z\frac{f''(z)}{f'(z)} = \frac{1}{2}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\,dm(t).$$

Also, since $f_\varrho(z) \in V_k(\varrho)$, satisfies (1.2), there exists a $p_\varrho(z) \in P_k(\varrho)$ such that

$$1+z\frac{f_\varrho''(z)}{f_\varrho'(z)} = p_\varrho(z).$$

From Lemma 1 we get

$$\left\{1+z\frac{f_\varrho''(z)}{f_\varrho'(z)} - \varrho\right\} = \frac{(1-\varrho)}{2}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\,dm(t) = (1-\varrho)\left\{1+z\frac{f''(z)}{f'(z)}\right\},$$

where $f(z) \in V_k$. Hence

$$\frac{f_\varrho''(z)}{f_\varrho'(z)} = (1-\varrho)\frac{f''(z)}{f'(z)},$$

which on integration gives the required result.

THEOREM 1. *$f_\varrho(z) \in V_k(\varrho)$ if and only if*

$$f_\varrho'(z) = \exp\left\{-(1-\varrho)\int_0^{2\pi} \log(1-ze^{-it})\,dm(t)\right\},$$

where $m(t)$ satisfies (2.2).

Proof. This is an immediate consequence of the above lemmas.

Remark. When $k = 2$ it is easy to see that $\operatorname{Re}\left\{1+z\frac{f_\varrho''(z)}{f_\varrho'(z)}\right\} > \varrho$ holds for the functions $f_\varrho(z) \in V_k(\varrho)$. Thus the class $V_k(\varrho)$ coincides with the class $K(\varrho)$ of convex functions of the order ϱ when $k = 2$.

COROLLARY 1. *$f_\varrho(z) \in V_k(\varrho)$ if and only if there exist $f_{\varrho i}(z) \in K(\varrho)$ for $i = 1, 2$, such that*

$$f_\varrho'(z) = \{f_{\varrho 1}'(z)\}^{(k/4+1/2)}/\{f_{\varrho 2}'(z)\}^{(k/4-1/2)}.$$

Proof. This follows immediately from the integral representation for $f_\varrho(z)$.

COROLLARY 2. *Suppose* $f_\varrho(z) \in V_k(\varrho)$. *Then*

$$|\arg f_\varrho'(z)| \leqslant (1-\varrho)k \sin^{-1} r, \quad \textit{where } |z| = r \textit{ and } k \geqslant 2.$$

Proof. This follows from Lemma 2 and the fact that if $f(z) \in V_k$, then

$$|\arg f_\varrho'(z)| \leqslant k \sin^{-1} r, \quad [5].$$

3. Some properties of the class $V_k(\varrho)$.

LEMMA 3. *Suppose* $f_\varrho(z) \in V_k(\varrho)$. *Then* $F_\varrho(z)$, *defined by*

$$F_\varrho'(z) = \frac{f_\varrho'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{f_\varrho'(a)(1+\bar{a}z)^{2(1-\varrho)}} \tag{3.1}$$

for $|a| < 1$ *and* $z \in E$, *also belongs to* $V_k(\varrho)$.

Proof. Since $f_\varrho(z) \in V_k(\varrho)$, from Lemma 2, there exists an $f(z) \in V_k$ such that $f_\varrho'(z) = \{f'(z)\}^{(1-\varrho)}$. It is well known [6] that if $f(z) \in V_k$, then $F(z)$, defined by

$$F(z) = \frac{f\left(\dfrac{z+a}{1+\bar{a}z}\right) - f(a)}{f'(a)(1-|a|^2)} \quad \text{for } |a| < 1,$$

also belongs to V_k. Therefore there exists an $F_\varrho(z) \in V_k(\varrho)$ such that

$$F_\varrho'(z) = \{F'(z)\}^{1-\varrho} = \frac{\left\{f'\left(\dfrac{z+a}{1+z\bar{a}}\right)\right\}^{(1-\varrho)}}{\{f'(a)\}^{(1-\varrho)}(1+z\bar{a})^{2(1-\varrho)}} = \frac{f_\varrho'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{f_\varrho'(a)(1+\bar{a}z)^{2(1-\varrho)}},$$

which proves the lemma.

THEOREM 2. *Suppose* $f_\varrho(z) = z + A_2 z^2 + A_3 z^3 + \ldots \in V_k(\varrho)$; *then*

$$|A_2| \leqslant \frac{k(1-\varrho)}{2} \quad \textit{for } k \geqslant 2, \tag{3.2}$$

and

$$|A_3 - A_2^2| \leqslant \begin{cases} \dfrac{(1-\varrho)}{12}\{(1-\varrho)k^2 - 4\} & \textit{for } k > \dfrac{4}{1-\varrho}, \\[2ex] \dfrac{(1-\varrho)(k-1)}{3} & \textit{for } k < \dfrac{4}{1-\varrho}. \end{cases} \tag{3.3}$$

Inequality (3.2) *is sharp; inequality* (3.3) *is sharp for* $k > 4/(1-\varrho)$.

Proof. Since $f_\varrho(z) \in V_k(\varrho)$ there exists an $f(z) = z + a_2z^2 + \ldots \in V_k$ such that $f'_\varrho(z) = \{f'(z)\}^{(1-\varrho)}$. Hence

$$1 + 2A_2z + 3A_3z^2 + \ldots = 1 + 2(1-\varrho)a_2z + [3a_3(1-\varrho) - 2\varrho(1-\varrho)a_2^2]z^2 + \ldots$$

Comparing the coefficients we get $|A_2| = (1-\varrho)|a_2| \leqslant (1-\varrho)k/2$, $k \geqslant 2$, [6]. Also

$$|A_3 - A_2^2| = \left|(1-\varrho)a_3 - (1-\varrho)\left(1-\frac{\varrho}{3}\right)a_2^2\right| = (1-\varrho)\left|a_3 - \left(1-\frac{\varrho}{3}\right)a_2^2\right|$$

$$\leqslant \begin{cases} \dfrac{(1-\varrho)}{3}\left\{(1-\varrho)\dfrac{k^2}{4} - 1\right\} & \text{for } k > \dfrac{4}{1-\varrho}, \\ \dfrac{(1-\varrho)(k-1)}{3} & \text{for } k < \dfrac{4}{1-\varrho}, \end{cases}$$

by applying a theorem of Moulis [2]. Taking the function $f_\varrho(z)$, defined by

$$f'_\varrho(z) = \frac{(1-z)^{(k/2-1)(1-\varrho)}}{(1+z)^{(k/2+1)(1-\varrho)}},$$

we can show that these inequalities (3.2) and (3.3) for $k > 4/(1-\varrho)$ are sharp.

Theorem 3. *Suppose* $f_\varrho(z) \in V_k(\varrho)$. *Then it is convex in*

$$|z| \leqslant R_\varrho = \frac{k(1-\varrho) - \sqrt{k^2(1-\varrho)^2 - 4(1-2\varrho)}}{2(1-2\varrho)}.$$

Also $f_\varrho(z)$ *is a convex function of the order* ϱ *for* $|z| \leqslant \dfrac{k - \sqrt{k^2-4}}{2}$. *These bounds are sharp.*

Proof. Suppose $f_\varrho(z) \in V_k(\varrho)$. Then by Lemma 3, $F_\varrho(z)$ defined by (3.1) is also in $V_k(\varrho)$. Hence

$$\left|\frac{F''_\varrho(0)}{2}\right| = \frac{1}{2}\left|\frac{f''_\varrho(a)}{f'_\varrho(a)}(1-|a|^2) - 2(1-\varrho)\bar{a}\right| \leqslant \frac{k(1-\varrho)}{2} \qquad \text{from (3.2)},$$

for $|a| < 1$. Hence

$$\left|\frac{f''_\varrho(a)}{f'_\varrho(a)} - \frac{2(1-\varrho)\bar{a}}{1-|a|^2}\right| \leqslant \frac{(1-\varrho)k}{1-|a|^2}.$$

Since a is any arbitrary complex number in E, we can replace a by z in the above inequality and write

$$(3.4) \qquad \left|z\frac{f''_\varrho(z)}{f'_\varrho(z)} - \frac{2(1-\varrho)|z|^2}{1-|z|^2}\right| \leqslant \frac{k(1-\varrho)|z|}{1-|z|^2} \qquad \text{for } |z| < 1.$$

Hence

$$\operatorname{Re}\left\{1+z\frac{f_\varrho''(z)}{f_\varrho'(z)}\right\}\geqslant\frac{1-k(1-\varrho)r+(1-2\varrho)r^2}{1-r^2},\qquad \text{where } |z|=r.$$

Thus

$$\operatorname{Re}\left\{1+z\frac{f_\varrho''(z)}{f_\varrho'(z)}\right\}>0$$

for

$$|z|=r<R_\varrho=\frac{(1-\varrho)k-\sqrt{k^2(1-\varrho)^2-4(1-2\varrho)}}{2(1-2\varrho)}.$$

Also $\operatorname{Re}\left\{1+z\frac{f_\varrho''(z)}{f_\varrho'(z)}\right\}>\varrho$ provided $1-kr+r^2>0$, i.e., $r\leqslant\frac{k-\sqrt{k^2-4}}{2}$.

These bounds are sharp for the function $f_\varrho(z)$, defined by

$$f_\varrho'(z)=\frac{(1-z)^{(k/2-1)(1-\varrho)}}{(1+z)^{(k/2+1)(1-\varrho)}}.$$

Corollary 3. *Suppose that* $f_\varrho(z)\in V_k(\varrho)$; *then it is univalent if* $\varrho\geqslant(k+1)/(k+2)$.

Proof. Since $f_\varrho(z)\in V_k(\varrho)$, we have from (3.4)

$$\left|\frac{f_\varrho''(z)}{f_\varrho'(z)}-\frac{2(1-\varrho)|z|}{1-|z|^2}\right|\leqslant\frac{k(1-\varrho)}{1-|z|^2}.$$

Hence

$$\left|\frac{f_\varrho''(z)}{f_\varrho'(z)}\right|\leqslant\frac{(k+2|z|)(1-\varrho)}{(1-|z|^2)}<\frac{(k+2)(1-\varrho)}{(1-|z|^2)}\qquad\text{since } |z|<1.$$

It is well known [1] that if

$$\left|\frac{F''(z)}{F'(z)}\right|\leqslant\frac{\beta}{1-|z|^2}\qquad\text{in } |z|<1$$

for some constant β, where β is at least 1, then $F(z)$ is univalent in E. Hence $f_\varrho(z)$ is univalent if $(k+2)(1-\varrho)\leqslant 1$, that is, if $\varrho\geqslant(k+1)/(k+2)$.

Theorem 4. *Suppose* $f_\varrho(z)\in V_k(\varrho)$; *then for* $k>4/(1-\varrho)$,

$$|\{f_\varrho,z\}|\leqslant\frac{(1-\varrho)\{(1-\varrho)k^2-4+4k\varrho r+4\varrho r^2\}}{2(1-r^2)^2},$$

where $\{f_\varrho,z\}$ *denotes the Schwarzian derivative of* f_ϱ *with respect to* z.

Proof. Suppose $f_\varrho(z) \in V_k(\varrho)$. Then $F_\varrho(z) = z + B_2 z^2 + \ldots$, defined by (3.1), also belongs to $V_k(\varrho)$. Then

$$6|B_3 - B_2^2| = \left|\left(\frac{F_\varrho''(z)}{F_\varrho'(z)}\right)' - \frac{1}{2}\left(\frac{F_\varrho''(z)}{F_\varrho'(z)}\right)^2\right| \qquad \text{at } z = 0.$$

Also

$$\left\{\frac{F_\varrho''(z)}{F_\varrho'(z)}\right\}'_{\text{at } z=0} = \left\{\frac{f_\varrho''(a)}{f_\varrho'(a)}\right\}' (1-|a|^2)^2 - \frac{f_\varrho''(a)}{f_\varrho'(a)} 2\bar{a}(1-|a|^2) + 2\bar{a}^2(1-\varrho),$$

and

$$\left\{\frac{F_\varrho''(z)}{F_\varrho'(z)}\right\}_{\text{at } z=0} = \frac{f_\varrho''(a)}{f_\varrho'(a)}(1-|a|^2) - 2(1-\varrho)\bar{a}.$$

Hence

$$\begin{aligned} 6|B_3 - B_2^2| &= \left|\{f_\varrho, a\}(1-|a|^2)^2 - 2\bar{a}\varrho(1-|a|^2)\left\{\frac{f_\varrho''(a)}{f_\varrho'(a)}\right\} + 2\varrho\bar{a}^2(1-\varrho)\right| \\ &= \left|\{f_\varrho, a\}(1-|a|^2)^2 - 2\bar{a}\varrho(1-|a|^2)\left\{\frac{f_\varrho''(a)}{f_\varrho'(a)} - \frac{\bar{a}(1-\varrho)}{1-|a|^2}\right\}\right|; \end{aligned}$$

that is,

$$|\{f_\varrho, a\}| \leqslant \frac{6|B_3 - B_2^2|}{(1-|a|^2)^2} + \frac{2\varrho|a|}{1-|a|^2}\left|\frac{f_\varrho''(a)}{f_\varrho'(a)} - \frac{\bar{a}(1-\varrho)}{1-|a|^2}\right|.$$

For $k > 4/(1-\varrho)$, from (3.3) we have

$$|\{f_\varrho, a\}| \leqslant \frac{(1-\varrho)\{(1-\varrho)k^2 - 4\}}{2(1-|a|^2)^2} + \frac{2\varrho|a|}{1-|a|^2}\left|\frac{f_\varrho''(a)}{f_\varrho'(a)} - \frac{\bar{a}(1-\varrho)}{1-|a|^2}\right|.$$

Also

$$\begin{aligned} \left|\frac{f_\varrho''(a)}{f_\varrho'(a)} - \frac{\bar{a}(1-\varrho)}{1-|a|^2}\right| &\leqslant \left|\frac{f_\varrho''(a)}{f_\varrho'(a)} - \frac{2\bar{a}(1-\varrho)}{1-|a|^2}\right| + \frac{|a|(1-\varrho)}{1-|a|^2} \\ &\leqslant \frac{k(1-\varrho)}{1-|a|^2} + \frac{|a|(1-\varrho)}{1-|a|^2}, \end{aligned}$$

by (3.4). Using this in the above inequality we get

$$\begin{aligned} |\{f_\varrho, a\}| &\leqslant \frac{(1-\varrho)\{(1-\varrho)k^2 - 4\}}{2(1-|a|^2)^2} + \frac{2(1-\varrho)(k+|a|)\varrho|a|}{(1-|a|^2)^2} \\ &= \frac{(1-\varrho)}{2(1-|a|^2)^2}\{(1-\varrho)k^2 - 4 + 4k\varrho|a| + 4\varrho|a|^2\}. \end{aligned}$$

Since a is an arbitrary complex number in E, we can replace a by z in the above inequality and write

$$|\{f_\varrho, z\}| \leqslant \frac{1-\varrho}{2(1-r^2)^2}\{(1-\varrho)k^2 - 4 + 4k\varrho r + 4\varrho r^2\}, \qquad \text{where } |z| = r.$$

This completes the proof of the theorem.

4. Properties of the class $U_k(\varrho)$.

THEOREM 5. *$f_\varrho(z) \in V_k(\varrho)$ if and only if $zf'_\varrho(z) \in U_k(\varrho)$.*

Proof. This follows at once from the definitions of the classes $V_k(\varrho)$ and $U_k(\varrho)$.

COROLLARY 4. *$f_\varrho(z) \in U_k(\varrho)$ if and only if*

$$(4.1) \qquad f_\varrho(z) = z\exp\Big\{-(1-\varrho)\int_0^{2\pi} \log(1-ze^{-it})\,dm(t)\Big\},$$

where $m(t)$ is a function of bounded variation on $[0, 2\pi]$, satisfying (2.2). *Also there exist $s_{\varrho_1}(z)$ and $s_{\varrho_2}(z)$ in $S^*(\varrho)$ such that*

$$(4.2) \qquad f_\varrho(z) = \frac{\{s_{\varrho_1}(z)\}^{k/4+1/2}}{\{s_{\varrho_2}(z)\}^{k/4-1/2}}.$$

Proof. This is a consequence of Theorem 1, Theorem 5 and Corollary 1.

Remark. When $k = 2$ it is easy to see that $\mathrm{Re}\left\{\frac{zf'_\varrho(z)}{f_\varrho(z)}\right\} > \varrho$ holds for functions $f_\varrho(z)$ in $U_k(\varrho)$. Thus the class $U_k(\varrho)$ coincides with the class of starlike functions of the order ϱ, $S^*(\varrho)$ when $k = 2$.

COROLLARY 5. *Suppose $f_\varrho(z) \in U_k(\varrho)$; then it is starlike for*

$$|z| \leqslant R_\varrho = \frac{(1-\varrho)k - \sqrt{k^2(1-\varrho)^2 - 4(1-2\varrho)}}{2(1-2\varrho)}.$$

Also $f_\varrho(z)$ is starlike of the order ϱ for $|z| \leqslant \frac{k-\sqrt{k^2-4}}{2}$. These bounds are sharp.

Proof. This follows from Theorem 3 and Theorem 5. For the function

$$F_\varrho(z) = z\left\{\frac{(1+z)^{k/2-1}}{(1-z)^{k/2+2}}\right\}^{(1-\varrho)}$$

we have

$$F'_\varrho(z) = \left\{\frac{(1+z)^{k/2-1}}{(1-z)^{k/2+1}}\right\}^{-\varrho}\left\{\frac{(1+z)^{k/2-2}}{(1-z)^{k/2+2}}\right\}\{1+k(1-\varrho)z+(1-2\varrho)z^2\}$$
$$= 0 \quad \text{for } z = -R_\varrho.$$

Hence the radius of univalence for $U_k(\varrho)$ coincides with the radius of starlikeness for $U_k(\varrho)$.

THEOREM 6. *If $f_\varrho(z) \in U_k(\varrho)$, then for $|z| = r < 1$*

$$r\left\{\frac{(1-r)^{k/2-1}}{(1+r)^{k/2+1}}\right\}^{(1-\varrho)} \leqslant |f_\varrho(z)| \leqslant r\left\{\frac{(1+r)^{k/2-1}}{(1-r)^{k/2+1}}\right\}^{(1-\varrho)}.$$

Proof. Since $f_\varrho(z) \in U_k(\varrho)$,

$$|f_\varrho(re^{i\theta})| = r\exp\left\{-\int_0^{2\pi}(1-\varrho)\log|1-ze^{-it}|\,dm(t)\right\}, \tag{4.3}$$

$$\int_0^{2\pi} dm(t) = 2, \qquad \int_0^{2\pi} |dm(t)| \leqslant k.$$

Now

$$-\int_0^{2\pi}\log|1-ze^{-it}|\,dm(t) = -\int_0^{2\pi}\log|1-ze^{-it}|\,dp(t) + \int_0^{2\pi}\log|1-ze^{-it}|\,dn(t),$$

where $p(t)$ and $n(t)$ are non-decreasing functions on $[0, 2\pi]$, satisfying $m(t) = p(t) - n(t)$,

$$\int_0^{2\pi} dp(t) \leqslant \left(\frac{k}{2}+1\right) \quad \text{and} \quad \int_0^{2\pi} dn(t) \leqslant \left(\frac{k}{2}-1\right).$$

$$-\int_0^{2\pi}\log|(1-ze^{-it})|\,dm(t) \leqslant -\left(\frac{k}{2}+1\right)\log(1-r) + \left(\frac{k}{2}-1\right)\log(1+r)$$

$$= \log\left\{\frac{(1+r)^{k/2-1}}{(1-r)^{\quad 2+1}}\right\}.$$

Further, we assert that

$$-\int_0^{2\pi}\log|(1-ze^{-it})|\,dm(t) \geqslant \log\left\{\frac{(1-r)^{k/2-1}}{(1+r)^{k/2+1}}\right\}.$$

To see this, we first assume that $\int_0^{2\pi}|dm(t)| = k$; then

$$\int_0^{2\pi} dp(t) = \left(\frac{k}{2}+1\right) \quad \text{and} \quad \int_0^{2\pi} dn(t) = \left(\frac{k}{2}-1\right).$$

Hence

$$-\int_0^{2\pi}\log|1-ze^{-it}|\,dm(t) \geqslant -\left(\frac{k}{2}+1\right)\log(1+r) + \left(\frac{k}{2}-1\right)\log(1-r)$$

$$= \log\left\{\frac{(1-r)^{k/2-1}}{(1+r)^{k/2+1}}\right\}.$$

Suppose $\int_0^{2\pi} |dm(t)| = k' < k$; then

$$-\int_0^{2\pi} \log|1-ze^{-it}|\,dm(t) \geqslant \log\left\{\frac{(1-r)^{k'/2-1}}{(1+r)^{k'/2+1}}\right\} > \log\left\{\frac{(1-r)^{k/2-1}}{(1+r)^{k/2+1}}\right\}$$

since $k' < k$. Therefore, from (4.3) we obtain

$$|f_\varrho(re^{i\theta})| \geqslant r\left\{\frac{(1-r)^{k/2-1}}{(1+r)^{k/2+1}}\right\}^{1-\varrho}.$$

Also for

$$f_\varrho(z) = z\left\{\frac{(1+z)^{k/2-1}}{(1-z)^{k/2+1}}\right\}^{(1-\varrho)}$$ these lounds are sharp.

This completes the proof of the theorem.

5. Radius of convexity for the class $U_k(\varrho)$.

LEMMA 3. *Suppose* $p_\varrho(z) \in P_k(\varrho)$. *Then*

(5.1) $$\operatorname{Re} p_\varrho(z) \geqslant \frac{1-k(1-\varrho)r+(1-2\varrho)r^2}{1-r^2} \quad \textit{for } |z| = r < 1;$$

and

(5.2) $$\operatorname{Re}\left\{\frac{zp_\varrho'(z)}{p_\varrho(z)}\right\} \geqslant \frac{(1-\varrho)\{-kr+4r^2-kr^3\}}{(1-r^2)\big(1-k(1-\varrho)r+(1-2\varrho)r^2\big)},$$

where $|z| = r < R_\varrho$, *as defined in Corollary* 6, $0 \leqslant \varrho < 1/2$ *and* $k \geqslant 4$. *These inequalities are sharp.*

For $2 \leqslant k \leqslant 4$,

(5.3) $$\operatorname{Re}\left\{\frac{zp_\varrho'(z)}{p_\varrho(z)}\right\} \geqslant \frac{(1-\varrho)\{-2kr+\{4+(k-2)^2\}r^2-2kr^3\}}{2(1-r^2)\big(1-k(1-\varrho)r+(1-2\varrho)r^2\big)}.$$

Proof. Given any $p_\varrho(z) \in P_k(\varrho)$, define $f_\varrho(z)$ in E such that

(5.4) $$1+z\frac{f_\varrho''(z)}{f_\varrho'(z)} = p_\varrho(z).$$

Then $f_\varrho(z) \in V_k(\varrho)$. Hence inequality (5.1) follows from (3.4). Since $f_\varrho(z) \in V_k(\varrho)$, there exists an $f(z) \in V_k$ such that $f_\varrho'(z) = \{f'(z)\}^{(1-\varrho)}$, that is,

$$\frac{f_\varrho''(z)}{f_\varrho'(z)} = (1-\varrho)\frac{f''(z)}{f'(z)}.$$

Also

$$\left|\left\{\frac{f''(z)}{f'(z)}\right\}' - \frac{1}{2}\left\{\frac{f''(z)}{f'(z)}\right\}^2\right| \leqslant \frac{(k^2-4)}{2(1-r^2)^2} \quad \text{for } k \geqslant 4,\ |z| = r.$$

Hence

$$\left|\frac{1}{1-\varrho}\left\{\frac{f_\varrho''(z)}{f_\varrho'(z)}\right\}-\frac{1}{2(1-\varrho)^2}\left\{\frac{f_\varrho''(z)}{f_\varrho'(z)}\right\}^2\right| \leqslant \frac{k^2-4}{2(1-r^2)^2}.$$

From (5.4) we have

$$\left(\frac{f_\varrho''(z)}{f_\varrho'(z)}\right)' = \frac{zp_\varrho'(z)-\left(p_\varrho(z)-1\right)}{z^2}; \quad \frac{f_\varrho''(z)}{f_\varrho'(z)} = \frac{p_\varrho(z)-1}{z}.$$

Hence

$$\left|zp_\varrho'(z)-\left\{\left(p_\varrho(z)-1\right)+\frac{1}{2(1-\varrho)}\left(p_\varrho(z)-1\right)^2\right\}\right| \leqslant \frac{(k^2-4)(1-\varrho)r^2}{2(1-r^2)^2},$$

$$\left|zp_\varrho'(z)-\left\{\frac{p_\varrho^2(z)}{2(1-\varrho)}-\frac{\varrho}{1-\varrho}p_\varrho(z)-\frac{(1-2\varrho)}{2(1-\varrho)}\right\}\right| \leqslant \frac{(k^2-4)(1-\varrho)r^2}{2(1-r^2)^2}.$$

For $r < R_\varrho$ (as defined in Corollary 6), $\operatorname{Re} p_\varrho(z) > 0$, and hence

$$\operatorname{Re}\left\{\frac{1}{p_\varrho(z)}\right\} < \frac{1}{\operatorname{Re}\left(p_\varrho(z)\right)};$$

also $|p_\varrho(z)| \neq 0$. Hence for $0 \leqslant \varrho < 1/2$, we have

$$\operatorname{Re}\left\{z\frac{p_\varrho'(z)}{p_\varrho(z)}\right\} \geqslant \frac{\operatorname{Re} p_\varrho(z)}{2(1-\varrho)}-\frac{\varrho}{1-\varrho}-\frac{(1-2\varrho)}{2(1-\varrho)\operatorname{Re} p_\varrho(z)}-\frac{(k^2-4)(1-\varrho)r^2}{2(1-r^2)^2|p_\varrho(z)|}.$$

Using (5.1),

$$\operatorname{Re}\left\{z\frac{p_\varrho'(z)}{p_\varrho(z)}\right\} \geqslant \frac{1-k(1-\varrho)r+(1-2\varrho)r^2}{2(1-\varrho)(1-r^2)}-\frac{\varrho}{1-\varrho}-$$

$$-\frac{(1-2\varrho)(1-r^2)}{2(1-\varrho)\{1-k(1-\varrho)r+(1-2\varrho)r^2\}}-$$

$$-\frac{(k^2-4)(1-\varrho)r^2}{2(1-r^2)\left(1-k(1-\varrho)r+(1-2\varrho)r^2\right)}$$

$$= \frac{(1-\varrho)(-kr+4r^2-kr^3)}{(1-r^2)\left(1-k(1-\varrho)r+(1-2\varrho)r^2\right)} \quad \text{for } 0 \leqslant \varrho < 1/2,\ k \geqslant 4$$

and $|z| \leqslant R_\varrho$. Consider the function

$$p_\varrho(z) = \{1-k(1-\varrho)z+(1-2\varrho)z^2\}/(1-z^2),$$

then we can show that equality is attained for $z = r$. Hence this is sharp.

For $2 \leqslant k \leqslant 4$ we have

$$\left|\left(\frac{f''(z)}{f'(z)}\right)^1-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right| \leqslant \frac{2(k-1)(1-\varrho)r^2}{(1-r^2)^2}, \quad [3].$$

Then we get

$$\left|zp_\varrho'(z)-\left\{\frac{p_\varrho^2(z)}{2(1-\varrho)}-\frac{\varrho}{(1-\varrho)}p_\varrho(z)-\frac{(1-2\varrho)}{2(1-\varrho)}\right\}\right|\leqslant\frac{2(k-1)(1-\varrho)r^2}{(1-r^2)^2}.$$

For $r<R_\varrho$, $\operatorname{Re}p_\varrho(z)>0$, and hence $\operatorname{Re}\frac{1}{p_\varrho(z)}<\frac{1}{\operatorname{Re}p_\varrho(z)}$; also $|p_\varrho(z)|\neq 0$. Hence for $0\leqslant\varrho<1/2$ we have

$$\operatorname{Re}\left\{z\frac{p_\varrho'(z)}{p_\varrho(z)}\right\}\geqslant\frac{\operatorname{Re}p_\varrho(z)}{2(1-\varrho)}-\frac{\varrho}{1-\varrho}-\frac{1-2\varrho}{2(1-\varrho)\operatorname{Re}p_\varrho(z)}-\frac{2(k-1)(1-\varrho)r^2}{|p_\varrho(z)|(1-r^2)^2}.$$

Using (5.1) we get

$$\operatorname{Re}\left\{z\frac{p_\varrho'(z)}{p_\varrho(z)}\right\}\geqslant\frac{1-k(1-\varrho)r+(1-2\varrho)r^2}{2(1-\varrho)(1-r^2)}-\frac{\varrho}{1-\varrho}-$$

$$-\frac{(1-2\varrho)(1-r^2)}{2(1-\varrho)\big(1-k(1-\varrho)r+(1-2\varrho)r^2\big)}-\frac{2(k-1)(1-\varrho)r^2}{(1-r^2)\big(1-k(1-\varrho)r+(1-2\varrho)r^2\big)}$$

$$=\frac{(1-\varrho)\{-2kr+\big(4+(k-2)^2\big)r^2-2kr^3\}}{2(1-r^2)\big(1-k(1-\varrho)r+(1-2\varrho)r^2\big)}.$$

THEOREM 7. *If $f_\varrho(z)\in U_k(\varrho)$, $0\leqslant\varrho<1/2$, then $f_\varrho(z)$ is convex in $|z|<R_c$, where R_c is the least positive root of the equation*

$$1-3k(1-\varrho)r+\big(6-8\varrho+k^2(1-\varrho)^2\big)r^2-k(1-\varrho)(3-4\varrho)r^3+(1-2\varrho)^2r^4=0$$

for $k\geqslant 4$.

Proof. Since $f_\varrho(z)\in U_k(\varrho)$, there exists a $p_\varrho(z)\in P_k(\varrho)$ such that

$$z\frac{f_\varrho'(z)}{f_\varrho(z)}=p_\varrho(z).$$

Hence, by applying (5.1) and (5.2),

$$\operatorname{Re}\left\{1+z\frac{f_\varrho''(z)}{f_\varrho'(z)}\right\}=\operatorname{Re}\left\{z\frac{p_\varrho'(z)}{p_\varrho(z)}\right\}+\operatorname{Re}\left\{z\frac{f_\varrho'(z)}{f_\varrho(z)}\right\}$$

$$\geqslant\frac{(1-\varrho)(-kr+4r^2-kr^3)}{(1-r^2)\big(1-k(1-\varrho)r+(1-2\varrho)r^2\big)}+\frac{1-k(1-\varrho)r+(1-2\varrho)r^2}{(1-r^2)}$$

for $k>4$, $r<R_\varrho$ and $0\leqslant\varrho<1/2$. That is,

$$\operatorname{Re}\left\{1+z\frac{f_\varrho''(z)}{f_\varrho'(z)}\right\}>0$$

provided

$$T(r)=1-3(1-\varrho)kr+\big(6-8\varrho+k^2(1-\varrho)^2\big)r^2-k(1-\varrho)(3-4\varrho)r^3+$$
$$+(1-2\varrho)^2r^4>0,$$

$T(0)>0$ and $T(R_\varrho)<0$. Hence the theorem is proved.

For the function

$$f_\varrho(z) = \left\{\frac{z(1-z)^{(k/2-1)}}{(1+z)^{(k/2+1)}}\right\}^{1-\varrho},$$

$$z\frac{f'_\varrho(z)}{f_\varrho(z)} = \frac{1-k(1-\varrho)z+(1-2\varrho)z^2}{1-z^2}$$

and

$$1+z\frac{f''_\varrho(z)}{f'_\varrho(z)} = 0 \quad \text{at } z = R_c.$$

Hence the bound is sharp.

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