ANNALES
POLONICI MATHEMATICI
XXIII (1970)

A relation between Mellin transform and Weyl (fractional) integral of two variables

by N. C. Jain (Indore, India)

1. Introduction. The author [1] has recently obtained a relation between Laplace and Stieltjes transform of two variables. In this paper, a relation between Mellin transform and Weyl (fractional) integral of two variables has been obtained. The result is important as it may be used in evaluating the Weyl (fractional) integral of two variables with the help of tables of Mellin transform of two variables. The technique is illustrated by an example.

The Mellin transform of a function $f(x, y)$, of two variables is given ([2], p. 565) in the form

$$F(p, q) = \int_0^\infty \int_0^\infty x^{p-1} y^{q-1} f(x, y)\, dx\, dy. \tag{1.1}$$

We shall denote (1.1) symbolically as

$$F(p, q) = M[f(x, y); p, q].$$

We define the Weyl (fractional) integral of order μ_1 and μ_2 of a function $f(x, y)$ of two variables as

$$(1.2)\quad W_{\mu_1,\mu_2}[f(x, y); p, q]$$
$$= \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \int_p^\infty \int_q^\infty (x-p)^{\mu_1-1}(y-q)^{\mu_2-1} f(x, y)\, dx\, dy, \quad R(\mu_1, \mu_2) > 0.$$

2. We shall require the following formula in the sequel, which can be proved easily

$$(2.1)\quad \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \int_0^s \int_0^t x^{p-1} y^{q-1} (s-x)^{\mu_1-1} (t-y)^{\mu_2-1}\, dx\, dy$$
$$= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+\mu_1)\Gamma(q+\mu_2)} s^{p+\mu_1-1} t^{q+\mu_2-1},$$

provided $R(\mu_1, \mu_2) > 0$, $R(p, q) > 0$.

3. Theorem. *If*

$$h(p, q) = W_{\mu_1, \mu_2}[f(x, y); p, q], \tag{3.1}$$

and

$$F(p, q) = M[h(x, y); p, q], \tag{3.2}$$

then

$$F(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+\mu_1)\Gamma(q+\mu_2)} M[f(x, y); p+\mu_1, q+\mu_2], \tag{3.3}$$

provided Weyl (fractional) integral of order μ_1 and μ_2 of $|f(x, y)|$ and Mellin transforms of $|h(x, y)|$ and $|f(x, y)|$ in (3.2) *and* (3.3) *exist, and* $R(\mu_1, \mu_2) > 0$, $R(p, q) > 0$.

Proof. From (3.1) and (3.2), we have

$$h(p, q) = \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \int_p^\infty \int_q^\infty (x-p)^{\mu_1-1}(y-q)^{\mu_2-1} f(x, y)\, dx\, dy, \tag{3.4}$$

and

$$F(p, q) = \int_0^\infty \int_0^\infty x^{p-1} y^{q-1} h(x, y)\, dx\, dy. \tag{3.5}$$

Substituting the value of $h(x, y)$ from (3.4) in (3.5), and changing the order of integration which is permissible under the conditions, stated in the theorem, we get

$$F(p, q) \tag{3.6}$$

$$= \int_0^\infty \int_0^\infty f(s, t)\, ds\, dt \left[\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)} \int_0^s \int_0^t x^{p-1} y^{q-1} (s-x)^{\mu_1-1} (t-y)^{\mu_2-1}\, dx\, dy \right].$$

Evaluating the inner double integral by (2.1), we obtain

$$F(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+\mu_1)\Gamma(q+\mu_2)} \int_0^\infty \int_0^\infty s^{p+\mu_1-1} t^{q+\mu_2-1} f(s, t)\, ds\, dt,$$

$$R(\mu_1, \mu_2) > 0, \quad R(p, q) > 0.$$

The result follows from (1.1).

4. Application. Let

$$f(x, y) = e^{-ax-by}, \quad R(a, b) > 0.$$

Then from (3.3), we have

$$F(p, q) = \frac{\Gamma(p)\Gamma(q)}{a^{p+\mu_1} b^{q+\mu_2}}, \quad R(p+\mu_1, q+\mu_2) > 0, R(a, b) > 0. \tag{4.1}$$

We know that

$$\frac{\Gamma(p)\Gamma(q)}{a^{p+\mu_1}b^{q+\mu_2}} = M\left[\frac{e^{-ax-by}}{a^{\mu_1}b^{\mu_2}}; p, q\right],$$

provided $R(\mu_1, \mu_2) > 0$, $R(p, q) > 0$.

Hence the theorem gives

$$W_{\mu_1,\mu_2}[e^{-ax-by}; p, q] = \frac{e^{-ap-bq}}{a^{\mu_1}b^{\mu_2}}$$

provided $R(p, q) > 0$, $R(\mu_1, \mu_2) > 0$ and $R(a, b) > 0$.

The author is extremely grateful to Dr. R. K. Saxena for his valuable guidance and help, throughout the preparation of this paper.

References

[1] N. C. Jain, *A relation between Laplace and Stieltjes transforms of two variables*, Ann. Polon. Math. 22 (1969), p. 313–315.

[2] I. S. Reed, *The Mellin type double integral*, Duke Math. Journ. 11 (1944), p. 565–572.

DEPARTMENT OF MATHEMATICS
SHRI G. S. TECHNOLOGICAL INSTITUTE
Indore (India)

Reçu par la Rédaction le 10. 1. 1969