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On some limit properties of the solution of the Riquier problem for the iterated Helmholtz equation in the half-plane

1. Let $R^2_+ = \{(x, y)\colon y > 0\}$. We consider in R^2_+ the following function

(1) $$u(f, g; x, y) = \frac{1}{2\pi}\int_R f(s)[2cr^{-1}yK_1(cr)+c^2yK_0(cr)]\,ds - \frac{1}{2\pi}\int_R g(s)\,yK_0(cr)\,ds,$$

where $r^2 = (x-s)^2+y^2$, c is a positive number, K_ν is the MacDonald function of index ν ([4]) and f, g are given functions defined in the set R of all real numbers.

In paper [6] it has been proved that the function u is a solution of the equation

(2) $$(\Delta - c^2)^2 u(x, y) = 0$$

in R^2_+ satisfying the following boundary conditions

$$u(x, y)|_{y=0} = f(x), \qquad \Delta u(x, y)|_{y=0} = g(x),$$

where f, g satisfy some assumptions.

Let us denote by L_p, $1 \leqslant p \leqslant \infty$, the class of functions with a finite norm defined as follows

$$\|f\|_p = \begin{cases} \left(\int_R |f(s)|^p\,ds\right)^{1/p} & \text{for } 1 \leqslant p < \infty, \\ \sup\operatorname{ess}_{s\in R} |f(s)| & \text{for } p = \infty. \end{cases}$$

In the present paper we shall prove a theorem concerning an estimation of $M_1(f, g; y) = \|u(f, g;\cdot, y)-f(\circ)\|_p$, $M_2(f, g; y) = \|\Delta u(f, g;\circ, y)-g(\circ)\|_p$ according to properties of the functions f, g. Moreover, for the case $g = 0$ in R we prove an converse theorem (in a sense).

This problem for the biharmonic equation was investigated in paper [2].

2. Let ω denote a function of the type of the second modulus of smootheness ([5]). Let $H_2^{\omega} L_p$ denote the set of all functions $f \in L_p$ for which

$$\|f(\circ - s) - 2f(\circ) + f(\circ + s)\|_p \leqslant \omega(|s|).$$

We prove

THEOREM 1. *If* $f \in H_2^{\omega} L_p$, $g \in L_p$, *then*

(a) $u \in C^{\infty}(R_+^2)$,

(b) $(\Delta - c^2)^2 u(x, y) = 0$ *in the domain* R_+^2,

(c) $M_1(f, g; y) \leqslant \frac{2}{\pi}\omega(y) + \frac{y}{\pi}\int\limits_y^{\infty} \frac{\omega(s)}{s^2} ds + ky$ *for* $y > 0$,

where $k = \frac{3}{2} c \|f\|_p + \frac{1}{2c}\|g\|_p$.

Proof. Applying the Hölder–Minkowski inequality, it is easy to verify that the following integrals

$$\int\limits_R f(s) D_x^m D_y^n (yK_0(cr))\, ds, \quad \int\limits_R f(s) D_x^m D_y^n (yr^{-1} K_1(cr))\, ds, \quad \int\limits_R g(s) D_x^m D_y^n (yK_0(cr))\, ds$$

are almost uniformly convergent in R_+^2 for arbitrary non-negative integers m, n.

Hence, by the formulas ([4])

$$\frac{d}{dt}(t^{\nu} K_{\nu}(t)) = -t^{\nu} K_{\nu-1}(t), \qquad \frac{d}{dt}(t^{-\nu} K_{\nu}(t)) = -t^{-\nu} K_{\nu+1}(t),$$

$$K_{\nu+2}(t) = \frac{2\nu}{t} K_{\nu+1}(t) + K_{\nu}(t), \qquad K_{-\nu}(t) = K_{\nu}(t), \tag{3}$$

we get (a) and (b).

We shall prove that (c) is true. In view of the properties of the convolution ([1]) the function $u(f, g; \circ, y) \in L_p$ for every $y > 0$. Moreover,

$$\Big\|\int\limits_R f(s)\, yK_0(cr)\, ds\Big\|_p \leqslant \|f\|_p \int\limits_R yK_0(c\varrho)\, ds,$$

$$\Big\|\int\limits_R g(s)\, yK_0(cr)\, ds\Big\|_p \leqslant \|g\|_p \int\limits_R yK_0(c\varrho)\, ds,$$

where $\varrho^2 = s^2 + y^2$.

By the formula ([7])

$$\int\limits_0^{\infty} t^{2m+1}(t^2 + z^2)^{-n/2} K_n(at^2 + z^2)\, dt = \frac{2^m \Gamma(m+1)}{a^{m+1} z^{n-m-1}} K_{n-m-1}(az), \tag{4}$$

where $m > -1$, $a > 0$, $z > 0$, and $n = 0, 1, \ldots$, we have

$$\int_0^\infty yK_0(c\varrho)\,ds = \frac{\pi}{2c}ye^{-cy}, \qquad y > 0.$$

Hence

$$\frac{1}{2\pi}\left\|c^2\int_R f(s)\,yK_0(cr)\,ds\right\|_p \leqslant \tfrac{1}{2}c\,\|f\|_p\,y, \qquad y > 0,$$

and

$$\frac{1}{2\pi}\left\|\int_R g(s)\,yK_0(cr)\,ds\right\|_p \leqslant \frac{1}{2c}\|g\|_p\,y, \qquad y > 0.$$

Further, by (4), we obtain

$$\frac{1}{\pi}\int_R cyr^{-1}K_1(cr)\,ds = e^{-cy}, \qquad y > 0,$$

and

$$M_1(f, g; y) \leqslant \left\|u(f, g; \circ, y) - \frac{1}{\pi}\int_R f(\circ)\,cyr^{-1}K_1(cr)\,ds\right\|_p + \|f\|_p\,|e^{-cy}-1|$$

$$\leqslant \frac{1}{\pi}\left\|\int_R [f(s)-f(\circ)]\,cyr^{-1}K_1(cr)\,ds\right\|_p + \frac{c^2}{2\pi}\left\|\int_R f(s)\,yK_0(cr)\,ds\right\|_p$$

$$+\frac{1}{2\pi}\left\|\int_R g(s)\,yK_0(cr)\,ds\right\|_p + \|f\|_p\,|e^{-cy}-1|$$

$$\leqslant A + \left(\tfrac{3}{2}c\,\|f\|_p + \frac{1}{2c}\|g\|_p\right)y, \qquad y > 0,$$

where

$$A = \frac{1}{\pi}\left\|\int_R [f(s)-f(\circ)]\,cyr^{-1}K_1(cr)\,ds\right\|_p.$$

We remark that

$$A = \frac{1}{\pi}\left\|\int_R [f(\circ - s)-f(\circ)]\,cy\varrho^{-1}K_1(c\varrho)\,ds\right\|_p$$

$$= \frac{1}{\pi}\left\|\int_0^\infty [f(\circ - s)-2f(\circ)+f(\circ + s)]\,cy\varrho^{-1}K_1(c\varrho)\,ds\right\|_p.$$

By Hölder–Minkowski inequality we get

$$A \leqslant \frac{c}{\pi} \int_0^\infty \|f(\circ - s) - 2f(\circ) + f(\circ + s)\|_p \, y\varrho^{-1} K_1(c\varrho)\, ds$$

$$\leqslant \frac{c}{\pi} \int_0^\infty \omega(s)\, y\varrho^{-1} K_1(c\varrho)\, ds, \qquad y > 0.$$

Using the formula ([4])

$$K_\nu(z) = \frac{2^\nu \Gamma(\nu + 1/2)}{z^\nu \sqrt{\pi}} \int_0^\infty \frac{\cos zt}{(1+t^2)^{\nu+1/2}}\, dt \tag{5}$$

we have $K_1(c\varrho) \leqslant (c\varrho)^{-1}$; hence

$$A \leqslant \frac{y}{\pi} \int_0^\infty \frac{\omega(s)}{\varrho^2}\, ds \leqslant \frac{y}{\pi} \int_0^y \frac{\omega(s)}{\varrho^2}\, ds + \frac{y}{\pi} \int_y^\infty \frac{\omega(s)}{s^2}\, ds, \qquad y > 0.$$

From the inequality

$$\omega(\lambda s) \leqslant (\lambda + 1)^2 \omega(s) \tag{6}$$

we get

$$y \int_0^y \frac{\omega(s)}{\varrho^2}\, ds \leqslant \int_0^y \frac{(s+y)^2}{y(s^2+y^2)}\, ds\, \omega(y) \leqslant 2\omega(y)$$

and

$$A \leqslant \frac{2}{\pi} \omega(y) + \frac{y}{\pi} \int_y^\infty \frac{\omega(s)}{s^2}\, ds$$

and finally (c).

THEOREM 2. *If* $f \in L_p$, $g \in H_2^\omega L_p$, *then*

(a) $u \in C^\infty(R_+^2)$,

(b) $(\Delta - c^2)^2 u(x, y) = 0$ *in* R_+^2,

(c) $M_2(f, g; y) \leqslant \frac{2}{\pi} \omega(y) + \frac{y}{\pi} \int_y^\infty \frac{\omega(s)}{s^2}\, ds + k_1 y$, $y > 0$,

where $k_1 = \frac{1}{2} c^3 \|f\|_p + \frac{3}{2} c \|g\|_p$.

Proof. From formulas (3) it follows that

$$u(f, g; x, y) = \frac{1}{2\pi} \int_R f(s) c^4 y K_0(cr)\, ds$$

$$+\frac{1}{2\pi} \int_R g(s) [2cr^{-1} y K_1(cr) - c^2 y K_0(cr)]\, ds,$$

hence, by the proof of Theorem 1 we get the assertion of Theorem 2.

3. Now we consider the case where $g(s) = 0$ for every $s \in R$. Let

$$u(f; x, y) = u(f, 0; x, y) = \frac{1}{2\pi} \int_R f(s) [2cr^{-1} y K_1(cr) + c^2 y K_0(cr)]\, ds.$$

From Theorem 1 we get

THEOREM 3. *If* $f \in H_2^{\omega} L_p$, *then*

$$\|u(f; \circ, y) - f(\circ)\|_p \leqslant \frac{2}{\pi} \omega(y) + \frac{y}{\pi} \int_y^{\infty} \frac{\omega(s)}{s^2}\, ds + \tfrac{3}{2} c \|f\|_p y$$

for $y > 0$.

In the sequel we consider (in a sense) the inverse problem to the problem of Theorem 3. First we prove a theorem of the Hardy–Littlewood type.

THEOREM 4. *If* $f \in H_2^{\omega} L_p$, *then*

$$\left\| \frac{\partial^2 u(f; \circ, y)}{\partial x^2} \right\|_p \leqslant 5 \frac{\omega(y)}{y^2}, \qquad y > 0.$$

Proof. Using formulas (3), we get

$$\frac{\partial^2 u(f; x, y)}{\partial x^2} = \frac{1}{2\pi} \int_R f(s) [2c^3 y (x-s)^2 r^{-3} K_3(cr) + c^4 y (x-s)^2 r^{-2} K_2(cr)$$

$$- 2c^2 y r^{-2} K_2(cr) - c^3 y r^{-1} K_1(cr)]\, ds$$

$$= \frac{1}{2\pi} \int_R f(x-s) [2c^3 y s^2 \varrho^{-3} K_3(c\varrho) + c^4 y s^2 \varrho^{-2} K_2(c\varrho)$$

$$- 2c^2 y \varrho^{-2} K_2(c\varrho) - c^3 y \varrho^{-1} K_1(c\varrho)]\, ds.$$

It is easy to verify, by formula (4), that

$$\int_R [2c^3 ys^2 \varrho^{-3} K_3(c\varrho) + c^4 ys^2 \varrho^{-2} K_2(c\varrho) - 2c^2 y\varrho^{-2} K_2(c\varrho) - c^3 y\varrho^{-1} K_1(c\varrho)]\, ds = 0,$$

hence

$$\frac{\partial^2 u(f;\, x,\, y)}{\partial x^2} = \frac{1}{2\pi} \int_0^\infty [f(x+s) - 2f(x) + f(x-s)]\, [2c^3 ys^2 \varrho^{-3} K_3(c\varrho) + c^4 ys^2 \varrho^{-2} K_2(c\varrho) - 2c^2 y\varrho^{-2} K_2(c\varrho) - c^3 y\varrho^{-1} K_1(c\varrho)]\, ds.$$

Further, by Hölder–Minkowski inequality and (6) we get

$$\left\| \frac{\partial^2 u(f;\, \circ,\, y)}{\partial x^2} \right\|_p \leqslant \frac{1}{2\pi} \int_0^\infty \omega(s)\, [|2c^3 ys^2 \varrho^{-3} K_3(c\varrho) + c^4 ys^2 \varrho^{-2} K_2(c\varrho) - 2c^2 y\varrho^{-2} K_2(c\varrho) - c^3 y\varrho^{-1} K_1(c\varrho)|]\, ds$$

$$\leqslant \frac{\omega(y)}{2\pi y^2} \int_0^\infty (s+y)^2\, [2c^3 ys^2 \varrho^{-3} K_3(c\varrho) + c^4 ys^2 \varrho^{-2} K_2(c\varrho) + 2c^2 y\varrho^{-2} K_2(c\varrho) + c^3 y\varrho^{-1} K_1(c\varrho)]\, ds.$$

Applying formula (4) and (3) we can calculate that

$$\int_0^\infty (s+y)^2\, [2c^3 ys^2 \varrho^{-3} K_3(c\varrho) + c^4 ys^2 \varrho^{-2} K_2(c\varrho) + 2c^2 y\varrho^{-2} K_2(c\varrho) + c^3 y\varrho^{-1} K_1(c\varrho)]\, ds = \pi e^{-cy}(6 + 4cy + c^2 y^2) + 6c^2 y^2 K_2(cy).$$

It follows from formula (5) that $c^2 y^2 K_2(cy) \leqslant 2$ for $y > 0$, $c > 0$. Since $e^{-cy}(6 + 4cy + c^2 y^2) \leqslant 6$ for $y > 0$, $c > 0$, then

$$\left\| \frac{\partial^2 u(f;\, \circ,\, y)}{\partial x^2} \right\|_p \leqslant 5 \frac{\omega(y)}{y^2}$$

which ends the proof of Theorem 4.

Using the results of paper [3], by Theorem 4 we get

THEOREM 5. *Let $f \in L_p$. If*

$$\|u(f;\, \circ,\, y) - f(\circ)\|_p \leqslant \omega(y), \quad y > 0,$$

then

$$\omega_2(f,\, t) \leqslant Mt^2 \int_t^1 \frac{\omega(s)}{s^2}\, ds, \qquad 0 < t < \tfrac{1}{2},$$

where $\omega_2(f, t)$ denotes the second modulus of smootheness of the function f and M is a positive number independent of t.

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