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A necessary and sufficient condition for the existence of a bundle in differential spaces

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Abstract. The notion of a differential space is due to R. Sikorski and S. Mac Lane who, working quite independently, introduced it in their works [1] and [2].
In the present paper a necessary and sufficient condition for the existance of a bundle in such a space is given, Steenrod's definition [3] of a bundle being followed. It occurred that the main difficulty while translating the construction of a bundle from the category of topological spaces and topological groups into the category of differential spaces and generalized Lie groups was to find conditions determining the division of a certain differential space by a suitable equivalence relation.

Preliminaries. Let $\mathscr{C}$ be a non-empty set of real functions defined on M. Following R. Sikorski we shall denote by $\tau_{\mathscr{C}}$ the weakest topology on M for which all functions $\alpha \in \mathscr{C}$ are continuous.

A set $\mathscr{C}$ of real functions defined on M is said to be a *differential structure* on M [2], if:

(*) The set $\mathscr{C}$ is closed with respect to localization, i.e. $\mathscr{C} = \mathscr{C}_M$, where $\mathscr{C}_M$ is the set of all functions $\alpha\colon M \to R$ such that for any point $p \in M$ there exists a set $A \in \tau_{\mathscr{C}}$ and $\beta \in \mathscr{C}$ with $\beta | A = \alpha | A$

and

(**) The set $\mathscr{C}$ is closed with respect to superposition with smooth functions, i.e. the following condition is satisfied:

if $\omega \in \mathscr{E}_n$ and $\varphi_1, \ldots, \varphi_n \in \mathscr{C}$, then $\omega(\varphi_1(\cdot), \ldots, \varphi_n(\cdot)) \in \mathscr{C}$,

where $\mathscr{E}_n$ is the set of real functions of class $\mathscr{C}^n(R^n)$.

By a differential space M we shall understand a couple $(M, \mathscr{F}(M))$, where $\mathscr{F}(M)$ is a differential structure on M [2]. M_A will denote the differential space $(A, (\mathscr{F}(M))_A)$, where $A \subset M$.

Let M and N be differential spaces. We say that f maps smoothly the differential space M into the differential space N, in symbols

$$f\colon M \to N,$$

if $f\colon M \to N$ and for an arbitrary function $a \in \mathscr{F}(N)$ we have $a \circ f \in \mathscr{F}(M)$.

The mapping $f\colon M \to N$ is called a *diffeomorphism* if f is a one-one mapping of the set M onto N and $f^{-1}\colon N \to M$.

Let M be a differential space and let $f\colon M \to N$. For an arbitrary real function β mapping the set N into R we put $f^*(\beta) = \beta \circ f$. Denoting by R^N the set of all such functions β, we obtain a mapping $f^*\colon R^N \to R^M$.

It can be proved (cf. [4]) that $f^{*-1}[\mathscr{F}(M)]$ is a differential structure on N; this structure is called the *structure coinduced by the mapping f from a differential space M.* It is also proved in [4] that $f^{*-1}[\mathscr{F}(M)]$ is the greatest of differential structures on N for which the mapping f is smooth.

Let $M = (M, \mathscr{F}(M))$ and $N = (N, \mathscr{F}(N))$ be non-empty differential spaces. Let $\mathscr{F}(M) \times \mathscr{F}(N)$ denote the smallest differential structure on $M \times N$ containing all functions $a \circ \pi_1$, $\beta \circ \pi_2$, where $a \in \mathscr{F}(M)$, $\beta \in \mathscr{F}(N)$ and $\pi_1\colon M \times N \to M$ and $\pi_2\colon M \times N \to N$ are the natural projections.

The differential space $(M \times N, \mathscr{F}(M) \times \mathscr{F}(N))$ is called the *product of the differential spaces* $(M, \mathscr{F}(M))$ and $(N, \mathscr{F}(N))$ and is denoted by $M \times N$.

Let $G = (G, \mathscr{F}(G))$ be a differential space and let $(G, \odot)$ be a group. If the mappings:

$$\big((g_1, g_2) \mapsto g_1 \odot g_2\big)\colon G \times G \to G \tag{1}$$

and

$$(g \mapsto g^{-1})\colon G \to G \tag{2}$$

are smooth, i.e. $\big((g_1, g_2) \mapsto g_1 \odot g_2\big)\colon G \times G \to G$ and $(g \mapsto g^{-1})\colon G \to G$, then the set G together with the differential structure $\mathscr{F}(G)$ and the group structure $\odot$ is called a *generalized Lie group.*

Let G be a generalized Lie group and let $F = (F, \mathscr{F}(F))$ be a differential space. G will be called a *group of transformations* of the space F acting smoothly on F by means of η if:

$$\eta\colon G \times F \to F, \tag{3}$$

$$\eta(e, y) = y \quad \text{for } y \in F, \text{ where } e \text{ is the unit of the group } G, \tag{4}$$

$$\eta(g_1 \odot g_2, y) = \eta\big(g_1, \eta(g_2, y)\big) \quad \text{for } y \in F,\ g_1, g_2 \in G. \tag{5}$$

We shall say that G acts effectively on F if the equality $\eta(g, y) = y$ holding for all $y \in F$ implies $g = e$. In the sequel we shall use the notation $g \cdot y$ instead of $\eta(g, y)$.

A bundle is a system of the form

$$\mathscr{B} = \big(B, \pi, M, G, \cdot, (\varphi_i;\ i \in I)\big),$$

where B, M, F are differential spaces, $\pi: B \to M$, G is a generalized Lie group acting effectively on F by means of an operation $\cdot$, $(V_i;\ i \in I)$ is an open covering of the space M and $(\varphi_i;\ i \in I)$ is a family of diffeomorphisms satisfying the following conditions:

(a) $$\varphi_i: \big(V_i, \mathscr{F}(M)_{V_i}\big) \times F \to \big(\pi^{-1}[V_i], \mathscr{F}(B)_{\pi^{-1}[V_i]}\big);$$

(b) $$\pi\big(\varphi_i(p, y)\big) = p \quad \text{for } p \in V_i,\ y \in F;$$

(c) For an arbitrary $i \in I$ and for any $p \in V_i$ the mapping

$$\varphi_{ip}: F \to B_{\pi^{-1}[\{p\}]}$$

defined by the formula

$$\varphi_{ip}(y) = \varphi_i(p, y) \quad \text{for } y \in F$$

is a diffeomorphism;

(d) For arbitrary elements $i, j \in I$ and $p \in V_i \cap V_j$ the mapping defined by the formula

$$\varphi_{ij}(p) \cdot y = \varphi_{i,p}^{-1}\big(\varphi_{j,p}(y)\big)$$

is smooth, i.e.

$$\varphi_{ij}: M_{V_i \cap V_j} \to G.$$

From the definition of φ_{ij} it follows that for arbitrary $i, j, k \in I$ and $p \in V_i \cap V_j \cap V_k$

(7) $$\varphi_{kj}(p) \odot \varphi_{ji}(p) = \varphi_{ki}(p).$$

Putting in (7) $i = j = k$, we see that $\varphi_{ii}(p)$ is the unity of the group, and further, putting $i = k$, we get

$$\varphi_{jk}(p) = \big(\varphi_{kj}(p)\big)^{-1}.$$

2. A necessary and sufficient condition for the existence of a bundle. Let $M = \big(M, \mathscr{F}(M)\big)$, $F = \big(F, \mathscr{F}(F)\big)$ be differential spaces and let $(V_i;\ i \in I)$ be an open covering of the space M. Let G be a generalized Lie group acting effectively on F by means of an operation $\cdot$. Further, let $(\varphi_{ij};\ i, j \in I)$ be an indexed family of functions satisfying the conditions:

(8) $$\varphi_{ij}: M_{V_i \cap V_j} \to G,$$

(9) $$\varphi_{ij}(p) \odot \varphi_{jk}(p) = \varphi_{ik}(p) \quad \text{for } p \in V_i \cap V_j \cap V_k.$$

For an arbitrary $i \in I$ we shall consider the differential space

(10) $$i^* = (\{i\}, \{c_i;\ c \in E\}), \quad \text{where } c_i(t) = c \text{ for } t = i.$$

Let T be the subset of $M \times F \times I$ defined by

(11) $$T = \bigcup_{i \in I} V_i \times F \times \{i\};$$

we define a differential space T by putting

(12) $$T = \bigoplus_{i \in I} M_{V_i} \times F \times i^*.$$

In the set T we now introduce a relation $\bar{\varphi}$:

(13) $$(p, y, i)\bar{\varphi}(p', y', j) \Leftrightarrow \big(p = p' \wedge \varphi_{ji}(p)\cdot y = y' \wedge i, j \in I\big).$$

It is easy to show that the relation defined above is an equivalence.

Let B denote the set of all cosets modulo the relation $\bar{\varphi}$ in the set T, i.e.

(14) $$B = T/\bar{\varphi} = \{[t]_{\bar{\varphi}};\ t \in T\}.$$

Let $\check{\varphi}$ be the mapping of the set T into the set B defined as follows:

(15) $$\check{\varphi}(t) = [t]_{\bar{\varphi}} \quad \text{for } t \in T.$$

It can easily be verified that

(16) $$\check{\varphi}^{*-1}[\mathscr{F}(\bigoplus_{k \in I} M_{V_k} \times F \times k^*)]_{\check{\varphi}[V_i \times F \times \{i\}]} \subset (\check{\varphi}|_{V_i \times F \times \{i\}})^{*-1}[\mathscr{F}(M_{V_i} \times F \times i^*)],$$

where $\check{\varphi}^{*-1}[\mathscr{F}(\bigoplus_{k \in I} M_{V_k} \times F \times k^*)]$ denotes the structure coinduced from the differential space T by the mapping $\check{\varphi}$.

We shall prove the following

THEOREM 2.1. *If M, F are differential spaces, $(V_i;\ i \in I)$ is an open covering of the space M, G is a generalized Lie group acting effectively on F by means of an operation $\cdot$ and $(\varphi_{ij};\ i, j \in I)$ is an indexed family of functions satisfying conditions* (8) *and* (9), *then in order that there exist a bundle $\mathscr{B} = \big(B, \pi, M, F, G, \cdot, (\varphi_i;\ i \in I)\big)$ such that*

(17) $$\varphi_{ij}(p)\cdot y = \varphi_{i,p}^{-1}\big(\varphi_{j,p}(y)\big) \quad \text{for } p \in V_i \cap V_j,\ y \in F$$

it is necessary and sufficient that the condition

(18) $$\check{\varphi}^{*-1}[\mathscr{F}(\bigoplus_{k \in I} M_{V_k} \times F \times k^*)]_{\check{\varphi}[V_i \times F \times \{i\}]} \subset (\check{\varphi}|_{V_i \times F \times \{i\}})^{*-1}[\mathscr{F}(M_{V_i} \times F \times i^*)]$$

be satisfied.

Proof. Assume that the differential spaces M, F are given. Let $(V_i;\ i \in I)$ be an open covering of the space M, G a generalized Lie group acting effectively on F by means of $\cdot$ and let $(\varphi_{ij};\ i, j \in I)$ be a family of functions satisfying (8) and (9). Assume, moreover, that the mapping $\check{\varphi}$ defined by (15) satisfies condition (18). Let B be a differential space defined by

$$B = (\boldsymbol{B}, \mathscr{F}(B)), \tag{19}$$

where $\boldsymbol{B} = \boldsymbol{T}/\bar{\varphi}$ and $\mathscr{F}(B) = \check{\varphi}^{*-1}[\mathscr{F}(T)]$.

Define a function $\pi\colon \boldsymbol{B} \to \boldsymbol{M}$ putting

$$\pi(\check{\varphi}(p, y, i)) = p \quad \text{for } (p, y, i) \in V_i \times \boldsymbol{F} \times \{i\}. \tag{20}$$

Assume that $\varphi(p, y, i) = \check{\varphi}(p_1, y_1, j)$. Then $(p, y, i)\bar{\varphi}(p_1, y_1, j)$. In accordance with the definition of $\bar{\varphi}$ we obtain $p = p_1$. Therefore the mapping π is well defined.

We now show that π is smooth. Consider the function $\lambda\colon \boldsymbol{T} \to \boldsymbol{M}$ defined by the formula $\lambda(p, y, i) = p$ for $(p, y, i) \in \boldsymbol{T}$. Since the family $(V_i \times \boldsymbol{F} \times \{i\};\ i \in I)$ is an open covering of the space T and $\lambda|_{V_i \times \boldsymbol{F} \times \{i\}}\colon M_{V_i} \times F \times i^* \to M$ for $i \in I$, we infer that λ maps smoothly T into M. From the facts that the differential structure $\mathscr{F}(B)$ is coinduced from the differential space T by the mapping $\check{\varphi}$ and $\pi \circ \check{\varphi} = \lambda$ we obtain the smoothness of π.

Let $(\varphi_i;\ i \in I)$ be an indexed family of functions defined as follows: for an arbitrary $i \in I$ we put

$$\varphi_i(p, y) = \check{\varphi}(p, y, i) \quad \text{for } p \in V_i,\ y \in \boldsymbol{F}. \tag{21}$$

The smoothness of $\check{\varphi}$ implies the smoothness if the mapping φ_i. Thus

$$\varphi_i\colon M_{V_i} \times F \to B.$$

We shall show that $\varphi_i[V_i \times \boldsymbol{F}] = \pi^{-1}[V_i]$. Let $b \in \varphi_i[V_i \times \boldsymbol{F}]$. Thus there exists a point $(p, y) \in V_i \times \boldsymbol{F}$ such that $b = \varphi_i(p, y)$. From (21) and (20) it follows that $\pi(b) = p$. Since $p \in V_i$, we have $b \in \pi^{-1}[V_i]$. Thus $\varphi_i[V_i \times \boldsymbol{F}] \subset \pi^{-1}[V_i]$. Let now $b = \check{\varphi}(p, y, k)$ and $b \in \pi^{-1}[V_i]$. Then $\pi(b) = p \in V_i$. From (20) it follows that $\pi(b) = p$ and $p \in V_k$. Hence $p \in V_i \cap V_k$. Let us put $y' = \varphi_{ik}(p) \cdot y$. By (13) we obtain $(p, y, k)\bar{\varphi}(p, \varphi_{ik}(p) \cdot y, i)$. Hence $b = \check{\varphi}(p, \varphi_{ik}(p) \cdot y, i)$. Using (21) we jet $b = \varphi_i(p, y')$. Since $p \in V_i \cap V_k$ and $y' \in \boldsymbol{F}$, we have $b \in \varphi_i[V_i \times \boldsymbol{F}]$. Thus $\pi^{-1}[V_i] \subset \varphi_i[V_i \times \boldsymbol{F}]$. We prove now that φ_i defined in (21) is a one-one mapping. To this end we assume that $\varphi_i(p, y) = \varphi_i(p', y')$ for $p, p' \in V_i$ and $y \in \boldsymbol{F}$. Then $\check{\varphi}(p, y, i) = \check{\varphi}(p', y', i)$, hence $(p, y, i)\bar{\varphi}(p', y', i)$ and so $(p, y) = (p', y')$. Therefore φ_i is a smooth one-one mapping of the differential space $M_{V_i} \times F$ onto the differential space $B_{\pi^{-1}[V_i]}$ and there exists a mapping $\varphi_i^{-1}\colon \pi^{-1}[V_i] \to V_i \times \boldsymbol{F}$. We are going to show that this mapping is smooth.

From the assumption and (16) it follows that

$$\check{\varphi}^{*-1}[\mathscr{F}(\bigoplus_{k \in I} M_{V_k} \times F \times k^*)]_{\check{\varphi}[V_i \times \boldsymbol{F} \times \{i\}]} = (\check{\varphi}|_{V_i \times \boldsymbol{F} \times \{i\}})^{*-1}[\mathscr{F}(M_{V_i} \times F \times i^*)]. \tag{22}$$

Thus the differential space $B_{\check{\varphi}[V_i\times F\times\{i\}]}$ has the differential structure identical with the one coinduced by the mapping $\check{\varphi}|_{V_i\times F\times\{i\}}$ from the differential space $M_{V_i}\times F\times i^*$. Let us put

$$r_i(p,y,i) = (p,y) \quad \text{for } (p,y,i)\in V_i\times F\times\{i\}. \tag{23}$$

Clearly $r_i\colon M_{V_i}\times F\times i^*\to M_{V_i}\times F$. Since $\varphi_i^{-1}\circ\check{\varphi}\,|_{V_i\times F\times\{i\}} = r_i$, and taking (22) into account, we see that the mapping φ_i^{-1} maps smoothly the differential space $B_{\pi^{-1}[V_i]}$ onto the differential space $M_{V_i}\times F$. Hence φ_i is a diffeomorphism for any $i\in I$.

Let p be a fixed point in $V_i\cap V_j$. Consider the mapping $\varphi_{j,p}^{-1}\circ\varphi_{i,p}\colon F\to F$, where $\varphi_{i,p},\varphi_{j,p}$ are defined by (6). If $y' = \varphi_{j,p}^{-1}(\varphi_{i,p}(y))$, then $\varphi_{j,p}(y') = \varphi_{i,p}(y)$; hence $\varphi_j(p,y') = \varphi_i(p,y)$ and so $(p,y',j)\bar{\varphi}(p,y,i)$. Therefore for an arbitrary $y\in F$ the equality $\varphi_{ji}(p)\cdot y = \varphi_{j,p}^{-1}(\varphi_{ip}(y))$ holds. Thus we have shown that under the given assumptions the system $\mathscr{B} = (B,\pi,M,F,G,\cdot,(\varphi_i;\ i\in I))$ is a bundle.

Now assume that there are given: differential spaces M, F, an open covering $(V_i;\ i\in I)$ of the space M, a generalized Lie group acting effectively on F by means of an operation $\cdot$, an indexed family of mappings $(\varphi_{ij};\ i,j\in I)$ satisfying conditions (8) and (9). Assume, further, that there exists a bundle $\mathscr{B} = (B,\pi,M,F,G,\cdot,(\varphi_i;\ i\in I))$ such that

$$\varphi_{ip}^{-1}(\varphi_{jp}(y)) = \varphi_{ij}(p)\cdot y.$$

For the given differential spaces M, F and the given covering $(V_i;\ i\in I)$ of the space M let T be the differential space defined by (12), $\bar{\varphi}$ the relation defined by (13), $\check{\varphi}$ the mapping defined by (15) and let B be the differential space

$$\check{B} = (\check{\boldsymbol{B}},\mathscr{F}(\check{B})), \tag{24}$$

where $\check{\boldsymbol{B}} = T/\bar{\varphi}$ and $\mathscr{F}(\check{B}) = \check{\varphi}^{*-1}[\mathscr{F}(T)]$.

If the mapping $\check{\pi}\colon \check{\boldsymbol{B}}\to M$ is given by

$$\check{\pi}(\check{\varphi}(p,y,i)) = p \quad \text{when } \check{\varphi}(p,y,i)\in\check{\boldsymbol{B}} \tag{25}$$

and the mapping $\check{\varphi}_i\colon V_i\times F\to\check{\boldsymbol{B}}$ by

$$\check{\varphi}_i(p,y) = \check{\varphi}(p,y,i), \tag{26}$$

then $\check{\pi}\colon \check{B}\to M$, $\check{\varphi}_i\colon M_{V_i}\times F\to\check{B}_{\check{\pi}^{-1}[V_i]}$ is a one-one mapping and

$$\check{\varphi}_i = \check{\varphi}\,|_{V_i\times F\times\{i\}}\circ r_i^{-1}, \tag{27}$$

where r_i is defined by (23).

Let b be an arbitrary point belonging to B. There exist $y\in F$, $i\in I$ and $p\in V_i$ such that $b = \varphi_i(p,y)$, where $\varphi_i\in(\varphi_k;\ k\in I)$. If $b = \varphi_j(p',y')$, where $p'\in V_j$, $y\in F$, then it follows from the properties of the bundle

$\mathcal{B}$ that $\varphi_i(p, y) = \varphi_j(p', y')$ and $p = p'$, or, in other words, $\varphi_{i,p}(y) = \varphi_{j,p}(y')$, where $\varphi_{i,p}, \varphi_{j,p}$ are defined by (6).

Hence $y' = \varphi_{j,p}^{-1}(\varphi_{i,p}(y))$. From the last equality and the assumption we have $y' = \varphi_{ji}(p)\cdot y$. Therefore $(p, y, i)\bar{\varphi}(p', y', j)$, that is, $\check{\varphi}(p, y, i) = \check{\varphi}(p', y', j)$. Thus we have

$$\varphi_i(p, y) = \varphi_j(p', y') \Leftrightarrow \check{\varphi}(p, y, i) = \check{\varphi}(p', y', j).$$

Hence it is justified to define a mapping $h\colon \boldsymbol{B} \to \check{\boldsymbol{B}}$ as follows:

$$h = (b \mapsto \check{b}), \tag{28}$$

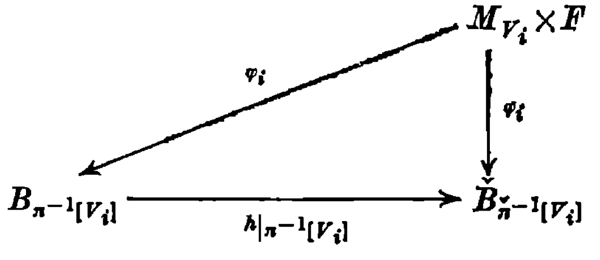
where $b = \varphi_i(p, y)$, $\check{b} = \check{\varphi}(p, y, i)$ and $p \in V_i$, $y \in \boldsymbol{F}$.

We shall show that h is a one-one mapping of the set $\boldsymbol{B}$ onto the set $\check{\boldsymbol{B}}$. Let $h(b) = h(b_1)$, where $b_1 = \varphi_j(p_1, y_1)$. Then we have $\check{\varphi}(p, y, i) = \check{\varphi}(p_1, y_1, j)$, that is, $(p, y, i)\bar{\varphi}(p_1, y_1, j)$. From the definition of the relation $\bar{\varphi}$ it follows that $p = p_1$ and $y_1 = \varphi_{ji}(p)\cdot y$ for $p \in V_i \cap V_j$. From this fact and from the assumption of the existence of a bundle we obtain $\varphi_{j,p}^{-1}(\varphi_{i,p}(y)) = y_1$. Therefore $\varphi_i(p, y) = \varphi_j(p, y_1)$ and so $b = b_1$.

Let $\check{b}$ be an arbitrary point belonging to $\check{\boldsymbol{B}}$. There exist $y \in \boldsymbol{F}$, $i \in \boldsymbol{I}$ and $p \in V_i$ such that $\check{b} = \check{\varphi}(p, y, i)$ and $\check{\pi}(\check{b}) = p$. Thus there exists a point $b \in \boldsymbol{B}$ such that $b = \varphi_i(p, y)$ and $\pi(b) = p$.

We shall show that $h[\pi^{-1}[V_i]] = \check{\pi}^{-1}[V_i]$. Let $b \in \pi^{-1}[V_i]$. Thus there exist $p \in V_i$, $y \in \boldsymbol{F}$ and a diffeomorphism $\varphi_i \in (\varphi_i;\ i \in I)$ such that $b = \varphi_i(p, y)$. Moreover, there exists a point $\check{b} = h(b)$ such that $\check{b} = \check{\varphi}(p, y, i)$. Since $\check{\pi}(\check{b}) = p$ and $p \in V_i$, we have $\check{b} \in \check{\pi}^{-1}[V_i]$. Hence $h[\pi^{-1}[V_i]] \subset \check{\pi}^{-1}[V_i]$. For the opposite inclusion take a $\check{b} \in \check{\pi}^{-1}[V_i]$. There exists $p \in V_i$ such that $\check{\pi}(\check{b}) = p$. Let $\check{b} = \check{\varphi}(p_1, y_1, j)$. Then $\check{\pi}(\check{b}) = p_1$, where $p_1 \in V_j$. Thus $p = p_1$ and $p \in V_i \cap V_j$. Put $y_1 = \varphi_{ji}(p)\cdot y$. Then $(p_1, y_1, j)\bar{\varphi}(p, y, i)$. From this and from the assumption we have $y_1 = \varphi_{j,p}^{-1}(\varphi_{i,p}(y))$, that is, $\varphi_{i,p}(y) = \varphi_{j,p}(y_1)$ for $p \in V_i \cap V_j$. Thus $\varphi_i(p, y) = \varphi_j(p_1, y_1) = b$. Since $\pi(b) = p \in V_i$, we jet $b \in \pi^{-1}[V_i]$. From the above considerations it follows that $h|_{\pi^{-1}[V_i]}$ is a one-one mapping of the set $\pi^{-1}[V_i]$ onto the set $\check{\pi}^{-1}[V_i]$. Since the triangle

(29)



is commutative, we obtain that $h|_{\pi^{-1}[V_i]}\colon B_{\pi^{-1}[V_i]} \to \check{B}_{\check{\pi}^{-1}[V_i]}$ is an epimorphism.

From what has been said above it follows that there exists a mapping $h^{-1}\colon \check{B}\to B$ such that $h^{-1}|_{\check{\pi}^{-1}[V_i]}\colon \check{\pi}^{-1}[V_i]\to\pi^{-1}[V_i]$ is an epimorphism.

Let $\check{b}$ be an arbitrary point belonging to $\check{\pi}^{-1}[V_i]$. Then $h^{-1}(\check{b}) = b$, that is, $h^{-1}(\check{\varphi}(p, y, i)) = b$. Since b can be written in the form

$$b = \varphi_i(r_i(p, y, i)),$$

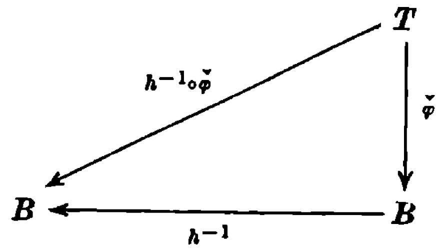
where r_i is defined by (23), we have

$$h^{-1}\circ\check{\varphi}\,|_{V_i\times F\times\{i\}} = \varphi_i\circ r_i. \tag{30}$$

From the fact that the family of sets $(V_i\times F\times\{i\},\ i\in I)$ is an open covering of the space T and from (30) it follows that

$$h^{-1}\circ\check{\varphi}\colon T\to B. \tag{31}$$

Further, from the fact that the differential structure of the differential space B is coinduced from T by the mapping $\check{\varphi}$ and from (30) and (31) it follows that the triangle



is commutative.

Therefore h is a diffeomorphism of the differential space B onto the differential space $\check{B}$.

Since $\check{\varphi}_i$ defined in (26) maps the set $V_i\times F$ onto the set $\pi^{-1}[V_i]$ and since $\check{\varphi}_i = h\circ\varphi_i$, we see that the mapping

$$\check{\varphi}_i\colon M_{V_i}\times F\to\check{B}_{\check{\pi}^{-1}[V_i]},$$

where

$$\check{B}_{\check{\pi}^{-1}[V_i]} = \big(\check{\pi}^{-1}[V_i],\ \check{\varphi}^{*-1}[\mathscr{F}(T)]_{\check{\varphi}[V_i\times F\times\{i\}]}\big),$$

is a diffeomorphism. Therefore the structure $\check{\varphi}^{*-1}[\mathscr{F}(T)]_{\check{\varphi}[V_i\times F\times\{i\}]}$ is identical with the structure coinduced by $\check{\varphi}_i$ from the differential space $M_{V_i}\times F$, that means

$$\varphi_i^{*-1}[\mathscr{F}(M_{V_i}\times F)] = \check{\varphi}^{*-1}[\mathscr{F}(T)]_{\check{\varphi}[V_i\times F\times\{i\}]}.$$

From the definition of the coinduced structure it follows that $a\circ\check{\varphi}_i^{*-1}[\mathscr{F}(M_{V_i}\times F)]$ if and only if $a\circ\check{\varphi}_i\in\mathscr{F}(M_{V_i}\times F)$. Since $\check{\varphi}_i = \check{\varphi}\,|_{V_i\times F\times\{i\}}\circ r_i^{-1}$, where r_i is defined in (23), we have

$$a\circ\check{\varphi}\,|_{V_i\times F\times\{i\}}\circ r_i^{-1}\in\mathscr{F}(M_{V_i}\times F).$$

Hence we obtain that

$$a \circ \check{\varphi} \,|_{V_i \times F \times \{i\}} \in (r_i^{-1})^{*-1}[\mathscr{F}(M_{V_i} \times F)].$$

Since r_i: $M_{V_i} \times F \times i^* \to M_{V_i} \times F$ is a diffeomorphism, it follows that the differential structure coinduced by r_i^{-1} from the differential space $M_{V_i} \times F$ is identical with the differential structure of the differential space $M_{V_i} \times \times F \times i^*$. Thus $a \circ \check{\varphi} \,|_{V_i \times F \times \{i\}} \in \mathscr{F}(M_{V_i} \times F \times i^*)$, i.e.

$$a \in (\check{\varphi} \,|_{V_i \times F \times \{i\}})^{*-1}[\mathscr{F}(M_{V_i} \times F \times i^*)].$$

Therefore

$$\check{\varphi}^{*-1}[\mathscr{F}(\bigoplus_{k \in I} M_{V_k} \times F \times k^*)]_{\check{\varphi}[V_i \times F \times \{i\}]} = (\check{\varphi} \,|_{V_i \times F \times \{i\}})^{*-1}[\mathscr{F}(M_{V_i} \times F \times i^*)],$$

which completes the proof.

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