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On the lifting of invariant measures

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*To the memory of Zdzisław Opial,
my first Professor*

Abstract. The Avez method is useful only for the construction of invariant measure for non-bijective transformations. In this paper, the author constructs the lifting of invariant measures. Using the lifting, we can also obtain an invariant measure for invertible transformations.

Introduction

The problem of an invariant measure was considered in a lot of papers. One of the methods was proposed by Avez [1] and used also by Lasota, Pianigiani [4] and the author [2], [3]. This method is based on a study of the right inverses of the transformation in question. Clearly, it cannot be used for bijective transformations. In this paper we propose a certain method, which extends the Avez method to invertible transformations.

1. Formulation of the results

Let X be an arbitrary set and let $T\colon X \to X$ be an invertible transformation. Let $(Y\colon \Sigma;\ \mu)$ be a probability space and $S\colon Y \to Y$ a transformation satisfying the condition

$$\forall E \in \Sigma \quad \mu(S^{-1}(E)) = \mu(E). \tag{1}$$

Finally, let $\Pi\colon X \to Y$ be a surjection such that $\Pi T = S\Pi$. Let $\mathscr{A}$ be the algebra of subsets of X defined by

$$\mathscr{A} = \{(\Pi T^n)^{-1}(E)\colon\ E \in \Sigma,\ n \in N\} \tag{2}$$

and let $\mathscr{B}$ be the σ-algebra generated by $\mathscr{A}$. To formulate the theorems we have to introduce two assumptions:

(A1) $\quad \forall x, y \in X\ \ [\forall n \in N,\ \Pi T^{-n}x = \Pi T^{-n}y] \Rightarrow x = y.$

(A2) For every sequence $\{E_n\}$ of sets from Σ, if

$$S(E_{n+1}) \subset E_n \text{ for all } n \text{ and } \bigcap_{n=1}^{\infty} T^n(\Pi^{-1}(E_n)) = \emptyset,$$

then $\lim_{n\to\infty} \mu(E) = 0$.

Now we can formulate the theorems.

THEOREM 1. *Under assumption* (A1), *there exists a finitely additive function* $\hat{\mu}$ *on* $\mathscr{A}$, *invariant with respect to* T *and such that*

$$\hat{\mu}(\Pi^{-1}(E)) = \mu(E).$$

THEOREM 2. *Under assumption* (A2), *the finitely additive function* $\hat{\mu}$ *can be extended to a probability measure on* $\mathscr{B}$.

2. Proofs of the theorems

Proof of Theorem 1. Define $\hat{\mu}$ by the formula $\hat{\mu}(\Pi T^n)^{-1}(E) = \mu(E)$. We must prove the correctness of the definition. Let $(\Pi T^n)^{-1}(E) = (\Pi T^r)^{-1}(E')$. It follows that $\Pi T^n x \in E$ iff $\Pi T^r x \in E'$. We may assume that $n > r$. By the bijectivity of T, $\Pi T^{n-r} y \in E$ iff $\Pi y \in E'$, and consequently, for every $y \in X$, we have $S^{n-r}\Pi y \in E$ iff $\Pi y \in E'$. From the surjectivity of Π it follows that $E' = (S^{n-r})^{-1}(E)$ and from (1) we get $\mu(E') = \mu(E)$. Hence, $\hat{\mu}$ is correctly defined.

The finite additivity and T-invariance follow from the definition. From (A1) we see that the algebra $\mathscr{A}$ separates points.

Proof of Theorem 2. Let $\{C_n\}$ be a decreasing sequence of sets from X, whose intersection is empty. We may assume that $C_n = T^n(\Pi^{-1}(E_n))$, where $S(E_{n+1}) \subset E_n$. By (A2), $\mu(E_n) \to 0$, and thus $\hat{\mu}(C_n) \to 0$. Hence, $\hat{\mu}$ is continuous at the empty set and can be extended to a measure on $\mathscr{B}$.

3. Examples and remarks

EXAMPLE 1. Let $X = \mathbf{R}$, $Tx = 2x$, $Y = C_1$, $\Pi x = \exp(2\pi i x)$, $Sz = z^2$. It is obvious that the normed standard Lebesgue measure λ on C_1 is S-invariant. We can lift this measure and obtain a T-invariant finitely additive function on the algebra $\mathscr{A} = \{E \in \mathfrak{M}_L \colon \exists n\ E + 2^n = E\}$ defined by the formula $\hat{\lambda}(E) = 2^{-n}\lambda(E \cap [0;\ 2^n])$ if $E + 2^n = E$.

Remark. This finitely additive function cannot be extended to a measure.

EXAMPLE 2. Let X be the set of locally Lipschitz functions on $\mathbf{R}^+$, vanishing at 0, and let $(Tv)(x) = 2^{\lambda} v(x/2)$ ([2], [3]). Let Y be the space of Lipschitz functions on $[0;\ 1]$ vanishing at 0; define $\Pi v = v|[0;\ 1]$ and $(Sv)(x) = 2^{\lambda} v(x/2)$. Let μ be the S-invariant measure defined as in [2], [3]. This system satisfies (A2) and gives rise to a T-invariant measure on X.

Proof of (A2). Let $\{E_n\}$ be a sequence of Y such that, for every n, $S(E_{n+1}) \subset E_n$. Assume that $\mu(E_n) > \varepsilon$ for some $\varepsilon > 0$. It is sufficient to prove that $\bigcap_{n=1}^{\infty} T^n(\Pi^{-1}(E_n)) \neq \emptyset$. Since the space Y is a subspace of $C[0; 1]$, the measure μ can be considered as a measure on a Polish space, and so, for every n there exists a compact set $K_n \subset E_n$ such that $\mu(E_n \setminus K_n) < 2^{-n}\varepsilon$. Let $K'_n = \bigcup_{i=1}^{n} S^{i-n}(K_i)$. Since K'_n is the intersection of a finite family of closed sets one of them being compact, K'_n is compact. Moreover,

$$\mu(E_n \setminus K'_n) = \mu\Big(\bigcap_{i=1}^{n} (E_n \setminus S^{i-n}(K_i))\Big) \leqslant \sum_{i=1}^{n} \mu(E_i \setminus K_i) \leqslant \sum_{i=1}^{n} 2^{-i}\varepsilon < \varepsilon.$$

Thus K'_n is nonempty. Let us consider a sequence $\{v_n\}$ of functions from X satisfying the condition $v_n \in T^n(\Pi^{-1}(K'_n))$, i.e., such that for every positive integer n the function $\bar{v}_n$ defined by $\bar{v}_n = 2^{-\lambda n} v_n(2^n x)$ for $x \in [0; 1]$ belongs to K'_n. It follows that for every $k \in N$ the functions $v_n|[0; 2^k]$ belong for all $n > k$ to a common compact subset of $C[0; 2^k]$. By the diagonal method we can choose a subsequence of $\{v_n\}$ convergent almost uniformly. The limit of this sequence belongs to $\bigcap_{n=1}^{\infty} T^n(\Pi^{-1}(E_n))$, which completes the proof.

By means of this method we can also construct an invariant measure in the case of continuous time. It would be interesting to find a simpler formulation of (A2).

References

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