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Doubly commuting operator representations of Dirichlet algebras

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Abstract. This paper deals with doubly commuting linear operator-valued mappings of some kinds of spaces of continuous functions on compacts. In the case where the mappings in question are representations of Dirichlet algebras one obtains analytic construction of representations of their tensor products.

We present in this paper a method of constructing some linear operator-valued mappings of tensor products of subspaces of spaces of continuous functions on compacts. The mappings of those subspaces are supposed to be doubly commuting. Our proofs are analytic and dilation free. The questions which we discuss in the present paper have been inspired by Sz.-Nagy's papers [3], [5]. As to the basic properties of semi-spectral measures and integrals we refer here to [1].

Suppose we are given a Hilbert space H with the inner product (f, g), $f, g \in H$, and the norm $|f| = \sqrt{(f, f)}$. Let $L(H)$ stand for the algebra of all linear, bounded operators in H. $|V|$ denotes the norm of $V \in L(H)$ and I is the identity operator. Let Ω be a fixed set and $\mathfrak{M}$ — the σ-field of its subsets. We say that the mapping $F\colon \mathfrak{M} \to L(H)$ is a *semispectral measure* on $\mathfrak{M}$ if $F(\Omega) = I$ and if for every $x \in H$ the function $\mu_x(\sigma) = \big(F(\sigma)x, x\big)$ defines a non-negative measure on $\mathfrak{M}$. Let $L_{(\infty)}(\Omega, \mathfrak{M}, F)$ stand for the class of measurable, almost everywhere F-bounded functions. The uniquely determined operator $\varphi(u) \in L(H)$, $u \in L_{(\infty)}(\Omega, \mathfrak{M}, F)$ such that for every $x, y \in H$ we have $\big(\varphi(u)x, y\big) = \int ud\big(F(\cdot)x, y\big)$ is called the *semispectral integral* of u. In this case we write $\varphi(u) = \int udF$.

In what follows Ω, Ω_1 etc. stand for compact Hausdorff spaces. If A is a subset of $C(\Omega)$, then $\operatorname{Re} A \overset{\text{df}}{=} \{v \mid v = \operatorname{Re} u, u \in A\}$. In further considerations we need the following lemma, which esentially belongs to Foiaş [2]:

LEMMA 1. *Let A be a subspace of $C(\Omega)$ such that $\overline{\operatorname{Re} A} = C_R(\Omega)$. Then every linear mapping $\varphi\colon A \to L(H)$ such that $\|\varphi(u)\| \leqslant \|u\|$ for $u \in A$ and $\varphi(1) = I$ determines a unique semispectral regular measure on Borel subsets of Ω such that $\varphi(u) = \int udF$ for $u \in A$.*

Two mappings $\varphi_i\colon A_i \subset C(\Omega_i) \to L(H_i)$ are called *doubly commuting* if $\varphi_1(u)\ \varphi_2(v) = \varphi_2(v)\ \varphi_1(u)$ and

$$\varphi_1(u)\varphi_2(v)^* = \varphi_2(v)^*\varphi_1(u) \quad \text{for every } u \in A_1,\ v \in A_2.$$

We can now formulate the following

PROPOSITION 1. *Let A_i $(i = 1, 2)$ be a linear subspace of $C(\Omega_i)$ such that $\overline{\mathrm{Re} A_i} = C_R(\Omega_i)$. Consider the mappings φ_i: $A_i \to \mathscr{L}(H)$ such that $\|\varphi_i(u_i)\| \leqslant \|u_i\|$ for $u_i \in A_i$ and $\varphi_i(1) = I$. Assume that φ_1 and φ_2 are doubly commuting. Then the values of semispectral measures F_1 and F_2, determined by φ_1 and φ_2 respectively, commute, i.e.*

$$F_1(\sigma)F_2(\gamma) = F_2(\gamma)F_1(\sigma) \quad \textit{for every } \sigma \in \mathscr{B}(\Omega_1), \gamma \in \mathscr{B}(\Omega_2)$$

($\mathscr{B}(\Omega)$ stands for the σ-field of Borel sets in Ω).

The proof of Proposition 1 is based on the following inequality, which holds for every regular semispectral measure F on compact Ω:

$$\left\|\left(\int u dF\right)f\right\|^2 \leqslant \int |u|^2 d\mu_f \tag{1}$$

for $u \in C(\Omega)$, $f \in H$, where $\mu_f(\sigma) \overset{\mathrm{df}}{=} (F(\sigma)f, f)$.

The above inequality results from a well-known Naimark dilation theorem and from the general properties of spectral integrals. However, we shall now give a simple direct proof.

Let u be a simple function of the form $u = \sum a_i \chi \sigma_i$, where $a_i \in C$, $\sigma_i \in \mathscr{B}(\Omega)$. (1) holds for simple functions, since, according to the Schwarz inequality, we have:

$$\begin{aligned} \left|\left(\int u dFf, g\right)\right| &= \left|\left(\sum a_i F(\sigma_i) f, g\right)\right| \\ &\leqslant \sum |a_i| \left|\left(F(\sigma_i)^{\frac{1}{2}} f, F(\sigma_i)^{\frac{1}{2}} g\right)\right| \leqslant \sum \|a_i F(\sigma_i)^{\frac{1}{2}} f\| \, \|F(\sigma_i)^{\frac{1}{2}} g\| \\ &\leqslant \sqrt{\sum |a_i|^2 (F(\sigma_i)f, f)} \cdot \sqrt{\sum (F(\sigma_i)g, g)} \leqslant \sqrt{\int |u|^2 d\mu_f} \cdot \|g\|. \end{aligned}$$

For any $u \in C(\Omega)$ we can find a sequence u_n of simple functions such that $u_n \to u$ uniformly on Ω, since Ω is compact. For every u_n we have

$$\left\|\sum a_k^{(n)} F(\sigma_k^{(n)}) f\right\|^2 \leqslant \sum |a_k^{(n)}|^2 (F(\sigma_k^{(n)})f, f) = \int |u_n|^2 d\mu_f.$$

By passing to the limit we complete the proof of (1). In further considerations we shall need also the following

LEMMA 2. *Let F be a regular semispectral measure on $\mathscr{B}(\Omega)$. If A is dense in $C(\Omega)$, then for every finite set of vectors $f_1, \ldots, f_k \in H$ and for every $\sigma \in \mathscr{B}(\Omega)$ there exists a sequence $\{u_n\}$, $u_n \in A$ such that*

$$\left(\int u_n dF\right) f_i \to F(\sigma) f_i, \quad i = 1, 2, \ldots, k,\ \sigma \in \mathscr{B}(\Omega).$$

Proof of Lemma 1. The formula $\mu(\sigma) = \sum_{i=1}^{k} \big(F(\sigma)f_i, f_i\big)$ defines a regular measure on $\mathscr{B}(\Omega)$. Since $C(\Omega)$ is dense in $L_2\big(\Omega, \mathscr{B}(\Omega), \mu\big)$, then for every $\sigma \in \mathscr{B}(\Omega)$ there exists a sequence $\{u_n\}$, $u_n \in A \subset C(\Omega)$, such that

$$\eta_n = \int |u_n - \chi_\sigma|^2 d\mu \to 0.$$

According to (1)

$$\Big\|\Big(\int u_n dF\Big) f_i - F(\sigma) f_i\Big\|^2 = \Big\|\Big(\int (u_n - \chi_\sigma) dF\Big) f_i\Big\|^2 \leqslant \int |u_n - \chi_\sigma|^2 d\mu_f \leqslant \int |u_n - \chi_\sigma|^2 d\mu = \eta_n \to 0,$$

which completes the proof.

Proof of Proposition 1. Since φ_i are doubly commuting, for every $u_1, v_1 \in A_1$, $u_2, v_2 \in A_2$, we have:

$$\int (u_1 + \bar{v}_1) dF_1 \int (u_2 + \bar{v}_2) dF_2 = \int (u_2 + \bar{v}_2) dF_2 \int (u_1 + \bar{v}_1) dF_1.$$

Since $\overline{\operatorname{Re} A_i} = C_R(\Omega_i)$, we infer that $A_i' = \{w \mid w = u_i + \bar{v}_i,\ u_i, v_i \in A_i\}$ is dense in $C(\Omega_i)$.

According to Lemma 2 for every $\sigma \in \mathscr{B}(\Omega_1)$, $v \in A_2$, $f_1 \in H$, $f_2 = \big(\int v dF_2\big) f_1$ there exists a sequence $\{u_n\}$, $u_n \in A_1'$ such that:

$$\Big(\int u_n dF_1\Big) f_1 \to F_1(\sigma) f_1,$$

$$\int u_n dF_1 \Big(\int v dF_2\Big) f_1 \to F_1(\sigma) \Big(\int v dF_2 f_1\Big).$$

For $u_n \in A_1'$, $v \in A_2'$ we have

$$\int u_n dF_1 \Big(\int v dF_2\Big) f_1 = \int v dF_2 \Big(\int u_n dF_1\Big) f_1$$

and, by passing to the limit, $F_1(\sigma)\big(\int v dF_2\big) f_1 = \big(\int v dF_2\big) F_1(\sigma) f_1$. Applying Lemma 2 to $f_1 \in H$, $f_2 = F_1(\sigma) f_1$ we can find a sequence $v_n \in A_2'$ such that:

$$\Big(\int v_n dF_2\Big) f_1 \to F_2(\gamma) f_1,$$

$$\Big(\int v_n dF_2\Big) F_1(\sigma) f_1 \to F_2(\gamma) F_1(\sigma) f_1.$$

It follows that

$$F_1(\sigma)\Big(\int v_n dF_2\Big) f_1 = \Big(\int v_n dF_2\Big) F_1(\sigma) f_1$$

and consequently

$$F_1(\sigma) F_2(\gamma) f_1 = F_2(\gamma) F_1(\sigma) f_1$$

for every $f_1 \in H$, i.e. $F_1(\sigma) F_2(\gamma) = F_2(\gamma) F_1(\sigma)$.

We can now formulate the following

THEOREM. *Consider* A_i, φ_i $(i = 1, 2)$ *as in Proposition* 1. *Assume that* φ_i *are doubly commuting. We claim that there exists a regular semispectral product measure* F *on* $\mathscr{B}(\Omega_1 \times \Omega_2)$ *such that* $\varphi_1(u_1)\varphi_2(u_2) = \int u_1 u_2 dF$ *for* $u_i \in A_i$, $i = 1, 2$.

Proof. We define

$$A = \{v(w_1, w_2) \,|\, v(w_1, w_2) = \sum_i u_i(w_1) z_i(w_2),\ u_i \in C(\Omega_1),\ z_i \in C(\Omega_2)\}.$$

According to Stone's theorem, A is dense in $C(\Omega_1 \times \Omega_2)$.

Let us define the mapping φ on A in the following way:

$$\varphi\Big(\sum_i u_i z_i\Big) = \sum_i \varphi(u_i z_i) = \sum_i \int u_i dF_1 \int z_i dF_2,$$

where F_i stands for the semispectral measure determined by φ_i, $i = 1, 2$.

The mapping φ is well defined. Indeed, suppose $\sum_i u_i z_i = \sum_i u_i' z_i'$, i.e.

$$\sum u_i(w_1) z_i(w_2) = \sum u_i'(w_1) z_i'(w_2) \quad \text{for every } w_i \in \Omega_i,\ i = 1, 2.$$

Let us fix w_1 and integrate the above equality with regard to measure F_2. We then have

$$\sum_i u_i(w_1) \int z_i(w_2) dF_2 = \sum_i u_i'(w_1) \int z_i'(w_2) dF_2.$$

Let us now fix w_2 and integrate with regard to F_2. We get

$$\sum_i \int u_i(w_1) dF_1 \int z_i(w_2) dF_2 = \sum \int u_i'(w_1) dF_1 \int z_i'(w_2) dF_2.$$

It follows that

$$\varphi\Big(\sum u_i z_i\Big) = \sum \int u_i dF_1 \int z_i dF_2 = \sum \int u_i' dF_1 \int z_i' dF_2 = \varphi\Big(\sum u_i' z_i'\Big).$$

The mapping φ is linear and $\varphi(u_0 z_0) = I$. We shall check that $\|\varphi(v)\| \leqslant \|v\|$ for $v = \sum u_i z_i$. We shall prove the inequality $\|\varphi(v)\| \leqslant \|v\|$ for $v = uz$. In the general case the reasoning is analogous. For every $\varepsilon > 0$ one can find $\delta > 0$ such that for every partition of $\overline{u(\Omega_1)} \times \overline{z(\Omega_2)}$ by a finite number of Borel sets of diameter smaller than δ we have

$$\Big\|\int u dF_1 \int z dF_2 - \sum_{i,k} u(\xi_i^1) z(\xi_k^2) F_1(\sigma_i^1) F_2(\sigma_k^2)\Big\| < \varepsilon.$$

Since, by Lemma 1, F_1 and F_2 commute, the following estimates hold true:

$$\Big|\Big(\sum_{i,k} u(\xi_i^1)z(\xi_k^2)F_1(\sigma_i^1)F_2(\sigma_k^2)f,\, g\Big)\Big|$$

$$= \Big|\Big(\sum_{i,k} u(\xi_i^1)z(\xi_k^2)\big(F_1(\sigma_i^1)F_2(\sigma_k^2)\big)^{\frac{1}{2}}f,\, \big(F_1(\sigma_i^1)F_2(\sigma_k^2)\big)^{\frac{1}{2}}g\Big)\Big|$$

$$\leqslant \|uz\| \sum_{i,k} \big\|\big(F_1(\sigma_i^1)F_2(\sigma_k^2)\big)^{\frac{1}{2}}\big\|\,\big\|\big(F_1(\sigma_i^1)F_2(\sigma_k^2)\big)^{\frac{1}{2}}g\big\|$$

$$\leqslant \|uz\| \sqrt{\sum_i \Big(\sum_k \big(F_2(\sigma_k^2)F_1(\sigma_i^1)f,f\big)\big(F_1(\sigma_k^1)F_2(\sigma_k^2)g,\, g\big)\Big)}$$

$$= \|uz\| \sqrt{\sum_i \Big(\sum_k F_2(\sigma_k^2)F_1(\sigma_i^1)f,f\Big)\big(F_2(\sigma_k^2)\,F_1(\sigma_i^1)\,g,\, g\big)}$$

$$= \|uz\| \sqrt{\sum_i \big(F_1(\sigma_k^1)f,f\big)\big(F_1(\sigma_i^1)g,\, g\big)} = \|uz\|\,\|f\|\,\|g\|.$$

By passing to the limit $|(\varphi(uz)f,\, g)| \leqslant \|uz\|\cdot\|f\|\cdot\|g\|$.

Going back to the general case, by the Foiaş lemma there exists a regular semispectral measure F on $\mathscr{B}(\Omega_1\times\Omega_2)$, such that for every $v = \sum u_i z_i$ we have

$$\varphi(v) = \int\!\!\int_{\Omega_1\times\Omega_2} v\,dF.$$

Moreover, $\varphi_1(u_1)\cdot\varphi_2(u_2) = \int u_1u_2\,dF$ for $u_1\in A_1$, $u_2\in A_2$. Notice that the measure F of our theorem satisfies

$$(2)\qquad F(\sigma_1\times\sigma_2) = F_1(\sigma_1)F_2(\sigma_2),\qquad \sigma_1\in\mathscr{B}(\Omega_1),\ \sigma_2\in\mathscr{B}(\Omega_2).$$

Indeed, let $\sigma_1\in\mathscr{B}(\Omega_1)$, $\sigma_2\in\mathscr{B}(\Omega_2)$. There are sequences $u_n\in C(\Omega_1)$, $v_n\in C(\Omega_2)$ such that

$$\Big(\int u_n\,dF_1\int v_n\,dF_2 f, f\Big) = \Big(\int v_n\,dF_2 f,\, \int \bar{u}_n\,dF_1 f\Big) \to \big(F_1(\sigma_1)F_2(\sigma_2)f, f\big).$$

Thus, it suffices to show that the sequence $\big(\int u_n\,dF_1\int v_n\,dF_2 f, f\big)$ has the limit $\big(F(\sigma_1\times\sigma_2)f, f\big)$.

We have

$$F(\sigma_1\times\sigma_2)f - \int u_n\,dF_1\int v_n\,dF_2 f = \int\!\!\int \chi_{\sigma_1}\chi_{\sigma_2}\,dFf - \int\!\!\int u_nv_n\,dFf$$

$$= \int\!\!\int (\chi_{\sigma_1}-u_n)\chi_{\sigma_2}\,dFf + \Big(\int\!\!\int (\chi_{\sigma_1}-u_n)\chi_{\sigma_2}\,dF\Big)f.$$

Consider $\Delta_n = \left(\int\int (\chi_{\sigma_1} - u_n)\chi_{\sigma_2} dF\right) f$.
According to (1) and (2) we have

$$\|\Delta_n\|^2 \leqslant \int\int |\chi_{\sigma_1} - u_n|^2 d(Ff, f) = \int |\chi_{\sigma_1} - u_n|^2 d(F_1 f, f)$$

and thus $\Delta_n \to 0$.

Similarly one can show that $\int\int (\chi_{\sigma_2} - v_n) u_n dFf \to 0$ which completes the proof of (2).

We can formulate Proposition 1 for the general case $n > 2$: Consider Ω_i, A_i and φ_i as in Proposition 1 for $i = 1, 2, \ldots, n$. Assume that φ_i are pairwise doubly commuting. Then the values of the semispectral measures $F_1, \ldots, F_n$, determined by $\varphi_1, \ldots, \varphi_n$ respectively commute also, i.e.

$$F_1(\sigma_1)F_2(\sigma_2) \ldots F_n(\sigma_n) = F_{i_1}(\sigma_{i_1})F_{i_2}(\sigma_{i_2}) \ldots F_{i_n}(\sigma_{i_n})$$

for every $\sigma_j \in \mathscr{B}(\Omega_j)$.

The proof is similar to that for $n = 2$.

Corollary. *If A_i are Dirichlet algebras and φ_i are representations, then $\varphi(u) = \int u d(F_1 \times F_2 \times \ldots \times F_n)$, when restricted to $A_1 \otimes A_2 \otimes \ldots \otimes A_n$, is a representation.*

The analytic method used in the present paper permits a slight extension of a well-known theorem of B. Sz. Nagy [5] in the case of n doubly commuting contractions.

Consider for example a finite set of operator sequences $\{T_n^{(i)}\}$, $i = 1, 2, \ldots, k$, $n = 0, \pm 1, \pm 2, \ldots$, such that

$$T_n^{(i)} = \int_K z^n dF_i,$$

where $F_1, \ldots, F_n$ are semispectral measures on $\mathscr{B}(K)$. K stands here for the unit circle on a complex plane. Moreover, assume that $T_n^{(i)}$ are doubly commuting, i.e.

$$\begin{aligned} T_n^{(i)} T_m^{(j)} &= T_m^{(j)} T_n^{(i)}, \\ T_n^{(i)} T_m^{(j)*} &= T_m^{(j)*} T_n^{(i)} \quad \text{for } n, m = 0, \pm 1, \pm 2, \ldots, n \neq m. \end{aligned} \tag{3}$$

The set of real parts of analytic polynomials is dense in $C(K_i)$. Define the mapping φ_i on A_i (a copy of the disc algebra) in the following way:

$$\varphi_i(z_i^k) \overset{\text{df}}{=} \int z_i^k dF_i, \quad i = 1, 2, \ldots, k;$$

and for polynomials let us set

$$\varphi_i\Big(\sum_k a_k z_i^k\Big) \overset{\text{df}}{=} \int_K \sum a_k z_i^k dF_i = \sum_k a_k \int z_i^k dF_i = \sum_k a_k \varphi(z_i^k).$$

Thus φ_i is linear and $\varphi_i(u_0) = F_i(K) = I$, where $u_0^i(z_i) = 1$.

It follows that

$$\Big\|\varphi_i\Big(\sum_k a_k z_i^k\Big)\Big\| \leqslant \Big\|\sum_k a_k z_i^k\Big\|.$$

Since $T_n^{(i)}$ are doubly commuting, so are φ_i, $i = 1, 2, \ldots, n$. By Lemma 2 this implies that F_i commute also and, according to our Theorem, there exists a product semispectral measure F.

We write $F = F_1 \times \ldots \times F_n$.

Consider the polynomial of k variables $(z_1, z_2, \ldots, z_k)$:

$$w(z_1, z_2, \ldots, z_{\tilde{k}}) = \sum a_{i_1 \ldots i_k} z_1^{i_1} \ldots z_k^{i_{\tilde{k}}}.$$

Define $w(\mathscr{T})$ in the following way:

$$w(\mathscr{T}) \stackrel{\mathrm{df}}{=} w(T^{(1)}, \ldots, T^{(k)}) = \sum a_{i_1 \ldots i_n} T_{a_1}^{(i_1)} \ldots T_{a_{\tilde{k}}}^{(i_{\tilde{k}})}.$$

We have, by (1),

$$\|w(\mathscr{T})\| \leqslant \sup_{K^{\tilde{k}}} |w(z_1, z_2, \ldots, z_k)|.$$

The above formula gives in particular the von Neumann inequality (see [4] for references) for n doubly commuting contractions.

Indeed, consider n contraction operators $T_1, T_2, \ldots, T_n$ such that $T_i T_k = T_k T_i$ and $T_k T_i^* = T_i^* T_k$.

By the Sz.-Nagy theorem ([4], Theorem 4.2, p. 13) for every k there is a unique semispectral measure F_k on $\mathscr{B}(K)$ such that

$$T_k^s = \int_K z^s dF_k \quad \text{for } s = 0, 1, 2.$$

Since $\{T_k\}$ doubly commute, we are in a position to apply the previous arguments.

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