ANNALES
POLONICI MATHEMATICI
XXVII (1972)

Some theorems on partial differential inequalities of parabolic type

by P. Besala (Gdańsk)

We treat a system of inequalities of the form

$$u_t^i \leqslant f^i(t, X, U, u_X^i, u_{XX}^i) \quad (i = 1, \ldots, m),$$

where $X = (x_1, \ldots, x_n)$, $U = (u^1, \ldots, u^m)$, $u_X^i = (u_{x_1}^i, \ldots, u_{x_n}^i)$ and $u_{XX}^i = (u_{x_j x_k}^i)$ $(j, k = 1, \ldots, n)$.

This note deals with a version of parabolic differential inequalities theorems in which weak differential inequalities and strong initial and boundary inequalities imply strong inequalities between functions involved in their existence domain. Theorems 1 and 2 established here, concerning bounded and unbounded domains respectively, are counterparts of a theorem proved in [2] for a system of differential inequalities of the first order.

1. Preliminary definitions. For any vectors $U = (u^1, \ldots, u^m)$, $V = (v^1, \ldots, v^m)$ we shall write

$$U \leqslant V \quad \text{if } u^j \leqslant v^j \ (j = 1, \ldots, m),$$

and

$$U < V \quad \text{if } u^j < v^j \ (j = 1, \ldots, m).$$

For a fixed i we write

$$U \overset{i}{\leqslant} V \quad \text{if } u^j \leqslant v^j \ (j = 1, \ldots, m) \text{ and } u^i = v^i.$$

Let D be a domain of the $(n+1)$-dimensional space of the variables $(t, X) = (t, x_1, \ldots, x_n)$. We assume that D is contained in the zone $0 < t < T \leqslant +\infty$, X arbitrary, and the intersection $S_{\bar{t}}$ of the closure of D with any plane $t = \bar{t}$, $0 \leqslant \bar{t} \leqslant T$, is non-empty. $\sum$ will stand for the part of the boundary of D, contained in the zone $0 < t < T$.

A vector-function $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ will be called *regular* in D if it is continuous in the closure of D and possesses continuous derivatives $\partial/\partial t$, $\partial/\partial x_j$, $\partial^2/\partial x_j \partial x_k$ in D.

Let functions $f^i(t, X, U, Q, R)$ $(i = 1, \ldots, m)$ be defined for $(t, X) \in D$ and arbitrary $U = (u^1, \ldots, u^m), Q = (q_1, \ldots, q_n), R = (r_{jk})$ $(j, k = 1, \ldots, n)$. We shall make use of the following definition of ellipticity given by J. Szarski.

DEFINITION 1. A function $f^i(t, X, U, Q, R)$ is called *elliptic* with respect to a function $U(t, X)$ of class $C^1(D)$ if for any two symmetric matrices $R = (r_{jk})$, $\tilde{R} = (\tilde{r}_{jk})$ $(j, k = 1, \ldots, n)$ such that

$$\sum_{j,k=1}^{n} (r_{jk} - \tilde{r}_{jk}) \lambda_j \lambda_k \leqslant 0$$

we have

$$f^i\big(t, X, U(t, X), u^i_X(t, X), R\big) \leqslant f^i\big(t, X, U(t, X), u^i_X(t, X), \tilde{R}\big)$$

for $(t, X) \in D$.

DEFINITION 2. A function $f^i(t, X, U, Q, R)$ will be said to *satisfy condition* W with respect to U if $U \overset{i}{\leqslant} V$ implies

$$f^i(t, X, U, Q, R) \leqslant f^i(t, X, V, Q, R).$$

2. Differential inequalities in bounded regions. In this section for the domain D introduced above it will be assumed that the intersection of its closure with any zone $0 \leqslant t \leqslant t_0$, $t_0 < T$, is bounded.

The index i being fixed let $\alpha^i(t, X)$, $\beta^i(t, X)$ be functions defined and positive on an open subset Σ_i of the set Σ and let $\beta^i(t, X)$ be bounded on Σ_i. Denote by $l^i(t, X)$, $(t, X) \in \Sigma_i$, a direction orthogonal to the t-axis and assume that some segment starting at (t, X) of the straight half-line from (t, X) in the direction l^i is contained in the closure of D. A vector-function $U(t, X)$ will be called *Σ-regular* if it is regular in D and the derivatives du^i/dl^i exist at points of Σ_i respectively $(i = 1, \ldots, m)$.

We recall the definition of Condition C introduced in [2].

CONDITION C. The index i being fixed the function $f^i(t, X, U, Q, R)$ will be said to *satisfy Condition* C with respect to u^i if $u^i \leqslant \tilde{u}^i$ implies

$$f^i(t, X, u^1, \ldots, u^{i-1}, u^i, u^{i+1}, \ldots, u^m, Q, R) - \\ - f^i(t, X, u^1, \ldots, u^{i-1}, \tilde{u}^i, u^{i+1}, \ldots, u^m, Q, R) \leqslant \sigma(t, u^i - \tilde{u}^i),$$

where the function $\sigma(t, z)$ has the following properties:

(a) $\sigma(t, z)$ is continuous and non-negative in the half-strip $t \in \langle 0, T)$, $z \leqslant 0$, and $\sigma(t, 0) \equiv 0$,

(b) the left-hand minimum solution of the equation

$$\frac{dz}{dt} = \sigma(t, z)$$

satisfying the condition $\lim_{t \to T-} z(t) = 0$ is $z(t) \equiv 0$.

THEOREM 1. *Suppose the functions* $U(t, X) = (u^1(t, X), \ldots, u^m(t, X))$, $V(t, X) = (v^1(t, X), \ldots, v^m(t, X))$ *are* Σ*-regular in* D *and satisfy the initial inequalities*

$$U(0, X) < V(0, X) \quad \text{for } X \in S_0 \tag{1}$$

and the boundary inequalities

$$\begin{gathered} u^i(t, X) < v^i(t, X) \quad \text{for } (t, X) \in \Sigma - \Sigma_i, \\ \beta^i(t, X)[u^i(t, X) - v^i(t, X)] - a^i(t, X)\frac{d[u^i - v^i]}{dl^i} \leqslant -\eta \end{gathered} \tag{2}$$

for $(t, X) \in \Sigma_i$ $(i = 1, \ldots, m)$, η *being a positive constant. Define*

$$G = \{(t, X) \in D\colon\ U(t, X) \leqslant V(t, X)\} \tag{3}$$

and let the inequalities

$$u^i_t(t, X) \leqslant f^i(t, X, U(t, X), u^i_X(t, X), u^i_{XX}(t, X)) \tag{4}$$

$$v^i_t(t, X) \geqslant f^i(t, X, V(t, X), v^i_X(t, X), v^i_{XX}(t, X)) \tag{5}$$

$$(i = 1, \ldots, m)$$

be satisfied whenever $(t, X) \in G$. *We assume that every function* $f^i(t, X, U, Q, R)$ $(i = 1, \ldots, m)$ *is elliptic with respect to* $V(t, X)$ *and satisfies condition* W *with respect to* U *and condition* C *with respect to* u^i.

Under these assumptions the inequality

$$U(t, X) < V(t, X) \tag{6}$$

holds true for $(t, X) \in D$.

Proof. It is well known ([3], Theorem 63.1) that if in (4) or (5) the weak inequality is replaced by the strong one, then Theorem 1 is valid. In this case condition C is superfluous. We shall show that the additional condition C enables us to reduce, similarly as in [2], the proof of Theorem 1 to the proof of the theorem concerning the strong differential inequalities. To this end we take advantage of the following lemma proved in [2].

LEMMA. *Let* z_0 *be any fixed positive number. If the function* $\sigma(t, z)$ *has properties* (a) *and* (b), *then for every* $\varepsilon > 0$ *there is* $\delta_0(\varepsilon) > 0$ *such that for any* $0 < \delta < \delta_0$ *the right-hand minimum solution* $\omega(t)$ *of the equation*

$$\frac{d\omega}{dt} = -\sigma(t, -\omega) - \delta \tag{7}$$

through $(0, z_0)$ *exists and is positive in* $\langle 0, T - \varepsilon)$.

Now, $\varepsilon > 0$ being chosen arbitrarily let Σ_i^ε, $(\Sigma - \Sigma_i)^\varepsilon$ be the parts of Σ_i, $\Sigma - \Sigma_i$ respectively, which are contained in the zone $0 < t < T - \varepsilon$. Put

$$z_1 = \min_{j,k}\{\inf_{S_0 \cup (\Sigma - \Sigma_j)^\varepsilon}[v^j(t, X) - u^j(t, X)],\ \inf_{\Sigma_k^\varepsilon}\eta[\beta^k(t, X)]^{-1}\}.$$

It follows from our assumptions that $z_1 > 0$. In the Lemma we choose $0 < z_0 < z_1$ and δ so that $\omega(t) > 0$ in $\langle 0, T-\varepsilon)$. Observe that $\omega(t) \leqslant z_0 < z_1 \leqslant \eta[\beta^i(t, X)]^{-1}$, $(t, X) \in \Sigma_i^\varepsilon$. Hence, denoting $\Omega(t) = (\underbrace{\omega(t), \ldots, \omega(t)}_{m})$, $\tilde{u}^i(t, X) = u^i(t, X) + \omega(t)$, $\tilde{U}(t, X) = U(t, X) + \Omega(t)$, we get from (1) and (2)

$$\tilde{U}(0, X) < V(0, X) \qquad \text{for } X \in S_0, \tag{8}$$

$$\tilde{u}^i(t, X) < v^i(t, X) \qquad \text{for } (t, X) \in \Sigma - \Sigma_i^\varepsilon \tag{9}$$

and

$$\beta^i(t, X)[\tilde{u}^i(t, X) - v^i(t, X)] - \alpha^i(t, X)\frac{d[\tilde{u}^i - v^i]}{dl^i} < 0 \tag{10}$$

$$\text{for } (t, X) \in \Sigma_i^\varepsilon \quad (i = 1, \ldots, m).$$

Let

$$\tilde{G}^i = \{(t, X) \in D:\ \tilde{U}(t, X) \overset{i}{\leqslant} V(t, X)\}.$$

We have $\tilde{G}^i \subset G$ $(i = 1, \ldots, m)$ and consequently inequalities (4), (5) hold for $(t, X) \in \tilde{G}^i$. Now adding (4) and (7) and applying successively conditions C and W we obtain

$$\begin{aligned} \tilde{u}_t^i &\leqslant f^i(t, X, U, u_X^i, u_{XX}^i) - \sigma(t, -\omega) - \delta \\ &\leqslant f^i(t, X, u^1, \ldots, u^{i-1}, \tilde{u}^i, u^{i+1}, \ldots, u^m, u_X^i, u_{XX}^i) - \delta \\ &\leqslant f^i(t, X, \tilde{U}, \tilde{u}_X^i, \tilde{u}_{XX}^i) - \delta \end{aligned}$$

that is, since $\delta > 0$,

$$\tilde{u}_t^i < f^i(t, X, \tilde{U}, \tilde{u}_X^i, \tilde{u}_{XX}^i) \qquad \text{for } (t, X) \in \tilde{G}^i. \tag{11}$$

Taking into account (8), (9), (10), (11) and (5) we see that all the assumptions of the theorem on strong differential inequalities are satisfied ([3], Theorem 63.1). Hence $\tilde{U}(t, X) < V(t, X)$ for $(t, X) \in D$, $0 \leqslant t < T - \varepsilon$. Since $\omega(t) > 0$ and ε is arbitrary, inequality (6) holds in D.

3. Differential inequalities in unbounded regions. In this section we retain all the definitions of section 1. For the domain D introduced there we assume that for any $t \in \langle 0, T\rangle$ the set S_t (see Section 1) is unbounded.

We introduce the following conditions.

CONDITION L. If the function $f^i(t, X, U, Q, R)$ satisfies the inequality

$$[f^i(t, X, U, Q, R) - f^i(t, X, \tilde{U}, \tilde{Q}, \tilde{R})]\operatorname{sgn}(u^i - \tilde{u}^i)$$

$$\leqslant L_0 \sum_{j,k=1}^{n} |r_{jk} - \tilde{r}_{jk}| + L_1(|X| + 1) \cdot \sum_{j=1}^{n} |q_j - \tilde{q}_j| + L_2(|X|^2 + 1) \sum_{l=1}^{m} |u^l - \tilde{u}^l|,$$

where $|X| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ and L_0, L_1, L_2 are some positive constants, then we say that f^i *satisfies condition* L.

CONDITION C′. If there is a positive function $K(X)$ defined in the n-space and bounded on any compact set of this space and such that $u^i \leqslant \tilde{u}_i$ implies

$$f^i(t, X, u^1, \ldots, u^{i-1}, u^i, u^{i+1}, \ldots, u^m, Q, R) - \\ -f^i(t, X, u^1, \ldots, u^{i-1}, \tilde{u}^i, u^{i+1}, \ldots, u^m, Q, R) \leqslant K(X)|u^i - \tilde{u}^i|,$$

then f^i is said to *satisfy condition* C′ with respect to u^i.

THEOREM 2. *Let vector-functions $U(t, X)$, $V(t, X)$ be regular in D and satisfy the initial-boundary inequality*

$$U(t, X) < V(t, X) \quad \textit{for } (t, X) \in S_0 \cup \Sigma \tag{12}$$

and let

$$u^i(t, X) - v^i(t, X) \leqslant M_1 \exp(M_2 |X|^2) \quad \textit{for } (t, X) \in D \tag{13}$$

$(i = 1, \ldots, m)$, M_1, M_2 being positive constants. Assume that the differential inequalities

$$u_t^i \leqslant f^i(t, X, U, u_X^i, u_{XX}^i) \tag{14}$$

$$v_t^i \geqslant f^i(t, X, V, v_X^i, v_{XX}^i) \qquad (i = 1, \ldots, m) \tag{15}$$

hold true for $(t, X) \in D$. Moreover, we assume that every function $f^i(t, X, U, Q, R)$ is elliptic with respect to $U(t, X)$ and satisfies condition W *with respect to U, condition* C′ *with respect to u^i and condition* L.

Then we have

$$U(t, X) < V(t, X) \quad \textit{in } D. \tag{16}$$

Proof. The assumptions of Theorem 2 imply all the assumptions of the theorem on weak differential inequalities (cf. [3], Theorem 65.1, or [1] Theorem 1), whence

$$U(t, X) \leqslant V(t, X) \quad \text{in } D. \tag{17}$$

To prove the strong inequality (16) take an arbitrary point $(\bar{t}, \bar{X}) \in D$. Define

$$h(t, X) = \bar{t} + 2 - t - (|X - \bar{X}|^2 + 1)^{a/2},$$

where a depending on the point $(\bar{t}, \bar{X})$ will be determined later. Consider the set of points $(t, X) \in D$ which satisfy the inequality $h(t, X) > 0$. Let Δ be that component of this set which contains $(\bar{t}, \bar{X})$ and let $\sigma = \partial\Delta \cap (S_0 \cup \Sigma)$, $\partial\Delta$ being the boundary of Δ. We introduce the function

$$\bar{h}(t, X) = [\bar{t} + 2 - t - (|X - \bar{X}|^2 + 1)^{a/2}] e^{-\bar{K}t},$$

where $\bar{K} = \sup_{\bar{\Delta}} K(X)$. Let us define

$$\bar{H}(t, X) = (\underbrace{\bar{h}(t, X), \ldots, \bar{h}(t, X)}_{m}), \quad \bar{U}(t, X) = U(t, X) + \varepsilon \bar{H}(t, X).$$

By (12) $\varepsilon > 0$ can be chosen so small that

$$\bar{U}(t, X) \leqslant V(t, X) \tag{18}$$

for $(t, X) \epsilon \sigma$. From (17) and the definition of Δ it follows that inequality (18) is also satisfied on the remaining part of $\partial\Delta$ except the points lying on the plane $t = T$ (if such points belong to $\partial\Delta$).

Now we show that $\bar{U}(t, X)$ satisfies, in Δ, inequalities (14). Evidently

$$\bar{u}_t^i \leqslant f^i(t, X, \bar{U} - \varepsilon\bar{H}, \bar{u}_X^i - \varepsilon\bar{h}_X, \bar{u}_{XX}^i - \varepsilon\bar{h}_{XX}) + \varepsilon\bar{h}_t. \tag{18}$$

Since $\bar{h} > 0$ in Δ, using condition W and then condition C$'$ we get

$$\begin{aligned} f^i(t, X, \bar{U} - \varepsilon\bar{H}, \bar{u}_X^i - \varepsilon\bar{h}_X, \bar{u}_{XX}^i - \varepsilon\bar{h}_{XX}) \\ \leqslant f^i(t, X, \bar{U}, \bar{u}_X^i - \varepsilon\bar{h}_X, \bar{u}_{XX}^i - \varepsilon\bar{h}_{XX}) + \varepsilon\bar{K}\cdot\bar{h}. \end{aligned} \tag{20}$$

Further, condition L implies the inequality

$$\begin{aligned} f^i(t, X, \bar{U}, \bar{u}_X^i - \varepsilon\bar{h}_X, \bar{u}_{XX}^i - \varepsilon\bar{h}_{XX}) \\ \leqslant f^i(t, X, \bar{U}, \bar{u}_X^i, \bar{u}_{XX}^i) + \varepsilon L_0 \sum_{j,k=1}^{n} |\bar{h}_{x_j x_k}| + \varepsilon L_1(|X|+1) \sum_{j=1}^{n} |\bar{h}_{x_j}|. \end{aligned} \tag{21}$$

By (19), (20), (21) it follows that if we show that

$$F(\bar{h}) \equiv L_0 \sum_{j,k=1}^{n} |\bar{h}_{x_j x_k}| + L_1(|X|+1) \sum_{j=1}^{n} |\bar{h}_{x_j}| + \bar{K}\bar{h} + \bar{h}_t \leqslant 0$$

in Δ, then $\bar{U}(t, X)$ will satisfy

$$\bar{u}_t^i \leqslant f^i(t, X, \bar{U}, \bar{u}_X^i, \bar{u}_{XX}^i) \quad \text{for } (t, X) \epsilon \Delta \ (i = 1, \ldots, m). \tag{22}$$

An easy computation shows that

$$\begin{aligned} F(\bar{h}) \leqslant e^{-\bar{K}t}\{L_0 a|a-2|n(|X-\bar{X}|^2+1)^{\frac{a}{2}-1} + L_0 an(|X-\bar{X}|^2+1)^{\frac{a}{2}-1} + \\ + L_1(|X|+1)a\sqrt{n}(|X-\bar{X}|^2+1)^{\frac{a}{2}-\frac{1}{2}} - 1\}. \end{aligned}$$

Since $|X| \leqslant |\bar{X}| + |X-\bar{X}|$ and $(|X-\bar{X}|^2+1)^{\frac{a}{2}} \leqslant \bar{t}+2-t \leqslant \bar{t}+2$ in Δ, we obtain

$$F(\bar{h}) \leqslant e^{-\bar{K}t}\{an(\bar{t}+2)[L_0(|a-2|+1) + L_1(|\bar{X}|+2)] - 1\}.$$

Hence it follows that a can be chosen so small that $F(\bar{h}) \leqslant 0$ in Δ.

The differential inequalities (22), (15), the initial-boundary inequality (18) and the other assumptions of Theorem 2 imply that for the domain Δ

all the assumptions of the theorem on weak differential inequalities of parabolic type proved in [3] (Theorem 64.1) are fulfilled. Thus we obtain

$$\overline{U}(t, X) \leqslant V(t, X) \quad \text{for } (t, X) \in \Delta .$$

In particular this inequality holds at $(\bar{t}, \bar{X})$. Since $\bar{h}(\bar{t}, \bar{X}) > 0$ the theorem follows.

References

[1] P. Besala, *On solutions of Fourier's first problem for a system of non-linear parabolic equations in an unbounded domain*, Ann. Polon. Math. 13 (1963), p. 247 - 265.

[2] — *On partial differential inequalities of the first order*, ibidem 25 (1971), p. 145-148.

[3] J. Szarski, *Differential inequalities*, Warszawa 1965.

Reçu par la Rédaction le 5. 12. 1970