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An additional note on entire functions represented by Dirichlet series

by A. R. REDDY (Madras, India)

1. Introduction. This note is concerned with two earlier notes, one by Rajagopal and the present author [2], the other by Rahman [1]. It seeks to prove certain theorems in [2] under conditions less restrictive than the ones assumed in [2], using two auxiliary results of [1] stated below as Lemmas 1, 2, and adopting the following notation, borrowed from [2].

$$f(s) = \sum_{1}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \quad 0 < \lambda_n < \lambda_{n+1} \ (n \geqslant 1), \quad \lambda_n \to \infty ,$$

is an entire function in the sense that the Dirichlet series representing it is absolutely convergent for all finite s. For this function,

$$M(\sigma) = \operatorname*{l.u.b}_{-\infty<t<\infty} |f(\sigma+it)|, \qquad \mu(\sigma) = \max_{n\geqslant 1} |a_n e^{(\sigma+it)\lambda_n}| ,$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} = \frac{\lambda}{\lambda'}, \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(\sigma)}{\sigma} = \frac{\varrho_*}{\lambda_*}.$$

As stated in [2], we may differentiate the Dirichlet series for $f(s)$ j times ($j \geqslant 1$) and obtain another such series absolutely convergent for all finite s to $f^{(j)}(s)$, which is thus another entire function. Following [2] again, we define $M^j(\sigma)$, $\mu^j(\sigma)$, ϱ^j, λ^j, ϱ_*^j, λ_*^j for $f^{(j)}(s)$ exactly as we have defined $M(\sigma)$, $\mu(\sigma)$, ϱ, λ, ϱ_*, λ_* for $f(s)$ with the understanding that we may sometimes write for convenience $M(\sigma) \equiv M^0(\sigma)$, $\mu(\sigma) \equiv \mu^0(\sigma)$, etc.

One conclusion which emerges from [2] is that some results involving either $M^j(\sigma)$ or $\mu^j(\sigma)$, $j \geqslant 0$, such as Lemmas 3, 4 of [2], can be proved without imposing any additional condition on $\{\lambda_n\}$.

In the present note there are some further results involving $M^j(\sigma)$, $j \geqslant 0$ proved without an additional condition on $\{\lambda_n\}$. These results are Theorem I, which gives without such a condition on $\{\lambda_n\}$, part of an earlier result ([2], Theorem 1), and Theorems II, II′, which give likewise two other earlier results ([2], Theorems 3, 3′). Theorem II contains, as its Corollary II, Rahman's main results ([1]; (10), (11) in p. 138), and as its case $\lambda_n = n$, two results proved by Shah by an entirely different method ([3], Theorems A, 1). On the other hand, Theorem II is itself contained in Theorem II'.

2. Lemmas. Of the lemmas given below, the first three are quoted from earlier papers and the remaining three are their extensions wholly or in part.

LEMMA 1 ([1]; (13), (14)). *Given $\eta > 0$, $\eta' > 0$, $s_0 = \sigma_0 + it_0$ and any $\delta = \delta(\sigma) > 0$ (which in our applications is either fixed or tends to 0 as $\sigma \to \infty$), we have*

$$M(\sigma) - \eta < (\sigma - \sigma_0) M^1(\sigma) + |f(s_0)| \quad (\sigma > \sigma_0), \tag{1}$$

$$M^1(\sigma) - \eta' < \frac{1}{\delta} M(\sigma + \delta). \tag{2}$$

LEMMA 2 ([1], p. 139). *$\log M(\sigma)$ is a monotonic increasing downward convex function of σ and hence there exists a monotonic increasing function $w(\sigma)$ such that*

$$\log M(\sigma) = \log M(\sigma_0) + \int_{\sigma_0}^{\sigma} w(x)\,dx \quad (\sigma > \sigma_0), \tag{3}$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log w(\sigma)}{\sigma} = \frac{\varrho}{\lambda}. \tag{4}$$

(4) is not given by Rahman [1], but is readily deducible from (3), just as the analogue of (4) for ϱ_* and λ_* is deducible from the analogue of (3) for $\log \mu(\sigma)$ ([2], Lemma 3).

Remarks. From (3) it is clear that $w(\sigma)$ tends to infinity with σ, since, if $w(\sigma)$ has finite limit, $\log M(\sigma)/\sigma$ will have the same limit, which is a possibility to be excluded as it can occur only when $f(s)$ has a finite number of terms. Save in this excluded case, it follows from (3) that

$$\log M(\sigma) \geqslant \int_{\sigma/2}^{\sigma} w(x)\,dx \geqslant \tfrac{1}{2}\sigma w(\tfrac{1}{2}\sigma),$$

or

$$\frac{\log M(\sigma)}{\sigma} \geqslant \frac{1}{2} w(\tfrac{1}{2}\sigma) \to \infty \quad \text{as} \quad \sigma \to \infty.$$

LEMMA 3 ([2], Lemma 4). *For all sufficiently large σ,*

$$\frac{M^1(\sigma)}{M(\sigma)} \geqslant \frac{\log M(\sigma)}{\sigma}.$$

LEMMA 1'. *Given a positive integer $j (\geqslant 1)$, $\eta' > 0$ and $\delta = \delta(\eta) > 0$, we have*

$$M^j(\sigma) - \eta' < \frac{j!}{\delta^j} M(\sigma + \delta). \tag{2'}$$

Proof. The proof, though exactly like that of (2), is given here for the sake of completeness. We first find $s_j = \sigma + it_j$ such that $|f^{(j)}(s_j)| > M^j(\sigma) - \eta'$. We then use the well-known formula

$$f^{(j)}(s_j) = \frac{j!}{2\pi i}\int_{\Gamma}\frac{f(s)}{(s-s_j)^{j+1}}\,ds, \quad \text{where} \quad \Gamma\colon |s-s_j| = \delta ,$$

to infer that

$$M^j(\sigma) - \eta' < |f^{(j)}(s_j)| \leqslant \frac{j!}{\delta^j} M(\sigma+\delta) .$$

LEMMA 2′. *In Lemma 2 we can evidently replace* $\log M(\sigma)$ *by* $\log M^j(\sigma)$, *where* j *is an integer* ($\geqslant 1$), *and* $w(\sigma)$ *by the monotonic increasing function* $w^j(\sigma)$ *associated with* $\log M^j(\sigma)$ *precisely as* $w(\sigma)$ *is associated with* $\log M(\sigma)$.

LEMMA 3′. *For any integer* $j \geqslant 1$, *and all sufficiently large* σ.

$$\frac{M^j(\sigma)}{M(\sigma)} \geqslant \operatorname{const}\left[\frac{\log M(\sigma)}{\sigma}\right]^j > 0 .$$

Proof. We have only to use induction on j involving the form of Lemma 3 with $M^{j+1}(\sigma)$ and $M^j(\sigma)$ instead of $M^1(\sigma)$ and $M(\sigma)$ as well as the following consequence of that form of Lemma 3:

$$\log M^j(\sigma) > \operatorname{const}\log M(\sigma) .$$

3. Theorems. The part $\varrho = \varrho^1$ of the theorem which follows is stated by Rahman ([1], (9)), apparently with the superfluous restriction $\varrho < \infty$.

THEOREM I. *If* $f(s)$ *is any entire Dirichlet series for which* $0 \leqslant \lambda \leqslant \varrho \leqslant \infty$, *and* $f^1(s)$ *is the differentiated series, then*

$$\varrho = \varrho^1, \quad \lambda = \lambda^1 . \tag{5}$$

Proof. From the remarks following Lemma 2 it is plain that, as $\sigma \to \infty$,

$$\frac{\log\sigma}{\log M^1(\sigma)} < \frac{\sigma}{\log M^1(\sigma)} = o(1) . \tag{6}$$

Using (1) and then (6), we get successively

$$\log M(\sigma) + \log\left[1 - \frac{\eta}{M(\sigma)}\right] < \log M^1(\sigma) + \log\sigma + \log\left[1 + \frac{|f(s_0)|}{M^1(\sigma)\sigma}\right],$$

$$\log M(\sigma) + o(1) < \log M^1(\sigma)[1+o(1)] + o(1) \qquad (\sigma\to\infty) ,$$

$$\log\log M(\sigma) + o(1) \leqslant \log\log M^1(\sigma) + o(1) ;$$

$$\lim_{\sigma\to\infty}\genfrac{}{}{0pt}{}{\sup}{\inf}\frac{\log\log M(\sigma)}{\sigma} \leqslant \lim_{\sigma\to\infty}\genfrac{}{}{0pt}{}{\sup}{\inf}\frac{\log\log M^1(\sigma)}{\sigma} . \tag{7}$$

On the other hand, (2) for a fixed δ gives us in succession:

$$\log M^1(\sigma)+o(1) < \log M(\sigma+\delta)-\log\delta \qquad (\sigma\to\infty)\,,$$

$$\log\log M^1(\sigma)+o(1) < \log\log M(\sigma+\delta)+o(1)\,;$$

$$(8)\qquad \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log M^1(\sigma)}{\sigma} \leqslant \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log M(\sigma+\delta)}{(\sigma+\delta)}\cdot\frac{\sigma+\delta}{\sigma} = \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log M(\sigma)}{\sigma}\,.$$

(7) and (8) together imply (5), the conclusion sought.

THEOREM II. *In Theorem* I *we have, in addition to* (5), *the following conclusion*:

$$(9)\qquad \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log[M^1(\sigma)/M(\sigma)]}{\sigma} = \frac{\varrho}{\lambda}\,.$$

Proof. The argument which follows is modelled on one originally used by Valiron ([4], pp. 15-19) to prove the ϱ-part of (9) in the case $\lambda_n = n$.

Using (2) and then (3), we obtain

$$(10)\qquad \log M^1(\sigma)+\log\left[1-\frac{\eta'}{M^1(\sigma)}\right] < \log M(\sigma+\delta)-\log\delta = \log M(\sigma)+\int_{\sigma}^{\sigma+\delta} w(x)\,dx-\log\delta\,,$$

where δ is to be finally chosen in terms of σ as follows. The last expression for σ fixed in the first instance and for varying δ, is least when δ satisfies the equation

$$(11)\qquad w(\sigma+\delta)-\frac{1}{\delta}=0\,.$$

$w(\sigma)$ being a monotonic increasing function of σ, we get from (11):

$$\frac{1}{\delta}\geqslant w(\sigma)\to\infty \qquad \text{as} \qquad \sigma\to\infty\,,$$

while we get

$$\log M^1(\sigma)+\log\left[1-\frac{\eta'}{M^1(\sigma)}\right] < \log M(\sigma)+\delta w(\sigma+\delta)-\log\delta = \log M(\sigma)+1+\log w(\sigma+\delta)\,,$$

choosing δ at this stage so as to satisfy (11). Hence as $\sigma\to\infty$

$$\log M^1(\sigma)-\log M(\sigma) < O(1)+\log w(\sigma+\delta)\,,$$

$$(12)\qquad \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log[M^1(\sigma)/M(\sigma)]}{\sigma} \leqslant \lim\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log w(\sigma+\delta)}{(\sigma+\delta)}\cdot\frac{\sigma+\delta}{\sigma} = \frac{\varrho}{\lambda}$$

where we use (4) in conjunction with the fact that $\delta \to 0$ as $\sigma \to \infty$. On the other hand, by Lemma 3,

$$\log[M^1(\sigma)/M(\sigma)] \geqslant \log\log M(\sigma) - \log\sigma ,$$

or

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log[M^1(\sigma)/M(\sigma)]}{\sigma} \geqslant \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log M(\sigma)}{\sigma} = \frac{\varrho}{\lambda} . \tag{13}$$

(12) and (13) together give us the desired conclusion (9).

Remark on Theorem II. A conjecture analogous to (9), for ϱ_* and λ_*, assumes the form

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log[\mu^1(\sigma)/\mu(\sigma)]}{\sigma} = \frac{\varrho_*}{\lambda_*},$$

where there is no additional condition on $\{\lambda_n\}$.

Without such a condition, but assuming $\varrho < \infty$ in the case "lim inf", the present author has established the above conjecture in a more general form, which will be published elsewhere.

Corollary II. *When ϱ is finite, we have*

$$\log M^1(\sigma) \sim \log M(\sigma) . \tag{14}$$

And when $0 < \varrho < \infty$, we have

$$\tau = \tau^1, \quad \omega = \omega^1, \tag{15}$$

where τ, ω are respectively the type and the lower type of $f(s)$ in the order ϱ, i.e.

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log M(\sigma)}{e^{\sigma\varrho}} = \frac{\tau}{\omega},$$

and τ^1, ω^1 are the type and the lower type respectively of $f^1(s)$.

Proof. The part $\tau = \tau^1$ of (15) is given by Rahman ([1], p. 138) as a deduction from (14). Here it may be pointed out that (15) as a whole is an obvious deduction from (14), while (14) itself is a deduction from Theorem II, though Rahman has proved it directly ([1], pp. 139-140). To effect the deduction last mentioned, it is enough to note that (9) gives us, for $\sigma > \sigma_0(\varepsilon)$ corresponding to any small $\varepsilon > 0$,

$$\frac{(\lambda - \varepsilon)\sigma}{\log M^1(\sigma)} < 1 - \frac{\log M(\sigma)}{\log M^1(\sigma)} < \frac{(\varrho + \varepsilon)\sigma}{\log M^1(\sigma)} .$$

Then letting $\sigma \to \infty$ and using the fact that $\sigma/\log M^1(\sigma) = o(1)$, which appears in (6), we immediately get (14).

Theorem II'. *If we extend* (9), *we have, for any positive integer $j \geqslant 1$,*

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log[M(\sigma)/M(\sigma)]^{1/j}}{\sigma} = \frac{\varrho}{\lambda} .$$

Proof. The proof is like that of Theorem II, but uses (2′) of Lemma 1′ instead of (2) of Lemma 1 and Lemma 2′ instead of Lemma 2. These changes in the proof of Theorem II result in the replacement of (12) by

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[M^j(\sigma)/M(\sigma)]}{\sigma} \leqslant \begin{matrix} j\varrho \\ j\lambda \end{matrix}. \tag{12′}$$

A further change in the proof of Theorem II, required now, consists in using Lemma 3′ instead of Lemma 3 and obtaining in place of (13):

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[M^j(\sigma)/M(\sigma)]}{\sigma} \geqslant \begin{matrix} j\varrho \\ j\lambda \end{matrix}. \tag{13′}$$

The proof is completed by combining (12′) and (13′).

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RAMANUJAN INSTITUTE OF MATHEMATICS
UNIVERSITY OF MADRAS, INDIA

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