ANNALES
POLONICI MATHEMATICI
XXX (1975)

On the existence of exactly one solution of integral equations in the space L^P with a mixed norm

by BOGDAN RZEPECKI (Poznań)

Abstract. In this paper we deal with the problem of the existence of a unique solution of integral equations in the space L^P with a mixed norm. Applying the Banach fixed-point theorem, we give some sufficient conditions for Fredholm second kind equations, Urysohn and Hammerstein equations and integral equations of a certain general type.

1. Introduction. The purpose of this paper is to transfer the problem of exactly one solution of integral equations of the fundamental type in the space L^2 to the case of space L^P with a mixed norm. We give sufficient conditions for the Fredholm equation of second order, for the Urysohn and Hammerstein equation and for the general integral equation.

For the definition and fundamental properties of the space L^P with mixed norm powers, see [2]. Let $P = (p_1, \ldots, p_n)$, $Q = (q_1, \ldots, q_n)$ be a given n-tuple with $1 \leqslant p_i$, $q_i \leqslant \infty$, $p_i^{-1}+q_i^{-1} = 1$ for $i = 1, \ldots, n$.

Let Ω denote the product of Ω_i, i.e. $\Omega = \Omega_1 \times \ldots \times \Omega_n$, $\operatorname{mes}\Omega$ — the product measure of the set Ω, where Ω_i denotes a measurable subset of the real line with a finite Lebesgue measure.

2. Fredholm equation. In this section we are going to consider the Fredholm operator

$$Au(\cdot) = \int_{\Omega} K(\cdot, y)u(y)dy,$$

under the following assumptions:

ASSUMPTION (A). *Suppose that*

1° *sets* Ω_i $(i = 1, \ldots, n)$ *are compact,*

2° $K\colon \Omega \times \Omega \to (-\infty, \infty)$ *is continuous.*

ASSUMPTION (B). *Suppose that*

1° $K\colon \Omega \times \Omega \to (-\infty, \infty)$ *is measurable,*

2° $\|K(\cdot, y)\|_Q \in L^P(\Omega)$ (*this means that the function* $\Phi(x) = \|K(x, y)\|_Q$ *belongs to the space* L^P).

Let assumption (A) *or* (B) *be satisfied. Then the Fredholm operator is defined in the space* $L^P(\Omega)$.

In the first case there exists a finite integral $\int_\Omega K(x, y)u(y)dy$ for every $x \in \Omega$ because K is bounded, the variable being fixed. When we fix the second variable, the function K is uniformly continuous and Au is continuous. In virtue of the Fubini Theorem, the integral $\int_{\Omega_1} |Au(x)|^{p_1} dx_1$ is defined and as finite almost everywhere on $\Omega_2 \times \ldots \times \Omega_n$, measurable and continuous. Applying the above argumentation to the function $\left(\int_{\Omega_1} |Au(x)|^{p_1} dx_1\right)^{p_2/p_1}$ integrable on $\Omega_2 \times \ldots \times \Omega_n$, we get consequently

$$\int_{\Omega_n} \left(\ldots \left(\int_{\Omega_2} \left(\int_{\Omega_1} |Au(x)|^{p_1} dx_1\right)^{p_2/p_1} dx_2\right)^{p_3/p_2} \ldots\right)^{p_n/p_{n-1}} dx_n < \infty.$$

When we assume (B) for $u \in L^P$, applying the Hölder inequality, we obtain

$$\int_\Omega |K(x, y)u(y)|dy \leqslant \|u\|_P \|K(x, y)\|_Q$$

and

$$\int_{\Omega \times \Omega} |K(x, y)u(y)|(dx, dy) = \int_\Omega \left(\int_\Omega |K(x, y)u(y)|dy\right) dx$$
$$\leqslant \|u\|_P \cdot \int_\Omega \|K(x, y)\|_Q dx \leqslant \|u\|_P \cdot \|1\|_Q \cdot \big\| \|K(x, y)\|_Q \big\|_P < \infty.$$

Hence we deduce that Au is defined almost everywhere on Ω, measurable and finite almost everywhere. Let us suppose the existence of $\|Au\|_P$. Then

$$\|Au\|_P \leqslant \|u\|_P \cdot \big\| \|K(x, y)\|_Q \big\|_P < \infty.$$

In order to prove the existence of $\|Au\|_P$ we consider $(n-1)$ functions

$$\Phi_i(x_{n-i+1}, x_{n-i+2}, \ldots, x_n)$$
$$= \int_{\Omega_{n-i}} \left(\int_{\Omega_{n-i-1}} \left(\ldots \left(\int_{\Omega_1} \|K(x, y)\|_Q^{p_1} dx_1\right)^{p_2/p_1} \ldots\right)^{p_{n-i-1}/p_{n-i-2}} \times \right.$$
$$\left. \times\, dx_{n-i-1}\right)^{p_{n-i}/p_{n-i-1}} dx_{n-i}$$

where $i = 1, \ldots, n-1$. Applying the Fubini Theorem (see [3]) we deduce that the function Φ_i is finite almost everywhere on $\Omega_{n-i+1} \times \ldots \times \Omega_n$. Let us notice that if f is a measurable function defined on the measurable space X, having finite real values, then for every $a > 0$ the function $|f|^a$ is measurable on X. Applying this remark to the function $|Au|^{p_1}$, we deduce that the integral $\int_{\Omega}|Au(x)|^{p_1}dx$ exists. Hence, according to the Fubini Theorem, the function

$$f(x_2, \ldots, x_n) = \int_{\Omega_1} |Au(x)|^{p_1}dx_1$$

is defined and measurable almost everywhere on $\Omega_2 \times \ldots \times \Omega_n$. We can apply our remark to the function f^{p_2/p_1} because

$$\int_{\Omega_1} |Au(x)|^{p_1}dx_1 \leqslant \|u\|_P^{p_1} \cdot \int_{\Omega_1} \|K(x, y)\|_Q^{p_1} dx_1 < \infty.$$

Repeating this argumentation $n-1$ times, we finally infer that $\|Au\|_P$ exists.

We shall now give Theorems on the existence of exactly one solution for the Fredholm equation of the second kind

$$\text{(I)} \qquad u(x) = \lambda \cdot \int_{\Omega} K(x, y) u(y) dy + f(x).$$

The following result is well known: If assumption (A) is satisfied and $f \in C(\Omega)$, then for λ such that

$$|\lambda| < (M \cdot \text{mes}\, \Omega)^{-1} \qquad (M = \max\{|K(x, y)|: x, y \in \Omega\})$$

there exists exactly one continuous solution of equation (I).

Considering the metric of the space L^P, we can obtain a solution of our equation for λ belonging to a larger interval of values than in the case of the space C. Let us prove

THEOREM 1. *Let assumption* (A) *be satisfied and* $f \in C(\Omega)$. *Then for*

$$|\lambda| < \big(\big\| \|K(x, y)\|_Q \big\|_P\big)^{-1}$$

there exists exactly one continuous solution of equation (I).

Proof. We deduce in a standard way that the assumptions imply that:

(a) the function $\varphi(x) = \|K(x, y)\|_Q$ is defined almost everywhere on Ω,

(b) the function

$$\Psi_i(x_{i+1}, x_{i+2}, \ldots, x_n)$$
$$= \int_{\Omega_i} \Big(\int_{\Omega_{i-1}} \Big(\ldots \Big(\int_{\Omega_1} \|K(x, y)\|_Q^{p_1} dx_1 \Big)^{p_2/p_1} \ldots \Big)^{p_{i-1}/p_{i-2}} dx_{i-1} \Big)^{p_i/p_{i-1}} dx_i,$$

is defined almost everywhere on $\Omega_{i+1} \times \ldots \times \Omega_n$, $i = 1, 2, \ldots, n-1$,

(c) $\big\| \|K(x, y)\|_Q \big\|_P$ exists and has a finite value,

(d) $\big\| \|K(x, y)\|_Q \big\|_P < M \cdot \text{mes}\, \Omega$, where $M = \max\{|K(x, y)|: x, y \in \Omega\}$.

Let $u \in L^P$ and let us consider the operator defined by the right-hand side of equation (I) with values in $C(\Omega)$. Since the assumptions ensure that our operator acts in L^P, we have proved that among the functions of L^P, only continuous functions may be solutions of equation (I). Now, let us take $u_1, u_2 \in L^P$ and consider the integral $\int_\Omega K(x, y)[u_1(y) - u_2(y)]dy$. Applying (c) and the Hölder inequality, we get

$$\Big\| \int_\Omega K(x, y)[u_1(y) - u_2(y)]dy \Big\|_P \leqslant \|u_1 - u_2\|_P \cdot \big\| \|K(x, y)\|_Q \big\|_P .$$

This means that the operator under consideration is a contraction for sufficiently small λ. The application of the Banach Principle completes the proof.

Remark 1. For every λ satisfying the inequality

$$(M \operatorname{mes} \Omega)^{-1} \leqslant |\lambda| < \big(\big\| \|K(x, y)\|_Q \big\|_P\big)^{-1}$$

the sequence of successive iterations converges in the space L^P to a continuous solution.

Remark 2. If $f \in L^P$, then equation (I) has a solution in L^P for the same interval of values of the parameter λ. The uniqueness of the solution now means uniqueness up to a set of measure zero.

Applying assumption (B), we obtain

THEOREM 2. *If assumption* (B) *is satisfied and* $f \in L^P(\Omega)$, *then for*

$$|\lambda| < \big(\big\| \|K(x, y)\|_Q \big\|_P\big)^{-1},$$

there exists exactly one solution of equation (I) *in the space* L^P.

3. Urysohn equation. In this part we shall set a theorem for the Urysohn equation

$$\text{(II)} \qquad u(x) = \int_\Omega K[x, y, u(y)]dy .$$

We introduce

ASSUMPTION (C). *Suppose that*

1° *the functions*

$$K\colon \Omega \times \Omega \times (-\infty, \infty) \to (-\infty, \infty), \qquad R\colon \Omega \times \Omega \times (-\infty, \infty) \to (-\infty, \infty)$$

satisfy the Carathéodory condition, i.e. they are functions measurable with respect to variables $(x, y) \in \Omega \times \Omega$ *for every* $u \in (-\infty, \infty)$, *and continuous with respect to the variable* $u \in (-\infty, \infty)$, *for almost all* $(x, y) \in \Omega \times \Omega$,

2° $|K(x, y, z)| \leqslant R(x, y, z)$ *for every* $x, y \in \Omega$, $z \in (-\infty, \infty)$,

3° *the integral operator*

$$Bu(\cdot) = \int_\Omega R[\cdot, y, u(y)]dy$$

acts in $L^P(\Omega)$.

Under assumption (C) *the Urysohn operator*

$$Au(\cdot) = \int_{\Omega} K[\cdot, y, u(y)]dy$$

acts in L^P.

Proof. Let $u \in L^P$. In virtue of the above assumptions, applying the Fubini Theorem and the Hölder inequality, we deduce that Au is defined almost everywhere and measurable on Ω, and has a finite value almost everywhere. We prove the existence of $\|Au\|_P$ applying assumptions 3° and 2°. This is proved in the same way as in the case of the Fredholm operator.

Now we shall apply the above result to the solution of equation (II) under the following assumption:

Assumption (D). *Suppose that*

1° *the function* $K: \Omega \times \Omega \times (-\infty, \infty) \to (-\infty, \infty)$ *satisfies*

(a) *Carathéodory condition,*

(b) *Lipschitz condition with respect to the last variable: there exists a constant* $C > 0$ *such that*

$$|K(x, y, z_1) - K(x, y, z_2)| \leqslant C|z_1 - z_2|$$

for every $x, y \in \Omega$ *and* $z_1, z_2 \in (-\infty, \infty)$,

2° $\int_{\Omega} |K(\cdot, y, 0)| dy \in L^P(\Omega)$.

Since

$$|K(x, y, z)| \leqslant C|z| + |K(x, y, 0)|,$$

it is convenient to take the operator B with the kernel

$$R(x, y, z) = C|z| + |K(x, y, 0)|.$$

Let $u \in L^P$; then $Bu \in L^P$. If we consider the operator defined by the right-hand side of equation (II), then assumption (C) is satisfied and this operator acts in L^P. For $u_1, u_2 \in L^P$

$$\left| \int_{\Omega} \{K[x, y, u_1(y)] - K[x, y, u_2(y)]\} dy \right| \leqslant C \cdot \int_{\Omega} |u_1(y) - u_2(y)| dy \leqslant C \|u_1 - u_2\|_P \cdot \|1\|_Q.$$

Hence

$$\left\| \int_{\Omega} \{K[x, y, u_1(y)] - K[x, y, u_2(y)]\} dy \right\|_P \leqslant C \cdot \operatorname{mes} \Omega \cdot \|u_1 - u_2\|_P.$$

Thus the operator considered is a contraction if

$$|\lambda| C \cdot \operatorname{mes} \Omega < 1.$$

We obtain the following

THEOREM 3. *Let assumption* (D) *be satisfied. Then for*

$$|\lambda| < (C \cdot \operatorname{mes} \Omega)^{-1},$$

there exists exactly one solution of equation (II) *in the space* $L^P(\Omega)$.

Remark 3. The existence of exactly one solution in L^2 of equation (II) was given in [5].

4. General integral equation. Let A denote an operator acting in L^P and let us take the non-linear operator of superposition

$$\mathfrak{f}u(\cdot) = f[\cdot, u(\cdot), Au(\cdot)].$$

ASSUMPTION (E). *Suppose that*

1° *the function* $f\colon \Omega \times (-\infty, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$ *satisfies the Carathéodory condition: it is measurable with respect to the variable* $x \in \Omega$ *at every fixed* $(y, z) \in (-\infty, \infty) \times (-\infty, \infty)$, *and continuous with respect to variables* $(y, z) \in (-\infty, \infty) \times (-\infty, \infty)$ *at almost every fixed* $x \in \Omega$,

2° $|f(x, y, z)| \leqslant M|y| + N|z| + a(x)$ *for any* $x \in \Omega$, $y, z \in (-\infty, \infty)$, *where* M, N *are non-negative constants,* $a \in L^P(\Omega)$.

Arguing as above and applying the Minkowski inequality, we infer

Under assumption (E) *the superposition operator* $\mathfrak{f}$ *acts in* L^P.

Setting for A an integral operator, we may generalize the results of Sections 2 and 3 to integral equations generated by a superposition operator. As an example we give the generalization obtained in the case of the Urysohn operator.

Let us consider the integral equation

$$u(x) = f\Big[x, u(x), \lambda \cdot \int_{\Omega} K\big(x, y, u(y)\big)dy\Big] \tag{III}$$

under the following assumptions:

ASSUMPTION (F). *Suppose that*

1° *the function* $f\colon \Omega \times (-\infty, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$ *satisfies:*

(a) *Carathéodory condition,*

(b) *Lipschitz condition with respect to the last two variables: there exist positive constants* M *and* N *such that*

$$|f(x, y_1, z_1) - f(x, y_2, z_2)| \leqslant M|y_1 - y_2| + N|z_1 - z_2|$$

for every $x \in \Omega$, $y_1, y_2, z_1, z_2 \in (-\infty, \infty)$,

2° $f(\cdot, 0, 0) \in L^P(\Omega)$,

3° *the function* $K\colon \Omega \times \Omega \times (-\infty, \infty) \to (-\infty, \infty)$ *satisfies assumption* (D).

THEOREM 4. *Let assumption* (F) *be satisfied. Then for*

$$|\lambda| < \frac{1-M}{NC\cdot \mathrm{mes}\,\Omega},$$

there exists in $L^P(\Omega)$ exactly one solution of equation (III).

Proof. Let A denote the Urysohn operator with kernel K. Let us consider the operator defined by the right-hand side of equation (III). Sinice

$$|f(x, y, z)| \leqslant M|y| + N|z| + |f(x, 0, 0)|,$$

assumption (E) is satisfied and our operator acts in L^P. We have for $u_1, u_2 \in L^P$

$$\Big|f\Big[x, u_1(x), \lambda\int_\Omega K\big(x, y, u_1(y)\big)dy\Big] - f\Big[x, u_2(x), \lambda\int_\Omega K\big(x, y, u_2(y)\big)dy\Big]\Big|$$
$$\leqslant M|u_1(x)-u_2(x)| + NC|\lambda|\cdot\|1\|_Q\cdot\|u_1-u_2\|_P.$$

Hence

$$\Big\|f\Big[x, u_1(x), \lambda\int_\Omega K\big(x, y, u_1(y)\big)dy\Big] - f\Big[x, u_2(x), \lambda\int_\Omega K\big(x, y, u_2(y)\big)dy\Big]\Big\|_P$$
$$\leqslant M\|u_1-u_2\|_P + |\lambda|NC\|1\|_Q\|u_1-u_2\|_P\cdot\|1\|_P = [M + NC|\lambda|\,\mathrm{mes}\,\Omega]\cdot\|u_1-u_2\|_P.$$

Since $M + NC|\lambda|\cdot\mathrm{mes}\,\Omega < 1$, the application of the Banach Principle completes the proof.

Remark 4. Sufficient conditions for the equation

$$\text{(IV)} \qquad u(x) = f\Big[x, \lambda\int_\Omega K\big(x, y, u(y)\big)dy\Big]$$

to have a continuous solution are given in [7]. These results were generalized to equation (III) in [1]. We introduce

ASSUMPTION (G). *Suppose that*

1° *the function f: $\Omega\times(-\infty, \infty)\to(-\infty, \infty)$ satisfies*

(a) *Carathéodory condition: it is measurable with respect of the variable $x\in\Omega$ for all $y\in(-\infty, \infty)$, continuous with respect to the variable $y\in(-\infty, \infty)$, at almost all $x\in\Omega$,*

(b) *Lipschitz condition with respect to the last variable: there exists a constant $L>0$ such that*

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2|$$

for every $x\in\Omega$, $y_1, y_2\in(-\infty, \infty)$,

2° $f(\cdot, 0)\in L^P(\Omega)$,

3° *the function K: $\Omega\times\Omega\times(-\infty, \infty)\to(-\infty, \infty)$ satisfies assumption* (D).

Then we get:

Under the above assumptions, there exists exactly one function in the class $L^P(\Omega)$ satisfying equation (IV) *for λ satisfying the inequality*

$$|\lambda| < (LC\cdot \operatorname{mes}\Omega)^{-1}.$$

Remark 5. Since (see e.g. [4]) every Hammerstein operator can be written in the form

$$A = K\mathfrak{f},$$

where K is the Fredholm operator generated by the kermel K and $\mathfrak{f}$ is the superposition operator

$$\mathfrak{f}u(\cdot) = f[\cdot, u(\cdot)],$$

the investigation of these operators can be reduced to the investigation of a linear operator K and a non-linear operator $\mathfrak{f}$. Since A acts in L^P if K and $\mathfrak{f}$ act in this space, we get

THEOREM 5. *Let assumption* (B) *and conditions* 1° *and* 2° *from assumntion* (G) *be satisfied. Then, for $|\lambda|$ sufficiently small, the Hammersteip equation*

$$u(x) = \lambda \int_{\Omega} K(x, y) f[y, u(y)]dy \tag{V}$$

has exactly one solution in $L^P(\Omega)$.

It is easy to verify that the values of λ for which the above theorem holds should satisfy the inequality

$$|\lambda| < L(\| \|K(x, y)\|_Q\|_P)^{-1}.$$

The existence of exactly one solution in L^2 for equation (V) is given in [6].

References

[1] J. Albrycht, *O pewnych układach nieliniowych równań całkowitych i całkowo-różniczkowych,* Zeszyty Naukowe UAM, Matematyka I (1957), p. 19–23.

[2] A. Benedek and R. Pazone, *The spaces L^P with mixed norm,* Duke Math. J. 28 (1961), p. 301–324.

[3] N. Dunford and J. T. Schwarz, *Linear operatores I,* New York 1958.

[4] М. А. Красносельский, *Топологические методы в теории нелинейных интегральных уравнений,* Москва 1956.

[5] В. В. Немыцкий, *Об одном классе нелинейных интегральных уравнений,* Мат. Сб. 41, № 4 (1934), p. 655–658.

[6] — *Теоремы существования и единственности для нелинейных интегральных уравнений,* ibidem 41, № 3 (1934), p. 438–452.

[7] О. Женхэн, *О существовани и единственности решений интегро-дифференцальных уравнений,* ДАН 86 (1952), p. 229–230.

Reçu par la Rédaction le 20. 5. 1972