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# On generalized periodic solutions of linear differential equations of order $n$

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**Abstract.** In this note the author proves the following theorems:

THEOREM 1. *Let $V^{(n-1)}$ be the set of all distributions whose derivatives (in the distributional sense) are functions of locally bounded variation in $R^1$. Moreover, let $p_i$ be periodic measures with the period $\omega$ $(\omega > 0)$ for $i = 1, 2, \ldots, n+1$ such that*

1° *$p_1$ is a locally integrable function in $R^1$,*

2° *$p_r \in V^{(n-r-1)}$ for $r = 1, 2, \ldots, n-1$ $(n \geqslant 2)$,*

3° $\sum_{i=1}^{n} (-1)^{n-i} p_i^{(n-i)} \geqslant 0$,

4° $\int_0^\omega p_n(t)\,dt > 0$, $p_n \geqslant 0$,

5° *for a fixed $\varepsilon > 0$*

$$\max_{1 \leqslant i \leqslant n} \int_0^{\omega+\varepsilon} |p_i(t)|\,dt < \Big(\sum_{i=0}^{n-1} (\omega+\varepsilon)^{n-i-1}\Big)^{-1}.$$

*Then there exists exactly one periodic solution with the period $\omega$ of the equation*

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x + p_{n+1}(t) = 0.$$

THEOREM 2. *If $p$ and $q$ are periodic measures with the period $\omega$ $(\omega > 0)$ such that*

1° $p \geqslant 0$, $\int_0^\omega p(t)\,dt > 0$,

2° *for a fixed $\varepsilon > 0$* $\int_0^{\omega+\varepsilon} p(t)\,dt < (\omega+\varepsilon)^{-n+1}$,

3° *$p$ is a locally integrable function in $R^1$ if $n = 1$, then there exists exactly one periodic solution with the period $\omega$ $(\omega > 0)$ of the equation*

$$x^{(n)} + p(t)x + q(t) = 0.$$

THEOREM 3. *If $p$ and $q$ are periodic measures with the period $\omega$ $(\omega > 0)$ such that*

$$p \neq 0, \quad \int_0^\omega p(t)\,dt \geqslant 0, \quad \int_0^\omega |p(t)|\,dt < 16/\omega,$$

*then there exists exactly one periodic solution with the period $\omega$ of the equation*

$$x'' + p(t)x + q(t) = 0.$$

The above theorems generalize some results for linear differential equations (see [5], [6]). All the solutions of the equations are considered in the class $V^{(n-1)}$ ($n$ — order of equation). The principal results of this note are based on the sequential theory of distributions (see [4]).

**1.** In this note we consider the equation

$$(*)\qquad x^{(n)}+p_1(t)x^{(n-1)}+\ldots+p_n(t)x+p_{n+1}(t)=0,$$

where $p_i$ for $i=1,2,\ldots,n+1$ are given periodic measures and $p_1$ is a locally integrable function. The derivative is understood in the distributional sense. By a solution of equation (*) we understand every distribution $(n-1)$ whose derivative (in the distributional sense) is a function of locally bounded variation in $R^1$. This class will be denoted by $V^{(n-1)}$. We prove some theorems on the existence and the uniqueness of periodic solutions of equation (*). Our results generalize some theorems for linear differential equations (see [5], [6]). The sequential theory of the distributions will be used (see [4]).

**2.** Our aim is now to show the principal results. We first introduce some notations.

A sequence of smooth, non-negative functions $\{\delta_k\}$ satisfying: $\int_{-\infty}^{\infty}\delta_k(t)dt=1$, $\delta_k(t)=\delta_k(-t)$, $\delta_k(t)=0$ for $|t|\geqslant a_k$, where $\{a_k\}$ is a sequence of positive numbers with $a_k\to 0$ is called a *δ-sequence* (see [3], [4], p. 75). By a regular sequence for a distribution $u$ we mean any sequence of the form $\varphi_k(t)=(u*\delta_k)(t)=\int_{-\infty}^{\infty}u(t-s)\delta_k(s)ds$, where $\{\delta_k\}$ is a $\delta$-sequence (see [4], p. 117, 153). If, for every regular sequence $\{\varphi_k\}$ of a distribution $u$, the sequence $\{\int_a^b\varphi_k(t)dt\}$ is convergent to some finite limit as $k\to\infty$, then the limit $\lim_{k\to\infty}\int_a^b\varphi_k(t)dt$ is called the *definite integral of* $u$ and is denoted by $\int_a^b u(t)dt$ (see [3], [4]). We say that the product of distributions $u$ and $v$ *exists* if the sequence $\{(u*\delta_k)(v*\delta_k)\}$ is distributionally convergent for every $\delta$-sequence $\{\delta_k\}$ (see [4], p. 242). If, for every regular sequence $\{\varphi_k\}$ for a distribution $u$, the sequence $\{|\varphi_k|\}$ is distributionally convergent, then we say that the modulus $|u|$ of $u$ exists and we put $|u|=\lim_{k\to\infty}(d)|\varphi_k|$.

The consistency of the last definitions follows from the fact that the interlaced sequence of two $\delta$-sequences is also a $\delta$-sequence.

By a non-negative distribution we understand a distribution for which there exists a fundamental sequence whose terms are non-negative functions. A distribution $u$ is a *measure* if there exists a function of locally bounded variation $\Phi$ such that $\Phi'=u$ (see [2]). We say that a distribution $p$ is *periodic* with the period $\omega$ ($\omega \neq 0$) if $p(t) = p(t+\omega)$ (see [4], p. 49).

Analogously to [5] (p. 72) we can prove (see [7]) the following

THEOREM 2.1. *If $p_i$ are periodic measures with the period $\omega$ for $i = 1, 2, \ldots, n+1$, $p_1$ is a locally integrable function in $R^1$ and zero-distribution is the only periodic solution with the period $\omega$ of the equation*

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x = 0, \tag{2.1}$$

*then there exists exactly one periodic solution with the period $\omega$ of equation* (*).

In the sequel we shall give some conditions which guarantee that the trivial solution is the only periodic solution of equation (2.1).

Let $\varepsilon$ and $\omega$ be positive numbers. We put

$$a(\omega, \varepsilon) = \Big[\sum_{i=0}^{n-1} (\omega + 3\varepsilon)^{n-i-1}\Big]^{-1}. \tag{2.2}$$

THEOREM 2.2. *Let $p_i$ ($i = 1, 2, \ldots, n$) be periodic measures with the period $\omega$ ($\omega > 0$) such that*

(2.3) $p_1$ *is a locally integrable function in* $R^1$,

(2.4) $p_r \in V^{(n-r-1)}$ *for* $r = 1, 2, \ldots, n-1$ ($n \geqslant 2$),

$$\sum_{i=1}^{n} (-1)^{n-i} p_i^{(n-i)} \geqslant 0, \tag{2.5}$$

$$\int_0^{\omega} p_n(t)\,dt > 0, \quad p_n \geqslant 0, \tag{2.6}$$

*for a fixed* $\varepsilon > 0$

$$\max_{1 \leqslant i \leqslant n} \int_0^{\omega+3\varepsilon} |p_i(t)|\,dt < a(\omega, \varepsilon). \tag{2.7}$$

*Then $x = 0$ is the unique $\omega$-periodic solution of equation* (2.1).

From Theorems 2.1 and 2.2 we infer that if $p_{n+1}$ is a periodic measure with the period $\omega$ and $p_i$ ($i = 1, 2, \ldots, n$) satisfy the assumptions of the least theorem, then equation (*) has exactly one periodic solution with the period $\omega$.

A real function of a single real variable is of class $C^n$ in $R^1$ if all its derivatives of order $\leqslant n$ exist and are continuous functions in $R^1$. Theorem 2.2 guarantees the existence and the uniqueness of periodic solutions in the class $C^n$ of the equation

$$c_1(t)x^{(n)} + \big(c_1'(t) + c_2(t)\big)x^{(n-1)} + \ldots + \big(c_n'(t) + c_{n+1}(t)\big)x = c_{n+2}(t) \tag{2.8}$$

in some cases to which the theorems of A. Lasota and Z. Opial (see [5], p. 81–85) cannot be applied.

EXAMPLE. Let $p$ be a periodic function defined as follows:

$$p(t) = \begin{cases} 0 & \text{for } n-1+\frac{1}{30} \leqslant t < n-\frac{1}{30}, \\ 900(t-n+\frac{1}{30}) & \text{for } n-\frac{1}{30} < t \leqslant n, \\ -900(t-n-\frac{1}{30}) & \text{for } n < t \leqslant n+\frac{1}{30}, \\ 0 & \text{for } n+\frac{1}{30} < t \leqslant n+1-\frac{1}{30}, \end{cases}$$

where $n$ is an arbitrary integer number. We take the following equation

$$x''+\tfrac{1}{4}x'+\tfrac{1}{3}p(t)x = 0. \tag{2.9}$$

Theorem 2.2 implies that $x = 0$ is the unique periodic solution with the period 1 of equation (2.9). However, Theorem 4 of [5] cannot be applied to (2.9) because $\lambda_1 c_1+\lambda_2 c_2 > 1$, where $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{8}$, $c_1 = \frac{1}{4}$, $c_2 = 10$.

THEOREM 2.3. *Let $p$ be a periodic measure with the period $\omega$ ($\omega > 0$) such that*

$$p \geqslant 0, \quad \int_0^{\omega} p(t)dt > 0, \tag{2.10}$$

*for a fixed $\varepsilon > 0$*

$$\int_0^{\omega+3\varepsilon} p(t)dt < (\omega+3\varepsilon)^{-n+1}, \tag{2.11}$$

*$p$ is a locally integrable function in $R^1$ if $n = 1$.*

*Then $x = 0$ is the unique periodic solution with the period $\omega$ of the equation*

$$x^{(n)}+p(t)x = 0. \tag{2.12}$$

By Theorem 2.1 we infer that if $p$ and $q$ are periodic measure with the period $\omega$ and $p$ satisfies the assumptions of the last theorem, then the equation

$$x^{(n)}+p(t)x+q(t) = 0 \tag{2.13}$$

has exactly one periodic solution with the period $\omega$.

If $p$ and $q$ are periodic locally integrable functions with the period $\omega$, then the results of A. Lasota and Z. Opial (see [5]) guarantee the existence and uniqueness of periodic solutions of equation (2.13) in some cases to which Theorem 2.3 cannot be applied.

THEOREM 2.4. *If $p$ is a periodic measure with the period $\omega$ $(\omega > 0)$ such that*

$$(2.14)\qquad \int_0^\omega |p(t)|\,dt < 16/\omega, \quad \int_0^\omega p(t)\,dt \geqslant 0, \quad p \neq 0,$$

*then $x = 0$ is the unique periodic solution with the period $\omega$ of the equation*

$$(2.15)\qquad x'' + p(t)x = 0.$$

Let $p$ and $q$ be periodic measures with the period $\omega$. Moreover, let $p$ satisfy the assumptions of Theorem 2.4. Then from Theorem 2.1 and 2.4 it follows that the equation

$$(2.16)\qquad x'' + p(t)x + q(t) = 0$$

has exactly one periodic solution with the period $\omega$.

If $p$ and $q$ are periodic locally integrable functions with the period $\omega$ and $\int_0^\omega |p(t)|\,dt \leqslant 16/\omega$, $\int_0^\omega p(t)\,dt \geqslant 0$, $p \neq 0$, then there exists exactly one periodic solution with the period $\omega$ of equation (2.16) (see [5]). In note [6] there are considered non-trivial periodic and sign-changing solutions of equation (2.15).

**3.** Before giving the proofs of Theorems 2.2, 2.3 and 2.4 we shall formulate some properties of a definite integral and introduce the notation of a smooth integral.

A distribution $u$ takes the mean value $a$ at a point $t_0$ if and only if $\lim_{k\to\infty} \varphi_k(t_0) = a$ for each regular sequence $\{\varphi_k\}$ of $u$. Then we put $u(t_0) = a$ (see [3]).

If $u$ is a measure, then for all $a$ and $b$ $(a, b \in R^1)$ the integral $\int_a^b u(t)\,dt$ exists and

$$(3.1)\qquad \left|\int_a^b u(t)\,dt\right| \leqslant \int_a^b |u(t)|\,dt \quad (a \leqslant b),$$

$$(3.2)\qquad U(t) = \int_a^t u(s)\,ds + U(a),$$

where $U' = u$ (see [3]).

We write $u \geqslant v$ if and only if the difference $u - v$ is a non-negative distribution (see [2]). Let $u$ and $v$ be measures and $u \geqslant v$; then

$$(3.3)\qquad \int_a^b v(t)\,dt \leqslant \int_a^b u(t)\,dt \quad (a \leqslant b).$$

Using (3.2), we obtain

$$\int_a^b u(t)q(t)\,dt = U(b)q(b) - U(a)q(a) - \int_a^b U(t)q(t)\,dt, \tag{3.4}$$

where $u$ is a measure, $U' = u$ and $q$ is an absolutely continuous function.

A periodic distribution $p$ is called *hereditarily periodic* if and only if there is a periodic distribution $q$ such that $q' = p$ (see [9]). One can show (see [9]) that for every hereditarily periodic distribution $p$ there exists a unique hereditarily periodic distribution $q$ such that $q' = p$ (see [9]).

Let $\varepsilon$ be a positive number and let the carrier of a smooth, non-negative function $\varphi$ equals $[\varepsilon, 2\varepsilon]$. Moreover, let $\int_{-\infty}^{\infty} \varphi(t)\,dt = 1/\omega$ $(\omega > 0)$ and let $\Pi$ be the characteristic function of the interval $[0, \omega]$. We define

$$F_k(t) = \int_0^{\omega+3\varepsilon} \lambda(r-c)\,dr \int_r^t f_k(s)\,ds \overset{\mathrm{df}}{=} \int_{c_\lambda}^t f_k(s)\,ds, \tag{3.5}$$

where $\lambda = \Pi * \varphi$, $c \in (0, \varepsilon)$ and $\{f_k\}$ is a fundamental sequence of $f$. If $c$ and $\lambda$ are fixed, then integral (3.5) is a primitive function of $f_k$. A primitive function which is of the form (3.5) will be called a *smooth integral of* $f_k$. One can show that the sequence $\{F_k\}$ defined by (3.5) is also fundamental (see [9]). Hence we can define

$$\int_{c_\lambda}^t f(s)\,ds \overset{\mathrm{df}}{=} \lim_{k\to\infty}(d) \int_{c_\lambda}^t f_k(s)\,ds. \tag{3.6}$$

The smooth integral of order $n$ we define by induction, letting

$$\int_{c_\lambda}^t f(s)\,ds^0 = f, \qquad \int_{c_\lambda}^t f(s)\,ds^n = \int_{c_\lambda}^t \Big(\int_{c_\lambda}^s f(r)\,dr^{n-1}\Big)ds. \tag{3.7}$$

One may prove (see [9]) that, for every hereditarily periodic distribution $f$, $\int_{c_\lambda}^t f(s)\,ds^n$ is also a hereditarily periodic distribution which does not depend on the choice of $c$ and $\lambda$.

Proof of Theorem 2.2. Let $x$ be a non-zero periodic solution with the period $\omega$ of equation (2.1). Then we consider two cases:

1° there exists a $t_0 \in [0, \omega]$ such that $x(t_0) = 0$,

2° $x(t) > 0$ $(x(t) < 0)$ for all $t \in R^1$.

If $n = 1$, then Theorem 2.2 is obvious. We assume that $n > 1$. Then $x', x'', \ldots, x^{(n-1)}$ are hereditarily periodic distributions. Hence we have

$$x(t) = -\int_{t_0}^t \Big[\int_{c_\lambda}^\tau \Big(\sum_{i=1}^n p_i(s)x^{(n-i)}(s)\Big)\,ds^{n-1}\Big]d\tau. \tag{3.8}$$

We put

$$(3.9)\qquad \varphi_{ir} = p_i * \delta_r, \quad \bar{\varphi}_{ir} = |p_i| * \delta_r, \quad \psi_{ir} = x^{(n-i)} * \delta_r,$$

$$M_i = \sup_{t\in[0,\omega]} |x^{(n-i)}|(t), \quad M = \sum_{i=1}^{n} M_i, \quad \Phi_{ik} = (p_i x^{(n-i)}) * \delta_k,$$

where $\{\delta_r\}$ is a $\delta$-sequence. We take the sequence $\{F_k^{(v)}\}$ defined as follows:

$$(3.10)\qquad F_k^{(v)}(t) = -\Big[\int_{c_\lambda}^{t} \Big(\sum_{i=1}^{n} \Phi_{ik}(s)\Big) ds^{n-1}\Big]^{(v)}, \quad v = 0, 1, \ldots, n-2.$$

By [9] we infer that $F_k^{(v)}$ is a hereditarily periodic distribution and the sequence $\{F_k^{(v)}\}$ is convergent to the function $x^{(v+1)}$. Since

$$(3.11)\qquad |\Phi_{ik}| = |\lim_{r\to\infty} (\varphi_{ir}\psi_{ir}) * \delta_k| \leqslant M_i \bar{\varphi}_{ik},$$

by (3.10) we have

$$(3.12)\qquad |F_k^{(n-2)}(t)| \leqslant M \max_{1\leqslant i\leqslant n} \int_0^{\omega+3\varepsilon} \bar{\varphi}_{ik}(t)\, dt.$$

Hence, we get

$$(3.13)\qquad \sup_{t\in[0,\omega]} |F^{(n-2)}|(t) \leqslant M \max_{1\leqslant i\leqslant n} \int_0^{\omega+3\varepsilon} |p_i(t)|\, dt.$$

Similarly

$$(3.14)\qquad \sup_{t\in[0,\omega]} |F^{(v)}|(t) \leqslant M(\omega+3\varepsilon)^{n-v-2} \max_{1\leqslant i\leqslant n} \int_0^{\omega+3\varepsilon} |p_i(t)|\, dt.$$

Using (3.8), (3.13) and (3.14), we obtain

$$(3.15)\qquad M \leqslant M\big(a(\omega, \varepsilon)\big)^{-1} \max_{1\leqslant i\leqslant n} \int_0^{\omega+3\varepsilon} |p_i(t)|\, dt.$$

Having integrated by parts the product $p_i x^{(n-i)}$, we obtain in case 2° the inequality

$$(3.16)\qquad 0 = \int_0^{\omega} x^{(n)}(t)\, dt = \Big|\int_0^{\omega} \Big(\sum_{i=1}^{n} (-1)^{n-i} p_i^{(n-i)}(t) x(t)\Big) dt\Big|$$

$$\geqslant \min_{t\in[0,\omega]} |x(t)| \int_0^{\omega} p_n(t)\, dt > 0,$$

which is of course impossible. Thus the proof is complete.

Proof of Theorem 2.3. Suppose that there exists a non-zero periodic solution $x$ with the period $\omega$ of equation (2.12). If there exists a $t_0$ such that $t_0 \in [0, \omega]$ and $x(t_0) = 0$, then by (3.8) we can write

$$\max_{t\in[0,\omega]}|x(t)| \leqslant \max_{t\in[0,\omega]}|x(t)|(\omega+3\varepsilon)^{n-1}\int_0^{\omega+3\varepsilon} p(t)\,dt, \tag{3.17}$$

which contradicts (2.11). In the second case Theorem 2.3 is obvious. Thus our assertion follows.

Proof of Theorem 2.4. Let $x$ be a non-zero periodic solution of (2.15) with the period $\omega$. Then we consider three cases:

I. $x(t) > 0$ $(x(t) < 0)$ for all $t \in R^1$,

II. there exist $t_1, t_2, t_3$ such that $0 \leqslant t_1 < t_2 < t_3 \leqslant \omega$,

$$x(t_1) = x(t_2) = x(t_3) = 0,$$

III. there exist only two point $t_1, t_2 \in [0, \omega]$ such that

$$0 \leqslant t_1 < t_2 \leqslant \omega, \quad x(t_1) = x(t_2) = 0.$$

By (2.15) and [1] we get in the case of I

$$x''\frac{1}{x} + p(t) = 0. \tag{3.18}$$

Since $x(0) = x(\omega)$, $x'(0) = x'(\omega)$, integrating by parts the left-hand side of (3.18), we infer that

$$\int_0^\omega \left[\frac{x'(t)}{x(t)}\right]^2 dt + \int_0^\omega p(t)\,dt = 0, \tag{3.19}$$

which contradicts (2.14).

In the case of II, we can write by [8]

$$\int_{t_1}^{t_2} |p(t)|\,dt \geqslant \frac{4}{t_2-t_1}, \quad \int_{t_2}^{t_3} |p(t)|\,dt \geqslant \frac{4}{t_3-t_2}. \tag{3.20}$$

Hence and from (2.14) we state that

$$\frac{16}{\omega} > \int_0^\omega |p(t)|\,dt \geqslant 4\left(\frac{1}{t_2-t_1} + \frac{1}{t_3-t_2}\right) \geqslant \frac{16}{\omega}, \tag{3.21}$$

which is of course impossible.

In the case of III, we can assume without loss of a generality

$$x(0) = x(\omega) = 0, \quad x(t) \geqslant 0 \quad (x(t) \leqslant 0) \quad \text{for all } t \in R^1. \tag{3.22}$$

If $x'(0) = 0$, then by [7] we have $x = 0$. Let $x'(0) \neq 0$ and let $\varphi_k = p * \delta_k$, $\psi_k = x * \delta_k$. Then there exists a point $t_k$ in the interval $[-2a_k, 2a_k]$ such that $\psi_k'(t_k) = 0$. Besides, the sequence $\{t_k\}$ converges to zero. Now we consider the sequence $\{Y_k^{(v)}\}$ defined as follows:

$$(3.23) \qquad Y_k^{(v)}(t) = -\Big[\int_{t_k}^{t} (t-s)\varphi_k(s)\psi_k(s)\,ds + \psi_k(t_k)\Big]^{(v)},$$

$v = 0, 1$.

From Helly's theorem it follows that a subsequence $\{Y_{k_r}^{(v)}\}$ of $\{Y_k^{(v)}\}$ is convergent to a function $Y^{(v)}$ of locally bounded variation in $R^1$. Without loss of a generality we can assume that sequences $\{Y_k^{(v)}\}$ are convergent to $Y^{(v)}$. Since

$$(3.24) \qquad Y_k''(t) = -\varphi_k(t)\psi_k(t),$$

$$(3.25) \qquad Y(t) = x(t) + ct + d,$$

where $c$ and $d$ are some constants. From the almost uniform convergence of $\{Y_k\}$ we have

$$(3.26) \qquad x(0) = \lim_{k\to\infty} \psi_k(t_k) = 0, \quad \lim_{k\to\infty} Y_k(t_k) = Y(0) = 0.$$

Hence, by (3.22) and (3.25), we get $d = 0$. We shall prove that $c = 0$. In fact, let $\beta_k \overset{\text{df}}{=} Y' * \delta_k$. Then by (3.25) we obtain $\beta_k(t_k) = c$. On the other hand, from the properties of convolution and (3.23) we have

$$(3.27) \qquad |\beta_k(t_k)| = \Big[\Big|\lim_{r\to\infty} \int_{-a_k}^{a_k} d\tau \int_{t_r}^{t-\tau} \varphi_r(s)\psi_r(s)\,\delta_k(\tau)\,ds\Big|\Big]_{t=t_k}$$
$$\leqslant [|\lim_{r\to\infty} (\Phi_r\psi_r) * \delta_k|]_{t=t_k} + |\lim_{r\to\infty} \Phi_r(t_r)\psi_r(t_r)| +$$
$$+ \Big[\Big|\lim_{r\to\infty} \int_{-a_k}^{a_k} d\tau \int_{t_r}^{t-\tau} \Phi_r(s)\psi_r'(s)\,\delta_k(\tau)\,ds\Big|\Big]_{t=t_k},$$

where $\varphi_r = p * \delta_r$, $\psi_r = x * \delta_r$, $\Phi_r(t) = \int_0^t \varphi_r(s)\,ds$.

Consequently $\lim_{k\to\infty} \beta_k(t_k) = 0$. Hence it follows that the sequence $\{Y_k\}$ is uniformly convergent to the function $x$. By (3.23) and Gronwall's inequality we get

$$(3.28) \qquad |Y_k(t)| \leqslant E_k(M)\exp\Big[|t - t_k|\Big|\int_{t_k}^{t} \varphi_k(s)\,ds\Big|\Big],$$

where $M$ is an arbitrary compact interval and

$$(3.29) \qquad E_k(M) = \max_{t \in M} \left| \int_{t_k}^{t} (t-s)\big(\psi_k(s) - Y_k(s)\big)\varphi_k(s)\,ds \right| + |\psi_k(t_k)|.$$

Evidently $\lim_{k\to\infty} Y_k = 0$. Thus our assertion follows.

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