ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVIII, 1 (1983), p. 19-25

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A GAME MODEL REFERRING TO THE CONTROL OF INDEPENDENT DISCRETE TIME STOCHASTIC PROCESSES

1. Introduction. There is a considerable number of papers dealing with decision models, control of stochastic processes, and stochastic games (see [4]-[7]). The aim of the present paper is to discuss a game model which incorporates some elements of the afore-said theories and was primarily stimulated by a "blackjack" type game solved in [1].

Let us consider $m \geqslant 2$ independent stochastic systems associated with m players. Each of the players observes subsequent states only in his system at discrete moments of time $n \in N = \{0, 1, \ldots\}$ and may control the transition probabilities of his system by using an action $a \in A$ provided the current state of his system is in the set of admissible states S. If the system leaves the set S, then the player has to stop the observation and control.

The player may stop the observation and control also before his system leaves the set S. It is assumed that for any sequence of actions an inadmissible state of a system is attained in a finite period of time. A player may either choose his actions in the set A or stop at random taking into account the information about the history of his system.

The game is over when all players stop their observations and controls. The random pay-off of a player depends on the sample functions of random sequences of states and actions generated in all m systems up to the corresponding stopping times.

In the next section we formulate the mathematical model and basic assumptions for the game. In Section 3 we discuss the method of solution based on the fixed point theorem [3] and on the results in the theory of decision models. The last section contains some examples of the game.

2. The game model and basic assumptions. First we describe a stochastic system observed and controlled by a given player. Let $(R^N, \mathfrak{R}^N)$, $N = \{0, 1, 2, \ldots\}$, be the linear topological space of real sequences with σ-algebra $\mathfrak{R}^N$ of Borel subsets in R^N. Let A and S be two given compact subsets of $R = (-\infty, \infty)$. We introduce the following mappings: for $a, x \in R^N$

$$T_A(a) = \inf\{n \in N \mid a_n \notin A\}, \quad T_S(x) = \inf\{n \in N \mid x_n \notin S\}.$$

The elements of R and S are called *states* and *admissible states* of the system, respectively. The elements of A constitue the actions to be taken by the given player and $T_A(a)$ is interpreted as the moment of stopping the observation and control of the system. $T_S(x)$ is the moment at which the system leaves the set S.

We denote by $\mathfrak{R}^{n+1}$ $(n \in N)$ the σ-algebra of cylinders with bases over $\{0, 1, \ldots, n\}$ and we put $\mathfrak{A}^n = \mathfrak{R}^n \cap A^n$. Let

$$\{p_n(dx_n \mid x_0, x_1, \ldots, x_{n-1};\ a_0, a_1, \ldots, a_{n-1})\}$$

be a given family of transition probabilities from $(R^n \times A^n, \mathfrak{R}^n \otimes \mathfrak{A}^n)$ into $(R, \mathfrak{R})$. It is known [2] that the family determines a family $\mathscr{M}'_A = \{\mu(\cdot \mid a), a \in A^N\}$ of probability measures on $(R^N, \mathfrak{R}^N)$. We consider a wider family of measures on $(R^N, \mathfrak{R}^N)$, denoted by $\mathscr{M}_A = \{\mu(\cdot \mid a), a \in R^N\}$, where in the case $n < T_A(a)+1$

$$\mu(C \mid a) = \int_{C_0} p_0(dx_0) \int_{C_1} p_1(dx_1 \mid x_0, a_0) \times \ldots \times$$
$$\times \int_{C_n} p_n(dx_n \mid x_0, \ldots, x_{n-1};\ a_0, \ldots, a_{n-1})$$

for $C = \{x \in R^N \mid x_0 \in C_0, \ldots, x_n \in C_n\}$, $C_k \in \mathfrak{R}$ $(k = 0, \ldots, n)$, and in the case $n \geqslant T_A(a)+1$

$$\mu(R^{n+1} \mid a) = 0.$$

In the sequel, we assume that for every $a \in A^N$

$$\mu\big(\{x \mid T_S(x) < \infty\}|a\big) = 1. \tag{1}$$

Evidently, if $T_A(a) < \infty$ $(a \notin A^N)$, then

$$\mu\big(\{x \mid T_S(x) = T_A(a)+1\}|a\big) = 1.$$

Thus, a stochastic system related to the given player is determined by the triple $(R^N, \mathfrak{R}^N, \mathscr{M}_A)$.

Now, we define a policy of the player. Let $a, x \in R^N$. We consider the family of functions

$$\{q_n(da_n \mid x_0, \ldots, x_n;\ a_0, \ldots, a_{n-1})\} \tag{2}$$

satisfying the following conditions:

(a) If $n \leqslant \min\big(T_S(x)-1, T_A(a)\big)$, then, for all $(x_0, \ldots, x_n; a_0, \ldots, a_{n-1})$ in $S^{n+1} \times A^n$, $q_n(\cdot \mid x_0, \ldots, x_n; a_0, \ldots, a_{n-1})$ is a measure on $\mathfrak{A} = \mathfrak{R} \cap A$ for which

$$0 \leqslant q_n(A \mid x_0, \ldots, x_n;\ a_0, \ldots, a_{n-1}) \leqslant 1,$$

and for every $B \in \mathfrak{A}$, $q_n(B \mid x_0, \ldots, x_n;\ a_0, \ldots, a_{n-1})$ is a $(\gamma^{n+1} \otimes \mathfrak{A}^n)$-measurable function where $\gamma^{n+1} = \mathfrak{R}^{n+1} \cap S^{n+1}$.

(b) If $n > \min\big(T_S(x)-1, T_A(a)\big)$, then

$$q_n(A \mid x_0, \ldots, x_n;\ a_0, \ldots, a_{n-1}) = 0.$$

Using Theorem 1.1 from [2] we have a family of probability measures $\Pi_S = \{\pi(\cdot \mid x), x \in R^N\}$ on $(R^N, \mathfrak{R}^N)$ satisfying the condition

$$\pi\big(\{a \mid T_A(a) \leqslant T_S(x)\} \mid x\big) = 1, \quad x \in R^N.$$

Now, for a given stochastic system $(R^N, \mathfrak{R}^N, \mathscr{M}_A)$ and for every policy $\pi \in \Pi_S$ there exists a unique probability P_π on $(R^N \times R^N, \mathfrak{R}^N \otimes \mathfrak{R}^N)$ and the probability measure is concentrated on $R_S^N \times R_A^N$, where

$$R_S^N = \{x \in R^N \mid T_S(x) < \infty\} \quad \text{and} \quad R_A^N = \{a \in R^N \mid T_A(a) < \infty\}.$$

Therefore, we have a random sequence $\zeta = (\xi_n, a_n)$, $n \in N$, with values in $R \times A$ and a random stopping time

$$\tau = \inf\{n \in N \mid a_n \notin A\},$$

where for $C_k \in \gamma$, $D_k \in \mathfrak{A}$ $(k = 1, \ldots, n-1)$ and $C_n \in \mathfrak{R}$, $D_n \in \mathfrak{A}$ $(n \in N)$

(3) $$P_\pi\{\xi_0 \in C_0, a_0 \in D_0, \ldots, a_{n-1} \in D_{n-1}, \xi_n \in C_n, \tau = n\}$$

$$= \int_{C_0} p_0(dx_0) \int_{D_0} q_0(da_0 \mid x_0) \times \ldots \times$$

$$\times \int_{C_n} \Big[1 - \int_A q_n(da_n \mid x_0, \ldots, x_n;\ a_0, \ldots, a_{n-1})\Big] \times$$

$$\times p_n(dx_n \mid x_0, \ldots, x_{n-1};\ a_0, \ldots, a_{n-1}).$$

It follows from (1) that $P_\pi\{\tau < \infty\} = 1$.

Now, let $(\mathscr{X}, \mathscr{B}, P_{\pi_1, \ldots, \pi_m})$ be the product space of probability spaces associated with m players. Thus, we assume that the random sequences ζ_i and the random stopping times τ_i, $i = 1, \ldots, m$, are mutually independent.

We define the pay-off functions for the game. First, let $F_i = \{f_i^{n_1, \ldots, n_m}\}$ be a family of measurable bounded functions defined in R^l, where $l = m + 2\sum_{i=1}^{m} n_i$, $n_i \in N$, $i = 1, \ldots, m$, satisfying the condition

(4) $$\sup_{(n_1, \ldots, n_m) \in N^m} \|f_i^{n_1, \ldots, n_m}\| < \infty,$$

where $\|\cdot\|$ denotes the supremum norm.

Now the pay-off function for the player i $(i = 1, \ldots, m)$ is defined by the relation

$$K_i(\pi_1, \ldots, \pi_m) = \mathrm{E}_{\pi_1,\ldots,\pi_m} f_i^{\tau_1,\ldots,\tau_m}(\chi_{\tau_1}, \ldots, \chi_{\tau_m}), \tag{5}$$

where $\chi_{\tau_i} = (\xi_0^i, \xi_1^i, \ldots, \xi_{\tau_i}^i, a_0^i, a_1^i, \ldots, a_{\tau_i-1}^i)$ describes the history of the stochastic system up to the random stopping time τ_i, and $\mathrm{E}_{\pi_1,\ldots,\pi_m}$ is the expectation operator with respect to the probability $P_{\pi_1,\ldots,\pi_m}$. We denote the considered game by

$$\Gamma = \big((S_i), (A_i), (\mathscr{M}_{A_i}), (F_i),\ i = 1, \ldots, m\big).$$

Remark 1. A similar construction of the game model could be based on the product of decision models described in [4] or on stopped decision models given in [7].

Remark 2. Instead of a fixed set of admissible states one could consider a sequence $\{S_n\}$, $n \in N$, as well as a sequence $\{A_n\}$, $n \in N$, of action sets.

3. The method of solution. We have already known that every strategy $\pi \in \Pi_S$ is represented by the family of transition probabilities given by (2). Let us introduce a sequence of measure spaces $(H_n, \mathfrak{H}_n, l_n)$, where $H_n = S^{n+1} \times A^n$, $\mathfrak{H}_n$ is the σ-algebra of Borel subsets in H_n, and l_n is the Lebesgue measure on $\mathfrak{H}_n$. Let $\mathfrak{M}(A)$ be the space of Radon measures on A and $\mathscr{C}(A)$ the space of continuous functions defined in A. We denote by $E_n = L^1_{\mathscr{C}(A)}(H_n)$ the space of all integrable functions defined in H_n and taking values in $\mathscr{C}(A)$. Let $\sigma(E_n', E_n)$ be the ω^*-topology in $E_n' = L^\infty_{\mathfrak{M}(A)}(H_n)$, the dual of E_n. We consider the space $\big(E', \prod_{n\in N} \sigma(E_n', E_n)\big)$, where

$$E' = \prod_{n\in N} E_n'.$$

Let Φ be a subset of E' defined by

$$\Phi = \prod_{n\in N} \{\varphi^{(n)} \in E_n' \mid \varphi^{(n)} \geqslant 0,\ \|\varphi^{(n)}\|_{E_n'} \leqslant 1\}.$$

An element of Φ determines a sequence of transition probabilities in (2) and, consequently, a strategy $\pi \in \Pi_S$. Using the Alouglou-Bourbaki theorem [8] we notice that Φ is a compact subset in E'. In order to apply the Kakutani fixed point theorem [3] one has to establish the continuity of the pay-off functions $K_i(\varphi_1, \ldots, \varphi_m)$ on $\prod_{i=1}^m \Phi_i$. Using (3) and (5) it is possible to find the expression for $K_i(\varphi_1, \ldots, \varphi_m)$, and then to study the continuity properties. The continuity properties depend, in general, on the families of functions F_i given by (4), on the subsets A_i and S_i as well as on the transition probabilities $p_n^i(\cdot \mid \cdot)$. It is an interesting problem to find some general sufficient conditions for the continuity of the pay-off functions $K_i(\varphi_1, \ldots, \varphi_m)$. The simplest case is that where A_i are one-element sets, p_n^i are Markov and homogeneous, and the random pay-offs depend only on the states of the m systems at the random stopping times.

If the existence of the optimal equilibrium strategies is established, then an associated decision problem [4] can be formulated as

$$G_i = \sup_{\pi_i \in \Pi_{S_i}} \mathrm{E}_{\pi_i} G_i^{\tau_i}(\chi_{\tau_i}), \tag{6}$$

where for $n_i \in N$ and $h_{n_i} \in S_i^{n_i+1} \times A_i^{n_i}$

$$G_i^{n_i}(h_{n_i}) = \mathrm{E}_{\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_m} f_i^{\tau_1, \ldots, n_i, \ldots, \tau_m}(\chi_{\tau_1}, \ldots, h_{n_i}, \ldots, \chi_{\tau_m}).$$

The problem stated in (6) can be solved by the application of the results presented in [2], [4], and [7]. Usually, a simple class of non-dominated strategies can be obtained, which enables the game to be reduced to a game over a compact subset of a finite-dimensional space with continuous pay-off functions. For the reduced game an analytical solution can be easily found. The method of solution described in this section was applied in several cases of two-person zero-sum games. We give some examples in the next section.

4. Examples. We consider here only some antagonistic symmetric games. Therefore, we give only the elements of the stochastic system for one of the players. We use the notation $\Gamma = (S, A, \mathscr{M}_A, F)$.

1. We generalize the model given in [1]. Let $S = [0, c]$ and $A = \{1, 2, \ldots, k\}$. The family of measures $\mathscr{M}_A$ is determined by the sequence

$$p_n(B \mid x_0, \ldots, x_{n-1}; a_0, \ldots, a_{n-1}) = \int_B g^{a_{n-1}}(u - x_{n-1})\,du,$$

where $n \geqslant 1$, $B \in \mathfrak{R}$, $(x_k, a_k) \in R \times A$ $(k = 1, \ldots, n-1)$, and $\{g^a(x), a \in A\}$ is a family of density functions satisfying the condition $g^a(x) = 0$ for $x < 0$, $a \in A$. The family F (see (4)) is given by

$$f^{n_1, n_2}(h_{n_1}, h_{n_2}) = r(x_{n_1}^1, x_{n_2}^2),$$

where

$$r(x, y) = \begin{cases} 1 & \text{if } 0 \leqslant y < x \leqslant c \text{ or } x \in S \text{ and } y > c, \\ 0 & \text{if } x = y \text{ or } x > c \text{ and } y > c, \\ -1 & \text{if } 0 \leqslant x < y \leqslant c \text{ or } x > c \text{ and } y \in S. \end{cases}$$

2. Let $S = A = \{1, 2\}$. The family $\mathscr{M}_A$ is described by two matrices $M(a) = \big(p_{ij}(a)\big)$, $a \in A$, $i, j \in S$, where $p_{ij}(a) > 0$ and $p_{i1}(a) + p_{i2}(a) < 1$.

The family F is given by

$$f^{n_1,n_2}(h_{n_1}, h_{n_2}) = w_{t(h_{n_1}),t(h_{n_2})},$$

where

$$t(h_{n_i}) = \begin{cases} \operatorname{card}\{0 \leqslant n \leqslant n_i - 1 \mid a_n = x_{n+1}\} & \text{if } x_{n_i} \in S, \\ -1 & \text{if } x_{n_i} \notin S, \end{cases}$$

and $(w_{r,s})$ is a given infinite matrix.

3. Here we take $S = [0, 1]$, $A = \{1\}$, and the family $\mathcal{M}_A$ described by

$$p_n(B \mid x_0, \ldots, x_{n-1}; 1, 1, \ldots, 1) = \int_B p(x_{n-1}, u)\, du,$$

where $n \geqslant 1$, $B \in \mathfrak{R}$, $x_k \in R$ $(k = 1, \ldots, n-1)$, and $p(x, y)$ is a Markov kernel with the condition $p(x, y) = 0$ if $y < x$. The pay-off is determined by

$$f^{n_1,n_2}(h_{n_1}, h_{n_2}) = k(x^1_{n_1}, x^2_{n_2}),$$

where

$$k(x, y) = \begin{cases} x - y + xy & \text{if } 0 \leqslant x < y \leqslant 1, \\ 0 & \text{if } 0 \leqslant x = y \leqslant 1, \\ -y + x - xy & \text{if } 0 \leqslant y < x \leqslant 1, \\ -y & \text{if } y \in S \text{ and } x > 1, \\ x & \text{if } x \in S \text{ and } y > 1. \end{cases}$$

The interpretation and analytical solutions of the examples given above will be published elsewhere [9].

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Received on 10. 2. 1979;
revised version on 9. 6. 1980