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On the order of natural differential operators and liftings

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Abstract. We start by proving that natural differential operators are functions of inifinite jets of sections. A concept of (k, s)-integrability is introduced which is shown to be more general than the integrability of geometric objects. Examples are given. We prove that if a section σ is (k, s)-integrable, then every σ-concomitant of rank s is of order at most k. Basing upon this fact, some general conclusions are drawn concerning the order of continuous n.d. operators and liftings.

It seems to be the first attempt to give a workable method of estimation of the order of natural operators which can be applied in general case. The estimate of the order of linear liftings of functions and vector fields was given by Gancarzewicz in his D. Sci. thesis [2]. In [5], Terng gave a bound on the order of linear differential operators between tensor bundles (to be 1) and of some particular polynomial operators on tensors. We shall determine ex-plicite bounds on the order of concomitants of geometric objects which are integrable or, more generally, which are (k, s)-integrable as defined below. Examples of exploiting the presented method to some particular cases will be given. Moreover, we shall prove that, in general, every natural operator depends at most on the infinite jets of section. This fact was proved by Epstein [1] for operators defined on metric tensors.

I. Definitions and basic facts. Let E and E' be two natural functors over the category of $C^\infty - n$-manifolds. A natural differential operator D is a family $\{D(M)\}$ of maps

$$D(M)\colon\ E(M) \to E'(M)$$

which assigns to each section σ of the bundle $E(M)$ a (smooth) section $D(M)\sigma$ of $E'(M)$ in such a way that for any diffeomorphism φ of M onto an open subset of N we have

$$\varphi_* D(M)\sigma = D(N)\varphi_*\sigma, \tag{I.1}$$

where $\varphi_*\sigma$ is a section of $E(N)$ equal to $E\varphi \circ \sigma \circ \varphi^{-1}$.

The natural operator D is clearly determined locally; $D(M)\sigma(x)$ depends only on the germ of σ at x. Therefore we can define D equivalently by considering $E(M)$ and $E'(M)$ as the sheafs of germs of sections of $E(M)$ and $E'(M)$, respectively. In this formulation σ and $\varphi_* \sigma$ will be considered as section-germs.

THEOREM I.1. *For each natural operator D, $D\sigma(x)$ depends only on $j^\infty_x \sigma$.*

We shall exploite the following Borel's lemma (see [4]).

LEMMA I.2. *If S_0 and S_1 are disjoint closed subsets of the unit sphere S^{n-1} of $\mathbf{R}^n$, of non-empty interior, then there is a function Φ: $\mathbf{R}^n \to \mathbf{R}$ such that:*

(i) *Φ is C^∞ on $\mathbf{R}^n - \{0\}$,*

(ii) *$D_\alpha \Phi(x) = O(|x|^{-|\alpha|})$ for all α, as $x \to 0$,*

(iii) *$\Phi(x) = t$ for $x/|x|$ in S_t $(t = 0, 1)$,*

(iv) *if f: $\mathbf{R}^n \to \mathbf{R}^m$ is C^∞ and $j^\infty_0 f = 0$, then ψ: $\mathbf{R}^n \to \mathbf{R}^m$ defined by $x \to \Phi(x) f(x)$ is C^∞ and $j^\infty_0 \psi = 0$.*

Proof of Theorem. I.1. Suppose that $j^\infty_x \sigma = j^\infty_x \sigma'$; we have to show that $D\sigma(x) = D\sigma'(x)$. Since the problem is local we may suppose that $M = \mathbf{R}^n$, $E(\mathbf{R}^n) = F \times \mathbf{R}^n$ and that $x = 0$, where $F = E(\mathbf{R}^n)_0$ (the fibre over 0); σ and σ' are given by C^∞ maps $\bar{\sigma}$ and $\bar{\sigma}'$, respectively

$$\bar{\sigma}, \bar{\sigma}': U \subset \mathbf{R}^n \to F,$$

such that $\bar{\sigma}(0) = \bar{\sigma}'(0) = y \in F$. Introducing local coordinates in F about y:

$$u: (W, y) \to (\mathbf{R}^m, 0)$$

we have that

$$f = u \circ \bar{\sigma}: U \to \mathbf{R}^m \quad \text{and} \quad f' = u \circ \bar{\sigma}': U \to \mathbf{R}^m$$

are C^∞.

Since $j^\infty_0 \bar{\sigma} = j^\infty_0 \bar{\sigma}'$, it follows that $j^\infty_0 f = j^\infty_0 f'$.

The function $g = f + \Phi(f' - f)$ is C^∞ (by Lemma I.1). On taking U sufficiently small, we may suppose that g takes values in $u(W)$. Write

$$V_t = \{x \in \mathbf{R}^n: \ x/|x| \in S_t\}.$$

Then

$$\bar{g} = u^{-1} \circ g: U \to F$$

is C^∞ and coincides with $\bar{\sigma}$ on V_0 and with $\bar{\sigma}'$ on V_1. So

$$D(\sigma| \ U \cap V_0) = D(\eta| \ U \cap V_0)$$

and

$$D(\sigma'| \ U \cap V_1) = D(\eta| \ U \cap V_1),$$

where η is the section corresponding to $\bar{g}$. Since 0 is in the closure of V_0 and V_1, it follows that $D\sigma(0) = D\sigma'(0)$, Q.E.D.

The *order* of a natural operator D is the smallest $k \leqslant \infty$ such that if $j_x^k \sigma = j_x^k \sigma'$ then $D\sigma(x) = D\sigma'(x)$ for each σ, σ' from $E(M)$ and $x \in \mathrm{Dom}(\sigma) \cap \mathrm{Dom}(\sigma')$. This is equivalent to the existence of a natural morphism Φ: $J^k E(M) \to E'(M)$ such that $D = \Phi \circ j^k$, that is

$$\Phi(j_x^k \sigma) = D\sigma(x).$$

Let E'' be a natural bundle over E'.

DEFINITION I.3. A *lifting* (natural lifting operator) L is a family $\{L(M)\}$ of maps $L(M)$: $E(M) \to E''(E')$ such that:

(i) if σ is a local section of the bundle $E \to M$ defined on an open set U, then $L(M)\sigma$ is a local section of the bundle $E'' \to E'$ defined on $E'|U$.

(ii) $L(N)\cdot\varphi_* = (E'\varphi)_* L(M)$ for each embedding φ: $M \to N$.

A lifting L is of order $k \leqslant \infty$ if k is the smallest integer (including ∞) such that $j_x^k \sigma = j_x^k \sigma'$ implies $L\sigma(z) = L\sigma'(z)$ for all z from the fibre $E'(M)_x$. Note that L is of order k if and only if there exists a natural morphism

$$\Phi\colon\ J^k E \times_M E' \to E''$$

such that

$$\Phi(j_{\pi(z)}^k \sigma, z) = L\sigma(z).$$

We can consider E'' as a natural functor on n-manifolds via E'. The proofs of our propositions on the order of natural operators remain valid, with only slight formal changes, for liftings. Therefore the results obtained hold also in this case.

Remark I.4. Let E be a natural functor of order r, $F = E(\mathbf{R}^n)_0$ its standard fibre. There is a canonical action of the r-th differential group $G(n, r)$ on F defined by

$$q(j_0^r f, y) = Ef(y) \tag{I.2}$$

for all local diffeomorphism f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ and $y \in F$. By the fundamental result of Palais and Terng [4] $E(M)$ can be identified with the associated bundle $H^r M \times_{G(n,r)} F$, to the r-th frame bundle $H^r M$. Then the section-jet bundle $J^k E(M)$ has as standard fibre the space $T_n^k F$ of k-jets at 0 of maps $\mathbf{R}^n \to F$ with the induced action of $G(n, k+r)$ defined by

$$J^k q(j_0^{r+k} f, j_0^k \sigma) = j_0^k q(\tilde{f}, \sigma \circ f^{-1}), \tag{I.3}$$

where $\tilde{f}$: $\mathbf{R}^n \to G(n, r)$, $\tilde{f}(x) = j_0^r(t_{-x} \circ f \circ t_{f^{-1}(x)})$, t_v translation of $\mathbf{R}^n$ by v.

Let E' be of order $s \leqslant r+k$, F' its standard fibre. Any natural morphism between two associated bundles in uniquely determined by the induced equivariant map between their standard fibres. Therefore the study of natural differential operators of order k between E and E' is equivalent to the study of equivariant maps between $T_n^k F$ and F'. This is true also for $k = \infty$. In this case the group $G(n, \infty)$ is defined to be the inverse limit of groups $G(n, p)$, $p = 1, 2, \ldots$, with canonical projections. Similarly, the action of $G(n, \infty)$ on $T_n^\infty F$ is the inverse limit of actions $J^p q$ defined in (I.3). In view of Theorem I.1 every natural differential operator can be studied in this way as far as it is defined for all section-germs of $E(M)$.

Clearly, in such a case, there is a little hope for the existence of a global bound on the order. For instance, if the action on F' is linear, it would require that there be no smooth and non-trival invariant functions on the fibres of the bundle $T_n^\infty F \to T_n^k F$. Otherwise, introducing such a function as a multiplicative factor to any equivariant map from $T_n^k F$ into F' (if it exists), we would get an equivariant map between $T_n^\infty F$ and F' which would be of higher order than k.

Most of n.d. operators occurring in differential geometry are not defined for all sections of $E(M)$ but on some subsets of them which, necessarily, are closed under the induced transformations $E\varphi$. In other words, $D(M)$ is defined on a $\mathrm{Diff}(M)$-invariant subset of $E(M)$. In particular, given a section σ, we shall denote by Orb σ the set of all sections $\varphi_* \sigma$ for $\varphi \in \mathrm{Diff}(M)$. A natural differential operator defined on Orb σ will be called a *concomitant* of σ.

The constructing of concomitants is what we do in research of geometric structures, which are defined by giving on a manifold one or several geometric objects, that is, sections of natural bundles.

LEMMA I.5. *A concomitant of σ is of order $\leqslant k$ if and only if for each $\varphi \in \mathrm{Diff}(M)$ such that $j_x^k \varphi_* \sigma = j_x^k \sigma$, it follows that $D(\varphi_* \sigma)(x) = D\sigma(x)$.*

It follows easily from the general definition of the order of n.d. operators and the naturality conditon.

Remark I.6. Since the naturality condition (I.1) "works" only within the orbits of sections and gives no relations between images of sections from two different orbits, there is much more chance to obtain some information about the order of concomitants than of operators defined for all sections. In the last case the restriction on the order may come from additional assumptions like the linearity of the operator or others.

When it is clear from the context, $\mathrm{Orb}\,\sigma$ will denote as well the set of germs of sections belonging to it.

DEFINITION I.7. A section σ will be called *k-determined* if for any section σ' of $E(M)$, $j_x^k \sigma' = j_x^k \sigma$ implies that the germ of σ' at x belongs to $\mathrm{Orb}\,\sigma$ (for the finite determinacy of geometric objects see [3] and [6]).

DEFINITION I.8. We shall say that a n.d. operator (a concomitant) is of *rank* s if s is the order of the image functor E'.

II. (k, s)-integrability. As before, we assume that E is a natural functor of order $r \geqslant 1$. Let $\Phi(M)$ be the set of germs of all local diffeomorphism of M and $\Phi^{r+k}(M)$ the set of their $(r+k)$-jets. The natural action of $\Phi(M)$ on the sheaf $E(M)$ factorizes by jet-projection to an action of $\Phi^{r+k}(M)$ on $J^k E(M)$. Given a section σ of $E(M)$, we denote by $G^{r+k}(\sigma)$ the collection of isotropy groups of section $j^k\sigma$ with respect to this action. Let $\operatorname{Aut}\sigma$ be the pseudogroup of all local automorphisms of σ, and $A^s(\sigma)$ the set of their s-jets. Denote by $G_s^{r+k}(\sigma)$ the image of $G^{r+k}(\sigma)$ by projection

$$j_s^{r+k}\colon \Phi^{r+k}(M) \to \Phi^s(M) \quad (1 \leqslant s \leqslant r+k).$$

DEFINITION II.1. A section σ will be called *(k, s)-integrable* if

$$G_s^{r+k}(\sigma) = A^s(\sigma). \tag{II.1}$$

If it holds for $s = r+k$ we shall say that σ is *k-integrable*.

Note that if σ is k-integrable then it is (k, s)-integrable for all $1 \leqslant s \leqslant r+k$.

PROPOSITION II.2. *σ is (k, s)-integrable if and only if for any $x \in \operatorname{Dom}(\sigma)$ and σ' from* Orb σ *such that $j_x^k\sigma' = j_x^k\sigma$, there exists $\varphi \in \Phi_x(M)$ with $j_x^s\varphi = \mathrm{id}$ (jet of identity map), such that the germ of $\varphi_*\sigma$ and the germ of σ' at x are equal.*

Proof. ($\Rightarrow$) Let $\sigma' = \psi_*\sigma$. From the assumption it follows that $j_x^{r+k}\psi$ is in the isotropy group of $j_x^k\sigma$. By (II.1) there exists a local automorphism $\bar{\varphi}$ of σ whose s-th jet at x is $j_x^s\psi$. Put $\varphi = \psi\circ\bar{\varphi}^{-1}$, then $\varphi_*\sigma = \psi_*\sigma = \sigma'$ (for germs at x) and $j_x^s\varphi = \mathrm{id}$, as required.

($\Leftarrow$) Let $j_x^{r+k}\psi \in G^{r+k}(\sigma)$. Take $\sigma' = \psi_*\sigma$ for any ψ with that jet at x. Then $j_x^k\sigma' = j_x^k\sigma$. By assumption, there is φ in $\Phi_x(M)$ such that $j_x^s\varphi = \mathrm{id}$ and $\sigma' = \varphi_*\sigma$ near x. It follows that in a neighbourhood of x, $\varphi^{-1}\circ\psi$ is a local automorphism of σ. Moreover, $j_x^s(\varphi^{-1}\circ\psi) = j_x^s\psi$, that is $j_x^s\psi$ is in $A^s(\sigma)$. The inverse inclusion is obvious.

THEOREM II.3. *Suppose that $r = 1$ and that σ is an integrable section. Then σ is k-integrable for all $k \geqslant 0$.*

Proof. We may suppose that σ is a constant section of $\boldsymbol{R}^n \times F$, say $\sigma\colon \boldsymbol{R}^n \to y \in F$. We have to show that every element of the isotropy group of $j_0^k\sigma$ with respect to the action of $G(n, 1+k)$ on $T_n^k F$ is the $(k+1)$-jet of an automorphism of σ at 0. We prove it by induction rel. k, using the notations of Remark I.4.

Let $k = 0$. Then the isotropy group $G^1(\sigma)$ at 0 is a group of matrices such that $q(a, y) = y.f \in \operatorname{Diff}(\boldsymbol{R}^n, 0)$ is an automorphism of σ iff its Jacobian matrix $Df(x)$ is in $G^1(\sigma)$ for all $x \in \operatorname{Dom}(f)$. For $a \in G^1(\sigma)$ it suffices to take the linear isomorphism of $\boldsymbol{R}^n$ determined by this matrix in order to see that $a \in A^1(\sigma)$.

Assume that our claim is true for $k-1$.

Let $j_0^{k+1}\varphi \in G^{k+1}(\sigma)$. Then the k-jet of φ at 0 is in $G^k(\sigma)$ and, by hypothesis for $k-1$, it coincides with such a jet of an automorphism $\bar{\varphi}$ of σ. Define $f = \varphi\circ\bar{\varphi}^{-1}$. Then $j_0^{k+1}f \in G^{k+1}(\sigma)$ and $j_0^k f = \mathrm{id}$.

Let g be the Lie algebra of $G^1(\sigma)$, $g^{(k)}$ its symmetric prolongation:

$$g^{(k)} = g \otimes S^k(R^{n*}) \cap R^n \otimes S^{k+1}(R^{n*}).$$

We claim that $D^{k+1}f(0)$ is in $g^{(k)}$. Since $j_0^{k+1} f$ does not change $j_0^k \sigma$, we have from formula (I.3), where we put $r = 1$ and $\sigma = y$ (= const),

$$j_0^k q(\tilde{f}, y) = j_0^k \hat{y} \quad \text{(jet of a constant map)}.$$

It follows

$$0 = d^k \hat{y}(0) = d^k q(\tilde{f}, y)(0) = (dq)(e, y) d^k \tilde{f}(0) = (dq)(e, y) d^{k+1} f(0),$$

where $e = \tilde{f}(0)$ is the unit matrix. But $(dq)(e, y)$ is the differential of the orbital projection from $GL(n)$ into F at y. Its kernel is the isotropy algebra g. The latter formula means that $D^{k+1}f(0)$ belongs to $g^{(k)}$.

Consider the vector field

$$X = \frac{1}{(k+1)!} \sum_{|\alpha| = k+1} \left(D_\alpha^{k+1} f^j(0) x^\alpha\right) \frac{\partial}{\partial x^i}.$$

We have

$$\left(\frac{\partial X^i}{\partial x^j}\right) = \sum \left(D_\alpha^{k+1} f^i(0) x^{\alpha - 1_j}\right),$$

so it is a matrix in g for all $x \in R^n$. It follows that X is an infinitesimal automorphism of σ. Consequently, $\bar{f} = \exp X \in \operatorname{Aut} \sigma$ and $j_0^{k+1} \bar{f} = j_0^{k+1} f$.

Now, $j_0^{k+1} \varphi = j_0^{k+1}(f \circ \bar{\varphi}) = j_0^{k+1}(\bar{f} \circ \bar{\varphi})$ and both $\bar{f}$ and $\bar{\varphi}$ are automorphisms of σ. Thus we proved that $j_0^{k+1} \varphi$ belongs to $A^{k+1}(\sigma)$, which completes the proof.

Remark II.4. The theorem is no longer true for $r > 1$, even for $k = 0$. This can be simply deduced from the fact that flat G-structures of higher order are, in general, not transitive.

The equality $G^{k+1}(\sigma) = A^{k+1}(\sigma)$ allows to determine the structure of the isotropy group $G^{k+1}(\sigma)$; we have namely

COROLLARY II.5. *Let G be the isotropy group of $y \in F$; then the isotropy group of $j_0^k \hat{y} \in T_n^k F$ is the k-th holonomic prolongation of G:*

$$p^k G = T_n^k G \cap G(n, k+1) = \{ j_0^k \alpha \in G(n, k+1);\ \alpha\colon R^n \to G \}.$$

Its Lie algebra is

$$p^k g = g + g^{(1)} + \ldots + g^{(k)}.$$

In fact, it suffices to put $\alpha = Df$ for arbitrary automorphism f of the constant section $\hat{y}\colon R^n \to y$. The second part follows from the fact that the elements of the Lie algebra of $p^k G$ are of form

$$\frac{d}{dt}(j_0^k \alpha_t)(0) = j_0^k\left(\frac{d}{dt}\alpha_t(0)\right) = j_0^k A \qquad \text{with } A\colon \boldsymbol{R}^n \to g,$$

and $A^i_{jj_1\ldots j_p} = \partial_{j_1\ldots j_p} A^i_j$ are symmetric in subindices.

LEMMA II.6. *Let f be a function defined on an open subset of M. Suppose that the differential of f is nowhere zero. Then f is k-integrable for all $k \geqslant 1$.*

Proof. In view of the assumption, every point in $\mathrm{Dom}(f)$ has a coordinate neighbourhood U, $x^1, \ldots, x^n$, which can be chosen in such a way that $x^1(x) = f(x)$ for $x \in U$. So, we can consider the case where $M = \boldsymbol{R}^n$ and $f = x^1$. According to Proposition II.2 we shall show that for each function-germ $g = \psi_* f = f \circ \psi^{-1}$ at a point $x \in \boldsymbol{R}^n$, such that $j_x^k g = j_x^k f$, there exists $\varphi \in \Phi_x(\boldsymbol{R}^n)$ with $j_x^k \varphi = \mathrm{id}$ such that $g = \varphi_* f$.

Assume that $x = 0$. We have $g = x^1 \circ \psi^{-1} = x^1(y^1, \ldots, y^n)$ and

$$j_0^k x^1(y^1, \ldots, y^n) = j_0^k y^1.$$

We can modify ψ to another local diffeomorphism of $\boldsymbol{R}^n$ defined by

$$\varphi^{-1}\{x^1 = x^1(y^1, \ldots, y^n),\ x^2 = y^2, \ldots, x^n = y^n\}.$$

Then $g = x^1 \circ \varphi^{-1}$ and $j_0^k \varphi = \mathrm{id}$, as required.'

Note that f is a non-integrable section of the bundle $M \times R$ of order $r = 0$.

LEMMA II.7. *Let π be a differential 1-form on M of constant pfaffian class $2r+1$ (i.e., $d\pi$ is of rank $2r$ and $\pi \wedge (d\pi)^r \neq 0$). Then π is (1, 1)-integrable.*

Proof. By Darboux theorem, there is a local chart U: $x^1, \ldots, x^n$, in which π has the canonical form

$$\sum_{i=1}^{r} x^i dx^{i+r} + dx^{2r+1}.$$

We have $\pi(0) = dx^{2r+1}$ and $d\pi(0) = \sum_{i=1}^{r} dx^i \wedge dx^{i+r}$. We are going to show that if $j_0^2 \varphi$ leaves $j_0^1 \pi$ invariant then $j_0^1 \varphi$ is the jet of an automorphism of π.

Let $\varphi = \{y^i = y^i(x^1, \ldots, x^n)\}$; by assumption about φ, there is

$$(*) \qquad (dy^{2r+1})_0 = dx^{2r+1} \quad \text{and} \quad \Big(\sum_1^r dy^i \wedge dy^{i+r}\Big)_0 = \sum_1^r dx^i \wedge dx^{i+r}.$$

Let

$$a^i_j = \frac{\partial y^i}{\partial x^j}(0).$$

We shall construct an automorphism of π whose Jacobian matrix at 0 is (a^i_j). We put $y^i = a^i_j x^j$ for $i \neq 2r+1$ and we determine $y^{2r+1}(x)$ as a solution of the differential equation

$$(**) \qquad dy^{2r+1} = dx^{2r+1} + \sum_1^r x^i dx^{i+r} - \sum_1^r y^i dy^{i+r},$$

where y^i are defined above. In view of (*) the differential of the right-hand side is zero identically, so the equation has a solution. Moreover, it follows from (**) that is satisfies

$$(dy^{2r+1})_0 = dx^{2r+1}, \quad \text{so} \quad \frac{\partial y^{2r+1}}{\partial x^j} = \delta_j^{2r+1} \quad \text{at } 0.$$

But this is exactly the $(2r+1)$-st row in (a_j^i), because of (*). On the other hand, equality (**) for functions $y^i(x)$ $(i = 1, \ldots, n)$, with non-singular Jacobian at 0, means that they define a germ of local automorphism of π. This completes the proof.

III. Estimations of the order.

THEOREM III.1. *If a section σ is (k, s)-integrable, then every σ-concomitant of rank $\leqslant s$ is of order $\leqslant k$.*

In particular, if σ is k-integrable for all $k \geqslant k_0$ and the rank of a σ-concomitant is s, such that $s-r \geqslant k_0$, then its order is $\leqslant s-r$.

Proof. Let D be a σ-concomitant of rank $\leqslant s$ and let σ' be a section-germ from $\operatorname{Orb}\sigma$, such that $j_x^k \sigma' = j_x^k \sigma$. By Proposition II.2, there exists $\varphi \in \Phi_x(M)$, $j_x^s \varphi = \mathrm{id}$, such that $\sigma' = \varphi_* \sigma$ near x. Then we have

$$D\sigma'(x) = D\varphi_* \sigma(x) = \varphi_* D\sigma(x) = D\sigma(x),$$

since $D\sigma$ is of rank $\leqslant s$ and $j_x^s \varphi = \mathrm{id}$. By Lemma I.5, D has order at most k.

For the proof of the second part of the theorem we put $k = s-r$ and apply the first part.

In view of what we proved in Part II, we conclude:

COROLLARY III.2.

(i) *Every concomitant of rank s of an integrable field of geometric objects of order 1 (e.g. tensors) is of order $\leqslant s-1$.*

In particular, the above is applicable to nowhere zero vector fields or a system of such vector fields with vanishing Poisson brackets, linearly independent.

(ii) *Any tensorial concomitant of an 1-form of constant odd pfaffian class has order at most one.*

(iii) *Any concomitant of rank s of a function with nowhere zero differential is of order $\leqslant s$ for $s \geqslant 1$.*

Remark III.3. Theorem III.1 is also true for n.d. operators defined on a set of (k, s)-integrable sections, which is $\mathrm{Diff}(M)$-invariant and any two of their germs at arbitrary point $x \in M$ are $\Phi_x(M)$-equivalent. The proof is analogous.

DEFINITION III.4. An n.d. operator D (a lifting L) will be said *continuous* if for each sequence σ_m of germs of sections, such that $j_x^\infty \sigma_m \to j_x^\infty \sigma$, $D\sigma_m(x) \to D\sigma(x)$ (resp., for any sequence $z_m \in E'_x(M)$, z_m tends to z implies that $L\sigma_m(z_m)$ tends to $L\sigma(z)$).

If we know that the operator D is of finite order, say p, then the convergence of infinite jets in the above definition should be replaced by that of p-jets. For instance, every polynomial n.d. operator of order $\leqslant p$ is continuous in this sense.

We shall say that a subset W of the sheaf $E(M)$ is *k-sufficient* if for any germ σ at x, $j_x^k \sigma \in j^k W$ implies that $\sigma \in W$ (roughly speaking W is defined by a condition on k-jets).

THEOREM III.5. *Let D be a continuous n.d. operator of order $p \leqslant \infty$, defined for all section-germs, and let W be a k-sufficient subset of $E(M)$, such that $j^p W$ is dense in $J^p E(M)$ (in the inverse limit topology if $p = \infty$). Suppose that D is of order $k < p$ on W. Then D is of order k on $E(M)$.*

Proof. Let σ and σ' be two germs such that $j_x^k \sigma = j_x^k \sigma'$. There exists a sequence of germs σ_m, such that $j_x^p \sigma_m$ is in $j^p W$ and tends to $j_x^p \sigma$. Since $p > k$, σ_m are in W. We can choose a sequence of germs σ'_m such that $j_x^p \sigma'_m$ tends to $j_x^p \sigma'$ and $j_x^k \sigma'_m = j_x^k \sigma_m$ (for instance, if $E(M)$ is a vector bundle, we can take $\sigma'_m = \sigma_m - \sigma + \sigma'$). Since σ_m are in W, so are σ'_m. As D is of order k on W, we have $D\sigma'_m(x) = D\sigma_m(x)$ and, by its continuity, we get $D\sigma'(x) = D\sigma(x)$, Q.E.D.

Let $C^\infty(M)$ be the set of germs of C^∞-functions on M; and let $\varepsilon(M)$ be the set of germs of functions with non-zero differential. Any two germs f, g from $\varepsilon(M)$ are $\Phi(M)$-equivalent, that is, there exists a germ φ of local diffeomorphism of M, such that $g = \varphi_* f = f \circ \varphi^{-1}$. The subset $\varepsilon(M)$ is evidently 1-sufficient in $C^\infty(M)$. Moreover, $j^\infty \varepsilon(M)$ is dense in $j^\infty C^\infty(M)$: if $df(x) = 0$ we may define, in local coordinates, the sequence of functions $f_m = f + x^1/m$ whose germs at x are in $\varepsilon(M)$, and we see that $j_x^\infty f_m \to j_x^\infty f$.

PROPOSITION III.6. *Any n.d. operator D of rank $s \geqslant 1$ defined on $\varepsilon(M)$ is of order $\leqslant s$. If, moreover, D is defined and continuous on $C^\infty(M)$, then it is of order $\leqslant s$ on $C^\infty(M)$.*

This follows immediately from Lemma II.6, Remark III.3 and Theorem III.5.

Let $V(M)$ denote the set of germs of local vector fields on M which are nowhere zero. Recall that such vector fields are integrable and any two of their germs are $\Phi(M)$-equivalent. Moreover, $j^\infty V(M)$ is dense in $J^\infty TM$. Similarly as for functions we obtain the following:

PROPOSITION III.7. *Any n.d. operator of rank s, defined on $V(M)$, is of order $\leqslant s-1$. If it is defined and continuous on TM, then it is of order $\leqslant s-1$ on TM.*

COROLLARY III.8. *Any continuous (in particular linear or polynomial) lifting of functions or vector fields to a natural bundle of order s is of order not greater than s.*

For functions this is immediate consequence of Proposition III.6; for vector fields, notice that it is a natural operator valued in $TE(M)$, so it has rank $s+1$, and we apply Proposition III.7.

References

[1] D. B. A. Epstein, *Natural tensors on Riemannian manifolds*, J. Diff. Geom. 10 (1975), 631–645.

[2] J. Gancarzewicz, *Liftings of functions and vector fields to natural bundles*, Inst. Math. Pol. Acad. Sci. (preprint, No. 207).

[3] H. Gollek, *Finitely determined germs of geometric objects*, Proc. Conf. Diff. Geom., Nove Mesto, September 1980.

[4] R. S. Palais and C. L. Terng, *Natural bundles have finite order*, Topology 16 (1978), 271–277.

[5] C. L. Terng, *Natural vector bundles and natural differential operators*, Ann. Math. J. 100 (1978), 775–828.

[6] A. Zajtz, *Canonical forms of geometric objects*, Potsdamer Forschungen, Reihe B Heft 43 (1984), 65–72.

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