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On the application of the Fourier–Legendre series for the Laplace equation

1. Let r, s, t be the spherical coordinates of points, $Q = \{(r, s, t)\colon 0 \leqslant r < 1;\ 0 \leqslant s \leqslant \pi;\ 0 \leqslant t < 2\pi\}$ and $\bar{Q} = \{(r, s, t)\colon 0 \leqslant r \leqslant 1;\ 0 \leqslant s \leqslant \pi;\ 0 \leqslant t < 2\pi\}$.

Denote by $X^m(Q)$ (m is a non-negative integer) the class of all real functions defined in Q, having the values independent of t and having the partial derivatives of the order $\leqslant m$ continuous in Q. Analogously will be interpreted the symbol $X^m(\bar{Q})$.

Let Δ be the Laplace operator, i.e.,

$$\Delta = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial s^2}+\frac{1}{r^2 \tan s}\frac{\partial}{\partial s}+\frac{1}{r^2 \sin^2 s}\frac{\partial^2}{\partial t^2},$$

$$\text{and} \qquad \Delta^n u = \Delta(\Delta^{n-1} u)$$

for $n = 2, 3, \ldots$ and $u \in X^\infty(Q)$ ($\Delta^1 \equiv \Delta$).

In this paper we shall give a solution of the Dirichlet problem for the equation $\Delta^n u(r, s, t) = 0$ in $\bar{Q}$. We shall construct the function u such that $u \in X^n(\bar{Q})$ ($n \geqslant 1$), $\Delta^{n+1} u(r, s) = 0$ in Q, $u(1, s) = f(s)$ and $(\partial^p u/\partial r^p)_{r=1} = 0$ for $s \in \langle 0, \pi\rangle$ and $p = 1, \ldots, n$, where f is a fixed function continuous in $\langle 0, \pi\rangle$.

The solution of Dirichlet's problem for the equation $\Delta u(r, s) = 0$ in $\bar{Q}$ was given in [3], p. 256. This problem for the Laplace equation of the order n in the unit disc was investigated in [1].

First we shall give some properties of the operator Δ^n.

Using the mathematical induction, we can prove

LEMMA 1. *If* $u \in X^{\infty}(Q)$ *and* $n = 1, 2, \ldots$, *then*

$$\Delta^n\left(r\frac{\partial u}{\partial r}\right) = r\frac{\partial}{\partial r}\Delta^n u(r, s) + 2n\Delta^n u(r, s);$$

$$\Delta^n\left(r^3\frac{\partial u}{\partial r}\right) = r\frac{\partial}{\partial r}\Delta^{n-1}\left(6u(r, s) + r^2\Delta u(r, s) + 4r\frac{\partial u}{\partial r}\right) + $$

$$+2(n-1)\Delta^{n-1}\left(6u(r, s) + r^2\Delta u(r, s) + 4r\frac{\partial u}{\partial r}\right)$$

and

$$\Delta^n(r^2\Delta u(r, s)) = r^2\Delta^{n+1}u(r, s) + 4nr\frac{\partial}{\partial r}\Delta^n u(r, s) + (6n + 4n(n-1))\Delta^n u(r, s)$$

for $(r, s, t) \in Q$.

Lemma 1 and linearity of the operator Δ^n yield

LEMMA 2. *Suppose that* $u \in X^{2n+2}(Q)$, $n \geqslant 1$. *If* $\Delta^n u(r, s) = 0$ *in* Q, *then the function* v,

$$v(r, s) = u(r, s) + \frac{r - r^3}{2(n+1)}\frac{\partial}{\partial r}u(r, s), \tag{1}$$

satisfies the equation $\Delta^{n+1}v(r, s) = 0$ *in* Q.

It is easy to verify the successive auxiliary result, too.

LEMMA 3. *Suppose that* $u \in X^{n+2}(\bar{Q})$, $n \geqslant 1$. *If* $(\partial^p u/\partial r^p)_{r=1} = 0$ *for* $s \in \langle 0, \pi\rangle$ *and* $p = 1, \ldots, n$, *then* v *defined by* (1) *satisfies the condition* $(\partial^p v/\partial r^p)_{r=1} = 0$ *for* $s \in \langle 0, \pi\rangle$ *and* $p = 1, \ldots, n+1$.

2. Let $L^2\langle -1, 1\rangle$ be the class of all real functions Lebesgue-integrable with 2-nd power in $\langle -1, 1\rangle$. Let Lip α, $0 < \alpha \leqslant 1$, be the set of functions $f \in L^2\langle -1, 1\rangle$ such that $|f(x) - f(y)| \leqslant M|x-y|^{\alpha}$ for $x, y \in \langle -1, 1\rangle$ ($M = \text{const} > 0$; see [4], p. 33).

The symbol $E_n^{(2)}(f)$ will denote the best approximation of $f \in L^2\langle -1, 1\rangle$ by algebraic polynomials of the order $\leqslant n$ ([5], p. 41).

Let

$$\sum_{n=0}^{\infty} c_n(f)P_n(x) \quad (x \in \langle -1, 1\rangle) \tag{2}$$

be the Fourier–Legendre series of $f \in L^2\langle -1, 1\rangle$, i.e.,

(3) $$P_n(x) = \frac{1}{n!\,2^n}\,\frac{d^n}{dx^n}(x^2-1)^n$$

and

$$c_n(f) = \frac{2n+1}{2}\int_{-1}^{1} f(x)\,P_n(x)\,dx \qquad \text{for } n = 0, 1, \ldots$$

([3], p. 64, 74).

It is known ([4], p. 26, 153, 386) that if f has the derivative $f^{(p)} \in \operatorname{Lip}\alpha$ (p is an integer $\geqslant 1$; $\alpha \in (0, 1\rangle$), then

(4) $$\max_{x\in\langle -1,1\rangle} \Big|f(x) - \sum_{k=0}^{n} c_k(f)\,P_k(x)\Big| \leqslant M_1\, n^{-p-\alpha+1/2}$$

and

$$\Big(\int_{-1}^{1} \big(f(x) - \sum_{k=0}^{n} c_k(f)\,P_k(x)\big)^2\,dx\Big)^{1/2} = E_n^{(2)}(f) \leqslant M_2\, n^{-p-\alpha}$$

(M_1, $M_2 = \text{const} > 0$). Arguing as in [2], p. 391, we obtain

LEMMA 4. *If f has the derivative $f^{(p)} \in \operatorname{Lip}\alpha$ ($p \geqslant 1$, $\alpha \in (0, 1\rangle$), then*

$$\sum_{k=1}^{\infty} k^{p-1}\,|c_k(f)| < \infty.$$

3. Let as in [1]:

(5) $$D^0(r^k) = r^k; \qquad D^n(r^k) = D^{n-1}(r^k) + \frac{r-r^3}{2n}\,\frac{d}{dr}\,D^{n-1}(r^k)$$

for $k = 0, 1, \ldots$, $n = 1, 2, \ldots$ and $r \in \langle 0, 1\rangle$.

Using the mathematical induction, we can prove

LEMMA 5. *If $k = 0, 1, \ldots$ and $n = 1, 2, \ldots$, then*

(6) $$D^n(r^k) = r^k + \sum_{q=1}^{n} W_q(r; n)\,\frac{d^q}{dr^q}\,r^k \qquad (r \in \langle 0, 1\rangle),$$

where W_q are some algebraic polynomials with coefficients depending on n only and such that

$$\left(\frac{d^p}{dr^p}\,W_q(r; n)\right)_{r=1} = \begin{cases} 0 & \text{if } p \neq q, \\ (-1)^p & \text{if } p = q \end{cases}$$

for $p = 0, 1, \ldots, n$ and $q = 1, \ldots, n$ (see [1]).

In [1] was given the following inequality:

$$|D^n(r^k - r^{k+1})| \leqslant M_3(n)(1-r)^{n+1}(k+1)^n r^k \tag{7}$$

($k, n = 0, 1, \ldots$; $r \in \langle 0, 1\rangle$) with $M_3(n) = \text{const} > 0$.

4. Let $f \in L^2\langle -1, 1\rangle$. Then the function $H_n(f)$,

$$H_n(r, x; f) = \sum_{k=0}^{\infty} D^n(r^k) c_k(f) P_k(x)$$

with P_k, $c_k(f)$ and D^n given by (2), (3) and (5), exists for $r \in \langle 0, 1)$ and $x \in \langle -1, 1\rangle$.

By the Abel transformation and the equality $\sum_{k=0}^{\infty} D^n(r^k - r^{k+1}) = 1$ for $r \in \langle 0, 1)$ and $n = 0, 1, \ldots$,

$$H_n(r, x; f) - f(x) = \sum_{k=0}^{\infty} D^n(r^k - r^{k+1})\Big(\sum_{q=0}^{k} c_q(f) P_q(x) - f(x)\Big) \tag{8}$$

($r \in \langle 0, 1)$, $x \in \langle -1, 1\rangle$) for $f \in L^2_{\langle -1,1\rangle}$.

Formula (8) and inequalities (4) and (7) lead to

THEOREM 1. *If f has the derivative $f^{(p)}$ (p is a integer $\geqslant 1$) continuous in $\langle -1, 1\rangle$ and $f^{(p)} \in \operatorname{Lip} \alpha$, $\alpha \in (0, 1\rangle$, then*

$$\max_{-1 \leqslant x \leqslant 1} |f(x) - H_n(r, x; f)| \leqslant M_4^* \begin{cases} (1-r)^{n+1} & \text{if } p+\alpha-\frac{3}{2} > n, \\ (1-r)^{n+1}|\log(1-r)| & \text{if } p+\alpha-\frac{3}{2} = n \end{cases}$$

for $r \in \langle 0, 1)$ ($M_4^* = M_4(n, p, \alpha) = \text{const} > 0$).

5. In this part we shall consider functions defined in $\langle 0, \pi\rangle$. In this case the symbol $\operatorname{Lip} \alpha$, $\alpha \in (0, 1\rangle$, will be defined analogously as in Section 2.

THEOREM 2. *Suppose that f has the derivative $f^{(2n+1)}$, with an integer $n \geqslant 0$, continuous in $\langle 0, \pi\rangle$ and $f^{(2n+1)} \in \operatorname{Lip} \alpha$, $\alpha \in (0, 1\rangle$. Then the function $U_n(f)$ defined by formula*

$$U_n(r, s; f) = \sum_{k=0}^{\infty} D^n(r^k) d_k(f) P_k(\cos s), \tag{9}$$

where P_k, D^n are defined by (3), (5) *and*

$$d_k(f) = \frac{2k+1}{2} \int_0^{\pi} f(s) P_k(\cos s) \sin s \, ds,$$

satisfies the conditions:

1° $U_n(f) \in X^n(\bar{Q})$,

2° $\Delta^{n+1} U_n(r, s; f) = 0$ *for* $(r, s, t) \in Q$,

3° $U_n(1, s; f) = f(s)$ *for* $s \in \langle 0, \pi \rangle$,

4° $\left(\frac{\partial^p}{\partial r^p} U_n(r, s; f)\right)_{r=1} = 0$ *for* $s \in \langle 0, \pi \rangle$ *and* $p = 1, \ldots, n$.

Proof. It is known ([3], p. 256) that the function $U_0(f)$ has properties 1°–3° if $\frac{df}{ds} \in \operatorname{Lip} \alpha$, $\alpha \in (0, 1\rangle$.

The inequality

$$|P_k(x)| \leqslant 1 \qquad (x \in \langle -1, 1 \rangle;\ k = 0, 1, \ldots)$$

([3], p. 69) and the Bernstein inequality ([5], p. 232) imply the estimation

$$\left|\frac{\partial^q}{\partial s^q} P_k(\cos s)\right| \leqslant (k+1)^q \qquad (s \in \langle 0, \pi \rangle;\ k, q = 0, 1, \ldots).$$

Hence, by Lemma 4, $U_0(f) \in X^\infty(Q)$ and $U_0(f) \in X^{2n}(\bar{Q})$.

By (6) and (9),

$$U_m(r, s; f) = U_0(r, s; f) + \sum_{q=1}^{m} W_q(r; m) \frac{\partial^q}{\partial r^q} U_0(r, s; f) \tag{10}$$

for $(r, s, t) \in \bar{Q}$ and $1 \leqslant m \leqslant n$.

Consequently,

$$U_m(f) \in X^\infty(Q) \quad \text{and} \quad U_m(f) \in X^{2n-m}(\bar{Q}) \tag{11}$$

for $0 \leqslant m \leqslant n$.

Clearly, by (5) and (9),

$$U_m(r, s; f) = U_{m-1}(r, s; f) + \frac{r - r^3}{2m} \frac{\partial}{\partial r} U_{m-1}(r, s; f) \tag{12}$$

for $(r, s, t) \in \bar{Q}$ and $1 \leqslant m \leqslant n$.

Applying the properties of $U_0(f)$, (11)–(12) and Lemma 2, we obtain 2° for $U_n(f)$.

By Lemma 5, (9) and (4),

$$U_n(1, s; f) = \sum_{k=0}^{\infty} d_k(f) P_k(\cos s) = f(s)$$

for $s \in \langle 0, \pi \rangle$. Hence, we have 3°.

Formulae (12) and (11) imply the equality $\left(\frac{\partial}{\partial r} U_1(r, s; f)\right)_{r=1} = 0$ for $s \in \langle 0, \pi \rangle$. Now, by (11)–(12), we can apply Lemma 3. For $U_n(f)$ we obtain 4°. Thus the proof is completed.

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