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On totally umbilical submanifolds in nearly Kählerian manifolds

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Abstract. Some properties of generic totally umbilical submanifolds in Kählerian and nearly Kählerian manifolds are proved.

Preliminaries. Let M' be an almost Hermitian manifold with an almost complex structure J' and an almost Hermitian metric $(\, , \,)$. Let M be a real submanifold of M' such that $\dim TM \cap J'TM$ is constant on M, i.e., M is a generic submanifold of M'. Since M is generic, $\mathscr{D} = TM \cap J'TM$ is a distribution on M. It is called the *holomorphic distribution* on M. Denote by:

N – the normal bundle of TM in $TM'|_M$,
$\mathscr{D}^\perp$ – the orthogonal complement to $\mathscr{D}$ in TM,
H – the holomorphic extension of TM in $TM'|_M$, i.e., $H = TM + J'TM$,
$\mathscr{D}_0$ – the orthogonal complement to TM in H,
NH – the orthogonal complement to H in $TM'|_M$,
p – the projection onto TM in $TM'|_M = TM \oplus N$,
n – the projection onto N in $TM'|_M = TM \oplus N$,

$$P = p \circ J'|_{TM}, \quad \psi = n \circ J'|_{TM},$$

∇, ∇' – the Riemannian connections on M and M', respectively,
D – the induced connection in N,
α – the second fundamental form of M in M',
A – the second fundamental tensor of M in M',

$$K_1(X, Y) = pJ'\alpha(X, Y) + A_{\psi Y} X,$$

$$K_2(X, Y) = nJ'\alpha(X, Y) - \alpha(X, PY) \quad \text{for } X, Y \subset T_x M, \ x \in M.$$

P is a (1, 1)-tensor field on M and ψ is a $\mathscr{D}_0$-valued 1-form on M. In the sequel we shall use the fact that $\ker P \cap \operatorname{im} P = \{0\}$, [6]. We put

$$\nabla_X \psi Y = D_X \psi Y - \psi \nabla_X Y$$

for any vector fields X, Y tangent to M. It is easy to check that if M' is Kählerian; then

$$(\nabla_X P) Y = K_1(X, Y) \tag{0.1}$$

and

(0.2) $$(\nabla_X \psi) Y = K_2(X, Y) \quad \text{for any } X, Y \in T_x M, \ x \in M.$$

Similarly, if M' is nearly Kählerian, i.e., $(\nabla'_X J') Y + (\nabla'_Y J') X = 0$ for any $X, Y \in T_x M'$, $x \in M'$, then

(0.3) $$(\nabla_X P) Y + (\nabla_Y P) X = K_1(X, Y) + K_1(Y, X)$$

and

(0.4) $$(\nabla_X \psi) Y + (\nabla_Y \psi) X = K_2(X, Y) + K_2(Y, X).$$

If M is a generic submanifold, then we can define an F-structure in the normal bundle in the following way

$$FX = \begin{cases} 0 & \text{for } X \in \mathscr{D}_0, \\ J'X & \text{for } X \in NH; \end{cases}$$

F is called the *induced F-structure* in the normal bundle. A generic submanifold is called

(i) *holomorphic* if $\mathscr{D}^\perp = 0$,

(ii) *purely real* if $\mathscr{D} = 0$,

(iii) a *CR-submanifold* if $J'\mathscr{D}_0 \subset \mathscr{D}^\perp$,

(iv) *totally real* if it is a purely real CR-submanifold,

(v) *proper* if it is neither holomorphic nor purely real.

If M is a totally umbilical submanifold, then

$$\alpha(X, Y) = (X, Y) L,$$

where L is the mean curvature vector. A totally umbilical submanifold is said to be *totally geodesic* if $L = 0$. If M is a totally umbilical submanifold, then, for any normal vector ξ, A_ξ is a multiple of identity. A normal vector ξ will be called *geodesic* if $A_\xi = 0$. A set of normal vectors will be called *geodesic* if it consists of geodesic vectors only. If M is a totally umbilical submanifold of M' and ξ is a normal vector at $x \in M$, then we define $\lambda(\xi)$ by the formula

$$A_\xi = \lambda(\xi) I_x,$$

where I_x is the identity endomorphism of $T_x M$.

1. Results. A totally umbilical generic submanifold of an almost Hermitian manifold is obviously mixed totally geodesic, i.e., $\alpha(X, Y) = 0$ for $X \in \mathscr{D}$ and $Y \in \mathscr{D}^\perp$. By Proposition 2.15 from [6], we obtain

PROPOSITION 1. *If M is a totally umbilical generic submanifold of a Kählerian manifold, then the distribution $\mathscr{D}^\perp$ is integrable and its leaves are totally geodesic in M.*

In the case where M' is nearly Kählerian, we have the following proposition:

PROPOSITION 2. *Let M be a totally umbilical CR-submanifold of a nearly Kählerian manifold. Then the distribution $\mathscr{D}^{\perp}$ is integrable and its leaves are totally geodesic in M.*

Proof. Let X and Y be vector fields on M belonging to the distributions $\mathscr{D}^{\perp}$ and $\mathscr{D}$, respectively. Using the equality $(\nabla'_X J')Y+(\nabla'_Y J')X = 0$ and the fact that M is CR and totally umbilical, we obtain

$$\nabla_X J' Y - A_{\psi X} Y + D_Y \psi X = J'(\nabla_X Y + \nabla_Y X)$$

and

$$\nabla_X J' Y \in TM, \qquad A_{\psi X} Y \in \mathscr{D}, \qquad D_Y \psi X \in N.$$

It follows that

$$\nabla_X J' Y - A_{\psi X} Y = P(\nabla_X Y + \nabla_Y X) \in \mathscr{D},$$

and consequently $\nabla_X J' Y \in \mathscr{D}$. Therefore, $\nabla_X Y \in \mathscr{D}$ for any vector fields X, Y belonging to $\mathscr{D}^{\perp}$ and $\mathscr{D}$, respectively. If, moreover, Z is a vector field on M belonging to $\mathscr{D}^{\perp}$, then

$$0 = Z(X, Y) = (\nabla_Z X, Y)+(X, \nabla_Z Y) = (\nabla_Z X, Y).$$

Consequently, $\nabla_Z X \in \mathscr{D}^{\perp}$ for any vector fields Z, $X \in \mathscr{D}^{\perp}$. It means that $\mathscr{D}^{\perp}$ is involutive and its leaves are totally geodesic in M.

Now we shall prove the following proposition.

PROPOSITION 3. *Let M' be a nearly Kählerian manifold and let M be a generic totally umbilical submanifold in M'. If $\mathscr{D} \neq 0$, then $L \in \mathscr{D}_0$.*

Proof. Since M' is nearly Kählerian, we have

$$(1.1) \qquad (\nabla_X \psi) X = nJ'\alpha(X, X) - \alpha(X, PX)$$

for any $X \in TM$. If X is a vector field belonging to $\mathscr{D}$, then $\psi X = 0$, and consequently

$$(1.2) \qquad (\nabla_X \psi) X = -\psi \nabla_X X = -nJ'(\nabla'_X X - \alpha(X, X)).$$

By comparing (1.1) and (1.2), we obtain $\alpha(X, J'X) = nJ' \nabla'_X X$; M is totally umbilical, so $\alpha(X, J'X) = 0$. It means that $J'\nabla'_X X \in TM$, i.e., $J'\nabla_X X + J'\alpha(X, X) \in TM$. Since $J'\nabla_X X \in H$, we have $J'\alpha(X, X) \in H$. Consequently, $\alpha(X, X) \in \mathscr{D}_0$. The proof is completed.

By this proposition and by Proposition 2.9 from [6] we obtain the following corollary.

COROLLARY 4. *If M' is Kählerian and M is a totally umbilical generic submanifold of M' with $\dim \mathscr{D} > 0$, then $DF = 0$.*

Assume now that M is a generic submanifold with $\dim \mathscr{D}^{\perp} = 1$. Then M is obviously a CR-submanifold and we have the following lemma due to B-Y Chen [3].

LEMMA 5. *Let M be a totally umbilical CR-submanifold of a Kählerian manifold with* $\dim \mathscr{D}^{\perp} = 1$. *Then*

(a) $D_X L \in NH$ *for* $X \in \mathscr{D}^{\perp}$.

(b) *If* $\dim M \geqslant 5$, *then* $D_X L = 0$ *for* $X \in \mathscr{D}$.

By using this lemma, Proposition 3 and Corollary 4 we get the following generalization of Theorem 4.2 from [4].

PROPOSITION 6. *Let M be a totally umbilical CR-submanifold with* $\dim \mathscr{D}^{\perp} = 1$ *in a Kählerian manifold M'. If* $\dim M \geqslant 5$, *then the mean curvature vector L is parallel with respect to $\mathscr{D}$.*

Proof. By (b) of Lemma 5 it is sufficient to verify that $D_X L = 0$ for $X \in \mathscr{D}^{\perp}$. But, by Proposition 3 and Corollary 4, $D_X L \in \mathscr{D}_0$. Hence, by (a) of Lemma 5, we obtain $D_X L = 0$.

Now we give the following proposition.

PROPOSITION 7. *Let M be a totally umbilical generic submanifold of a Kählerian manifold. If the distribution $\mathscr{D}$ is non-zero and integrable, then M is totally geodesic.*

Proof. Since the distribution $\mathscr{D}$ is integrable, we have

$$(\nabla_X \psi) Y - (\nabla_Y \psi) X = -\psi \nabla_X Y + \psi \nabla_Y X = -\psi [X, Y] = 0$$

for any vector fields X and Y belonging to $\mathscr{D}$. Consequently,

$$(\nabla_X \psi) Y = (\nabla_Y \psi) X \quad \text{for any } X, Y \text{ belonging to } \mathscr{D}. \tag{1.3}$$

From (0.2) we find that $\alpha(X, J'Y) = \alpha(J'X, Y)$. It follows that, for any X, $(X, X)L = -(X, X)L$, i.e., $L = 0$. The proof is completed.

In the "nearly Kählerian" case we have the proposition.

PROPOSITION 8. *Let M be a totally umbilical CR-submanifold of a nearly Kählerian manifold and the distribution $\mathscr{D}$ is non-zero and integrable. Then M is totally geodesic.*

Proof. Let $X, Y \in \mathscr{D}$. Since $\mathscr{D}$ is integrable, $(\nabla_X \psi) Y = (\nabla_Y \psi) X$ (cf. (1.3)). Therefore, by using equality (0.4), we have $(\nabla_X \psi) Y = (X, Y) nJ'L$. We have also the following equalities

$$(\nabla_X \psi) Y = -\psi \nabla_X Y = -nJ' \nabla_X Y = (X, Y) nJ'L - nJ' \nabla_X Y.$$

Therefore, $nJ' \nabla'_X Y = 0$. It means that $J' \nabla'_X Y \in TM$, i.e., $J' \nabla_X Y + J'\alpha(X, Y) \in TM$. By Proposition 3, $\alpha(X, Y) \in \mathscr{D}_0$. M is a CR-submanifold, so $J'\alpha(X, Y) \in TM$. Consequently, $J' \nabla_X Y \in TM$, i.e., $\nabla_X Y \in \mathscr{D}$. Let X be a unit vector field belonging to $\mathscr{D}$. Since M is nearly Kählerian,

$$J'(\alpha(X, X) + \nabla_X X) = \alpha(X, J'X) + \nabla_X J'X. \tag{1.4}$$

The vector fields $\nabla_X X$, $\nabla_X J'X$ belong to $\mathscr{D}$ and $J'\alpha(X, X) = J'L \in \mathscr{D}^{\perp}$. Hence, by equality (1.4), $L = 0$.

The above statement generalizes the following well-known fact.

COROLLARY 9. *A holomorphic totally umbilical submanifold of a nearly Kählerian manifold is totally geodesic.*

Now we shall prove the following proposition.

PROPOSITION 10. *Let M be a totally umbilical submanifold of a Kählerian manifold. If* $\dim\ker P_x = \mathrm{const} > 1$, *then* $J'(\ker P)$ *is geodesic and* $\ker P$ *is integrable.*

Proof. By using (0.1), we find

$$(\nabla_X P) Y - (\nabla_Y P) X = A_{\psi Y} X - A_{\psi X} Y = \lambda(\psi Y) X - \lambda(\psi X) Y.$$

Let X and Y be arbitrary vector fields belonging to $\ker P$. Then

$$(\nabla_X P) Y - (\nabla_Y P) X = -P[X, Y].$$

Therefore,

$$P([Y, X]) = \lambda(\psi Y) X - \lambda(\psi X) Y = \lambda(J'Y) X - \lambda(J'X) Y.$$

The left-hand side of this equality belongs to $\mathrm{im} P$, the right-hand side to $\ker P$, so both are zero. In particular, if X and Y are linearly independent at x, then $\lambda(J'X_x) = 0$. Integrability of $\ker P$ follows from the fact that $P([X, Y]) = 0$ and from the Frobenius theorem.

In the "nearly Kählerian" case we have the following proposition.

PROPOSITION 11. *Let M be a totally umbilical CR-submanifold of a nearly Kählerian manifold M'. If* $\dim \mathscr{D}^{\perp} > 1$, *then $\mathscr{D}_0$ is geodesic.*

Proof. Formula (0.3) implies

$$(1.5) \qquad (\nabla_X P) Y + (\nabla_Y P) X = 2(X, Y) pJ'L + \lambda(\psi Y) X + \lambda(\psi X) Y.$$

By taking vector fields X, Y, Z belonging to $\mathscr{D}^{\perp} = \ker P$, we obtain

$$(1.6) \qquad ((\nabla_X P) Y, Z) = (\nabla_X PY, Z) - (P\nabla_X Y, Z) = 0.$$

Therefore, by (1.5) and (1.6), we get

$$(2(X, Y) pJ'Y, Z) = -(\lambda(J'Y) X, Z) - (\lambda(J'X) Y, Z).$$

Let $Y \neq 0$ and let $0 \neq X = Z$ be perpendicular to Y. Then the last equality implies

$$\lambda(J'Y)(X, X) = 0, \quad \text{i.e.,} \quad \lambda(J'Y) = 0.$$

This completes the proof.

As an immediate consequence of this proposition and Proposition 3, we obtain the following corollary.

COROLLARY 12. *Let M be a totally umbilical CR-submanifold of a nearly Kählerian manifold. If* $\dim \mathscr{D} > 0$ *and* $\dim \mathscr{D}^{\perp} > 1$, *then M is totally geodesic.*

Corollary 12 is a "nearly Kählerian" analogue of Theorem 2.1 from [2] or Theorem 6.1 from [7], which state that a totally umbilical proper *CR*-submanifold of a Kählerian manifold is totally geodesic if $\dim \mathscr{D}^{\perp} > 1$.

By using Proposition 11 we also obtain the following corollary.

COROLLARY 13. *Let M be a totally umbilical CR-submanifold of a nearly Kählerian manifold M. If $TM + J'TM = TM'|_M$ and* $\dim \mathscr{D}^{\perp} > 1$, *then M is totally geodesic.*

The above statement is a generalization of the following assertion due to Ludden, Okumura and Yano [5].

Let M be an n-dimensional, $n > 1$, totally real submanifold of a nearly Kählerian 2n-dimensional manifold M'. If M is totally umbilical, then M is totally geodesic.

The following theorem is also known, [7].

Let M be a totally umbilical, totally real submanifold of a Kählerian manifold. If the induced F-structure in the normal bundle is parallel and $\dim M > 1$, *then M is totally geodesic.*

By virtue of Proposition 11 we obtain the following generalization of this theorem.

COROLLARY 14. *Let M be a totally umbilical totally real submanifold in a nearly Kählerian manifold. If the distribution NH (equiv. $\mathscr{D}_0$) is parallel with respect to D and* $\dim M > 1$, *then M is totally geodesic.*

Proof. By Proposition 11 it is sufficient to check that $L \in \mathscr{D}_0$. With the aim of proving this, we choose tangent vector fields X, Z on M and a normal vector field ξ belonging to NH. By the assumption, $(D_X J'Z, \xi) = 0$. But $(D_X J'Z, \xi) = (\nabla'_X J'Z, \xi)$, i.e., $\nabla'_X J'Z \in H$. If $Z = X \neq 0$, then $\nabla'_X J'X = J'\nabla'_X X = J'\nabla_X X + J'\alpha(X, X)$. Of course, $J'\nabla_X X \in H$, so $J'\alpha(X, X) \in H$. The bundle H is J'-invariant; hence $\alpha(X, X) \in H$ and consequently $\alpha(X, X) \in \mathscr{D}_0$. The proof is completed.

Notice that this theorem is not true in the case where M is assumed to be purely real only even if M' is Kählerian. In fact we have the following example. Let $M = \boldsymbol{R}^{2n}$, $n \geqslant 4$ and let J' be the canonical complex structure on $\boldsymbol{R}^{2n}$. Let V be the linear subspace in $\boldsymbol{R}^{2n}$ spanned by vectors $e_1, e_2, e_3, J'e_1$, where $(e_1, \ldots, e_n, J'e_1, \ldots, J'e_n)$ is the canonical base in $\boldsymbol{R}^{2n}$. Consider the 3-dimensional unit sphere S^3 in V defined by

$$S^3 = \{x = (x_1, x_2, x_3, 0, \ldots, 0, y_1, 0, \ldots, 0) \in \boldsymbol{R}^{2n}: \; x_1^2 + x_2^2 + x_3^2 + y_1^2 = 1\}.$$

Let M be an open submanifold in S^3 defined by

$$M = \{x \in S^3 \colon \; x_1 > 0, \; x_2 > 0, \; x_3 > y_1 > 0\}.$$

Of course, M is totally umbilical but not totally geodesic in M'. M is purely real. Indeed, choose a linear base E_1, E_2, E_3 in $T_x M$ as follows:

$$E_1 = x_2 e_1 - x_1 e_2, \quad E_2 = x_3 e_2 - x_2 e_3, \quad E_3 = y_1 e_3 - x_3 J' e_1.$$

Let $X \in T_x M \cap J' T_x M$. We have

$$X = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3, \quad J' X = \gamma_1 E_1 + \gamma_2 E_2 + \gamma_3 E_3$$

for some real numbers $\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, \gamma_3$. Hence

$$J' X = \alpha_1 x_2 J' e_1 + (\alpha_2 x_3 - \alpha_1 x_1) J' e_2 + (\alpha_2 y_1 - \alpha_2 x_2) J' e_3 + \alpha_3 x_3 e_1,$$

and

$$J' X = \gamma_1 x_2 e_1 + (\alpha_2 x_3 - \gamma_1 x_1) e_2 + (\gamma_3 y_1 - \gamma_2 x_2) e_3 - \gamma_3 x_3 J' e_1.$$

By comparing these equalities and using the fact that x_1, x_2, x_3, y_1 are positive, we obtain $X = 0$. M is not totally real, because

$$(J' E_1, E_3) = -x_3 x_2 (J' e_1, J' e_1) = -x_2 x_3 \neq 0,$$

where (,) denotes the standard scalar product in $\boldsymbol{R}^{2n}$. It is clear that NH_x is a space spanned by $e_4, \ldots, e_n, J' e_4, \ldots, J' e_n$ for every $x \in M$. It is also obvious that α has values in V, whence $\alpha(X, Y) \in \mathscr{D}_{0_x}$ for any $X, Y \in T_x M$, $x \in M$. Consequently, the induced F-structure in the normal bundle is parallel.

Remark. Corollary 14 was proved in [1] under stronger assumption, i.e., F was assumed to be parallel.

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