ANNALES
POLONICI MATHEMATICI
XXXII (1976)

On a class of starlike functions

by SANFORD S. MILLER* (Brockport, N. Y.)

Abstract. Let $f(z)$ be a regular function defined in the unit disc for which

$$\mathrm{Re}\,(zf'(z)/f(z))^{\mu}(zf''(z)/f'(z)+1)^{\nu} > 0,$$

for μ and ν real. The author shows that for certain values of μ and ν the function is univalent and starlike.

We wish to define some new classes of regular functions which will prove to be starlike.

DEFINITION. Let $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ be regular in the unit disc D, with $f(z)/z$, $f'(z)$, $zf''(z)/f'(z)+1 \neq 0$ for $z \in D$. If μ and ν are fixed real numbers and

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(\frac{zf''(z)}{f'(z)}+1\right)^{\nu} > 0 \tag{1}$$

for $z \in D$, where the powers appearing in (1) are meant as principal values, then we say that $f(z)$ *belongs to the class* $S(\mu, \nu)$.

This class of functions contains many classes of univalent functions. In fact, $S(1, 0) = S^*$, $S(0, 1) = C$, the class of convex functions, $S(\mu, 0)$ with $|\mu| \geqslant 1$ corresponds to strongly starlike functions [1, 3], $S(0, \nu)$ with $|\nu| \geqslant 1$ corresponds to strongly convex functions, and $S(1-\gamma, \gamma)$ with γ real corresponds to gamma-starlike functions [2]. We will show that for many more values of μ and ν condition (1) implies univalence and starlikeness.

Note that condition (1) is equivalent to the following condition:

$$\left|\mu \arg\left(\frac{zf'(z)}{f(z)}\right)+\nu \arg\left(\frac{zf''(z)}{f'(z)}+1\right)\right| < \frac{\pi}{2}. \tag{2}$$

---

* This work was carried out while the author was an I.R.E.X. Scholar in Poland.

In what follows we will have reference to the following region K of the (μ, ν)-plane:

$$K \equiv \{(\mu, \nu)\,|\,4n+1 \leqslant \mu+\nu \leqslant 4n+3,\ n \in I\} \cup \{(\mu, 0)\,|\,|\mu| \geqslant 1\} \cup \cup \{(0, \nu)\,|\,|\nu| \geqslant 1\},$$

where I is the set of integers. Region K is pictured below.

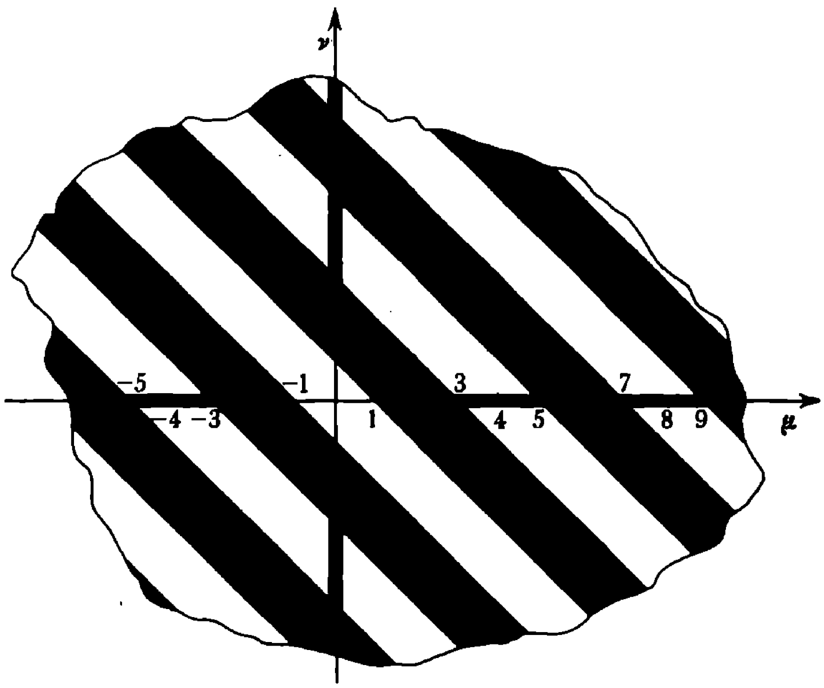

Fig. 1

We now show that if $(\mu, \nu) \in K$ and $f(z) \in S(\mu, \nu)$, then $f(z)$ is univalent and starlike.

THEOREM 1. *$S(\mu, \nu) \subset S^*$ if $(\mu, \nu) \in K$.*

Proof. The cases $|\mu| \geqslant 1$, $\nu = 0$ and $|\nu| \geqslant 0$, $\mu = 0$ are trivial and we only need to prove the result for the region

$$4n+1 \leqslant \mu+\nu \leqslant 4n+3, \tag{3}$$

where n is an integer. The technique we use is similar to that used in [2].

If $f(z) \in S(\mu, \nu)$ and if we set

$$\frac{1+w}{1-w} = \frac{zf'(z)}{f(z)} \tag{4}$$

for $z \in D$, then $w(0) = 0$, $w(z) \neq \pm 1$ and $w(z)$ is defined as a meromorphic function. To complete the proof of the theorem we need to show that

$|w(z)| < 1$ for $z \in D$. Let

$$w(z) = R(z)e^{i\Phi(z)} \quad \text{for } z = re^{i\theta},$$

and suppose that $z_0 = r_0 e^{i\theta_0}$ is a point of D such that

$$\max_{|z| \leq r_0} |w(z)| = |w(z_0)| = 1. \tag{5}$$

Then $\partial R(z_0)/\partial\theta = 0$, and since

$$\frac{zw'(z)}{w(z)} = \frac{\partial\Phi}{\partial\theta} - i\frac{1}{R}\frac{\partial R}{\partial\theta},$$

we must have

$$\frac{z_0 w'(z_0)}{w(z_0)} = \frac{\partial\Phi(z_0)}{\partial\theta}$$

and hence $z_0 w'(z_0)/w(z_0)$ is a real number. A simple geometric argument can show even more. If we assume $\partial\Phi(z_0)/\partial\theta < 0$, then $w(z)$ would be locally univalent at z_0 and this would lead to a contradiction of (5).

Thus we see that $\partial\Phi(z_0)/\partial\theta$ must be non-negative and so we set

$$\frac{z_0 w'(z_0)}{w(z_0)} \equiv B, \tag{6}$$

where $B \geq 0$.

Since $|w(z_0)| = 1$ and $w(z_0) \neq \pm 1$, we have

$$\frac{1+w(z_0)}{1-w(z_0)} = Ai, \tag{7}$$

where A is real and $A \neq 0$.

From (1) and (4) we obtain

$$\begin{aligned} \operatorname{Re} I(\mu, \nu, f(z)) &\equiv \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(\frac{zf''(z)}{f'(z)}+1\right)^{\nu} \\ &= \operatorname{Re}\left(\frac{1+w}{1-w}\right)^{\mu}\left(\frac{1+w}{1-w}+\frac{zw'}{w}\left[\frac{w}{1+w}+\frac{w}{1-w}\right]\right)^{\nu}. \end{aligned}$$

Thus at $z = z_0$, by using (6) and (7) we obtain

$$\operatorname{Re} I(\mu, \nu, f(z_0)) = \operatorname{Re}(Ai)^{\mu}\left(Ai+\frac{B}{2}\left[A+\frac{1}{A}\right]i\right)^{\nu}.$$

If we let $C = A + B[A+1/A]/2$, then since $B \geq 0$ and $A \neq 0$, we have $AC > 0$ and we obtain

$$\operatorname{Re} I(\mu, \nu, f(z_0)) = |A|^{\mu}|C|^{\nu}\cos(\mu+\nu)\pi/2.$$

Since $\mu+\nu$ is restricted by condition (3) we have $\operatorname{Re} I(\mu, \nu, f(z_0)) \leqslant 0$, and since this contradicts $f(z) \in S(\mu, \nu)$ we must have $|w(z)| < 1$, and thus $f(z) \in S^*$.

In Theorem 1 we proved the inclusion relationship $S(\mu, \nu) \subset S(1, 0)$. We can generalize this result as we do in the following theorem.

THEOREM 2. *If* $0 \leqslant t \leqslant 1$ *and* $(\mu, \nu) \in K$, *then* $S(\mu, \nu) \subset S((\mu-1)t+1, \nu t)$.

Proof. If $f(z) \in S(\mu, \nu)$, then

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(\frac{zf''(z)}{f'(z)}+1\right)^{\nu} = P_1(z), \tag{8}$$

where $P_1(0) = 1$ and $\operatorname{Re} P_1(z) > 0$. By Theorem 1

$$\frac{zf'(z)}{f(z)} = P_2(z), \tag{9}$$

where $P_2(0) = 1$ and $\operatorname{Re} P_2(z) > 0$. If we raise both sides of (8) to the t power and both sides of (9) to the $(1-t)$ power and multiply these two equations we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{(\mu-1)t+1}\left(\frac{zf''(z)}{f'(z)}+1\right)^{\nu t} = [P_1(z)]^t[P_2(z)]^{1-t} \equiv P_3(z).$$

Now $P_3(0) = 1$ and since $0 \leqslant t \leqslant 1$,

$$|\arg P_3(z)| \leqslant t\,|\arg P_1(z)| + (1-t)\,|\arg P_2(z)| \leqslant \pi/2.$$

Hence $\operatorname{Re} P_3(z) > 0$ and $f(z) \in S((\mu-1)t+1, \nu t)$.

Remarks. (i) Theorem 2 has the following geometric interpretation in terms of Fig. 1. If (μ, ν) is a point in the shaded region K, then all points on the line L between (μ, ν) and $(1, 0)$ will satisfy (1). Note that there may be points on L that are not in K and for such points, $S((\mu-1)t+1, \nu t)$ need not be a subclass of S^*. However, if we restrict (μ, ν) to the band $1 \leqslant \mu+\nu \leqslant 3$, then $L \subset K$ and $S((\mu-1)t+1, \nu t) \subset S^*$.

(ii) If $0 \leqslant \delta \leqslant \nu$ (or $\nu \leqslant \delta \leqslant 0$) and $(1, \nu) \in K$, then we have $S(1, \nu) \subset S(1, \delta)$.

(iii) If $0 \leqslant \delta \leqslant \nu$ (or $\nu \leqslant \delta \leqslant 0$), then $S(1-\nu, \nu) \subset S(1-\delta, \delta) \subset S^*$ for any real ν.

By comparing the coefficients of z and z^2 in equation (8) we obtain the following theorem:

THEOREM 3. *If* $f \in S(\mu, \nu)$, *then*

$$|a_2(\mu+2\nu)| \leqslant 2, \tag{i}$$

and

$$|a_3(4\mu+12\nu) - a_2^2(3\mu-\mu^2-4\mu\nu+12\nu-4\nu^2)| \leqslant 4. \tag{ii}$$

The author has not been able to prove that (i) and (ii) are sharp. They would be sharp if it is possible to show that the differential equation

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(\frac{zf''(z)}{f'(z)}+1\right)^{\nu}=\frac{1+z}{1-z}$$

with initial conditions $f(0) = 0$, $f'(0) = 1$ has a solution that is regular in D. This formidable task we leave to future work.

References

[1] D. A. Brannan and W. E. Kirwan, *On some classes of bounded univalent functions*, J. London Math. Soc. (2), 1 (1969), p. 431–443.

[2] Z. Lewandowski, S. Miller and E. Złotkiewicz, *Gamma-starlike functions*, Ann. Univ. Mariae Curie-Skłodowska (to appear).

[3] J. Stankiewicz, *On a family of starlike functions*, Ann. Univ. Mariae Curie-Skłodowska, Sec. A, 24 (1970), p. 175–181.

DEPARTMENT OF MATHEMATICS
STATE UNIVERSITY OF NEW YORK, BROCKPORT
NEW YORK, U.S.A.
INSTITUTE OF MATHEMATICS
MARIA CURIE-SKŁODOWSKA, LUBLIN

Reçu par la Rédaction le 9. 4. 1974