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On generalized solutions of some differential non-linear equations of order n

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Abstract. The paper contains some theorems on the existence and uniqueness of the solution of the equation

$$(*)\qquad x^{(n)} = f_0(t, x, x', \ldots, x^{(n-1)}) + \sum_{i=1}^{m} p_i(t) f_i(t, x, x', \ldots, x^{(n-2)})$$

in the class of all distributions for which $(n-1)$ derivatives $(n \geqslant 2)$ in the distribution sense are functions of finite variation in the interval (a, b).

In $(*)$ the function $f_0(t, x_0, \ldots, x_{n-1})$ is known and Lebesgue locally integrable in the set

$$D\colon\ a < t < b, \qquad -\infty < x_0, \ldots, x_{n-1} < \infty,$$

$p_i(t)$ for $i = 1, 2, \ldots, m$ are given measures in (a, b) (see [1], [2]), and $f_i(t, x_0, \ldots, x_{n-1})$ are continuous functions in D. The derivatives are understood in the distribution sense.

The principal results of this paper generalize some theorems for linear dfferential equations (see [6], [7], [8]).

In this note the sequential theory of distributions is used (see [5], [9]).

1. Let $p_r(t)$ for $r = 1, 2, \ldots, m$ be given measures defined in an interval (a, b), and $f_0(t, x_0, \ldots, x_{n-1})$ a locally integrable function in the set

$$D\colon\ a < t < b, \qquad -\infty < x_0, \ldots, x_{n-1} < \infty.$$

Moreover, let $f_r(t, x_0, \ldots, x_{n-2})$ be given continuous functions in D. We put

$$g(t, x, x', \ldots, x^{(n-1)}) = f_0(t, x, x', \ldots, x^{(n-1)}) + \sum_{r=1}^{m} p_r(t) f_r(t, x, x', \ldots, x^{(n-2)})$$

for $n \geqslant 2$.

In this note the equation

$$(*)\qquad x^{(n)} = g(t, x, x', \ldots, x^{(n-1)})$$

will be examined, where the derivatives are understood in the distribution sense. We prove some theorems on the existence and uniqueness of solution of equation $(*)$ in the class of all distributions whose $(n-1)$

derivatives in the distribution sense are functions of finite variation in the interval (a, b). This class will be denoted by $V_{(a,b)}^{(n-1)}$. The principal results of this paper generalize some results for linear differential equations (see [6], [7], [8]).

The sequential theory of the distributions (see [5], [9]) will be used. All distributions in this paper are real distributions. Moreover, all distributions of a single variable are defined in the interval (a, b). The measurability and integrability of functions is understood in Lebesgue's sense.

2. DEFINITION 1. We say that a function $f(t, x_0, \ldots, x_{n-1})$ defined in the set D *satisfies condition* C (*Carathéodory's condition*) if

1° for every fixed t, $f(t, x_0, \ldots, x_{n-1})$ is continuous with respect to $(x_0, \ldots, x_{n-1})$;

2° for fixed $x_0, \ldots, x_{n-1}$, $f(t, x_0, \ldots, x_{n-1})$ is measurable with respect to t.

DEFINITION 2. A sequence of smooth, non-negative and even functions $\{\delta_k(t)\}$ for which

1° $\int\limits_{-\infty}^{\infty} \delta_k(t)\,dt = 1$;

2° there is a sequence of positive numbers $\{a_k\}$ convergent to zero such that

$$\delta_k(t) = 0 \quad \text{for } |t| \geqslant a_k;$$

3° there are numbers $M_0, M_1, \ldots$ such that

$$a_k^s \int\limits_{-\infty}^{\infty} |\delta_k^{(s)}(t)|\,dt < M_s,$$

holds for $k = 1, 2, \ldots$ and every order s is called a *delta sequence* (see [4], [10]).

DEFINITION 3. We say that a distribution $f(t)$ is a *measure* if there exists a fundamental sequence $\{f_k(t)\}$ for f such that, for each compact interval $I \subset (a, b)$, the sequence of numbers $\{\int\limits_I |f_k(t)|\,dt\}$ is bounded (see [1], [2]).

DEFINITION 4. Under a regular sequence $\{f_k(t)\}$ of a given distribution $f(t)$ we understand every sequence (see [10])

$$f_k(t) = (f * \delta_k)(t) = \int\limits_{-\infty}^{\infty} f(t-s)\,\delta_k(s)\,ds.$$

DEFINITION 5. If for every regular sequence $\{f_k(t)\}$ of the distribution $f(t)$ the sequence $\{f_k(t_0)\}$ is convergent to some finite limit as $k \to \infty$, then the limit $\lim\limits_{k\to\infty} f_k(t_0)$ is called the *mean value of the distribution* $f(t)$ in t_0 and denoted by $f(t_0)$ (see [4]).

The consistence of the definition follows from the fact that an interlaced sequence of two delta sequences is another delta sequence.

One may prove (see [4], [8]) the following

THEOREM 1. *If distribution $f(t)$ is a function of finite variation, then it has a mean value at every point $t_0 \in (a, b)$ and*

$$f(t_0) = \tfrac{1}{2}[f(t_0^+) + f(t_0^-)],$$

where $f(t_0^+)$ and $f(t_0^-)$ denote, respectively the right- and the left-hand side limits of the function $f(t)$ in t_0.

3. Now we shall prove the principal results. At first we assume the following hypotheses:

HYPOTHESIS H_1.

1° *The function $f_0(t, x_0, \ldots, x_{n-1})$ satisfies condition* C.

2° *The functions $f_r(t, x_0, \ldots, x_{n-2})$ are continuous in D with respect to $(t, x_0, \ldots, x_{n-2})$.*

3° $|f_0(t, x_0, \ldots, x_{n-1})| \leqslant \sum_{i=0}^{n-1} q_{0i}(t)|x_i| + q_{0n}(t)$, *where $q_{0i}(t)$ are non-negative, locally integrable functions in (a, b) for $i = 0, 1, \ldots, n$.*

4° $|f_r(t, x_0, \ldots, x_{n-2})| \leqslant \sum_{i=0}^{n-2} q_{ri}(t)|x_i| + q_{rn-1}(t)$, *where $q_{ri}(t)$ are non-negative and continuous functions in (a, b) for $i = 0, 1, \ldots, n-1$ and $r = 1, 2, \ldots, m$.*

HYPOTHESIS H_2.

1° *Assumptions* H_1: 1°, 3°, 4° *are fulfilled.*

2° *The functions $f_r(t, x_0, \ldots, x_{n-2})$ have continuous partial derivatives of the first order in D with respect to $(t, x_0, \ldots, x_{n-2})$ for $r = 1, 2, \ldots, m$.*

HYPOTHESIS H_3.

1° *Assumptions* H_1: 1°, H_2: 2° *are satisfied.*

2° $|f_0(t, x_0, \ldots, x_{n-1}) - f_0(t, \bar{x}_0, \ldots, \bar{x}_{n-1})| \leqslant \sum_{i=0}^{n-1} q_{0i}(t)|x_i - \bar{x}_i|$, *where $q_{0i}(t)$ are non-negative, locally integrable functions in (a, b) for $i = 0, 1, \ldots, n-1$.*

3° $|f_r(t, x_0, \ldots, x_{n-2}) - f_r(t, \bar{x}_0, \ldots, \bar{x}_{n-2})| \leqslant \sum_{i=0}^{n-2} q_{ri}(t)|x_i - \bar{x}_i|$, *where $q_{ri}(t)$ are non-negative and continuous functions in (a, b) for $i = 0, 1, \ldots, n-2$ and $r = 1, 2, \ldots, m$.*

4° $|f_0(t, 0, \ldots, 0)| \leqslant u(t)$, *where $u(t)$ is a locally integrable function in (a, b).*

We put

$$W_{n-1}(t) = \sum_{\mu=0}^{n-1} \frac{\varkappa_\mu (t - t_0)^\mu}{\mu!}$$

for arbitrary $\varkappa_\mu \in R^1$ and $t_0 \in (a, b)$,

$$g_k(t, x, x', \ldots, x^{(n-1)})$$
$$= f_0(t, x, x', \ldots, x^{(n-1)}) + \sum_{r=1}^{m} p_{rk}(t) f_r(t, x, x', \ldots, x^{(n-2)}),$$

where

$$p_{rk}(t) = (p_r * \delta_k)(t) \quad \text{for } k = 1, 2, \ldots$$

THEOREM 2. *Let Hypothesis* H_2 *be fulfilled. Then the problem*

$$(**) \qquad \begin{aligned} & x^{(n)} = g(t, x, x', \ldots, x^{(n-1)}), \\ & x^{(d)}(t_0) = \varkappa_d, \quad t_0 \in (a, b), \; d = 0, 1, \ldots, n-1 \end{aligned}$$

has at least one solution in the class $V_{(a,b)}^{(n-1)}$.

Proof. Let $x_k(t)$ be a solution of the equation

$$(2.0) \quad x(t) = \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} g_k\big(s, x(s), x'(s), \ldots, x^{(n-1)}(s)\big) ds + W_{n-1}(t).$$

Then

$$(2.1) \qquad |Y_k(t)| \leqslant \Big[|Y_k(t_0)| + \Big|\int_{t_0}^{t} |B_k(s)|\, ds\Big|\Big] \exp\Big|\int_{t_0}^{t} \|A_k(s)\|\, ds\Big|,$$

where

$$|Y_k(t)| = \sum_{d=0}^{n-1} |x_k^{(d)}(t)|, \qquad |B_k(t)| = q_{0n}(t) + \sum_{r=1}^{m} |p_{rk}(t)|\, q_{rn-1}(t)$$

and

$$\|A_k(t)\| = (n-1) + q_{0n-1}(t) + \sum_{i=0}^{n-2} \Big(q_{0i}(t) + \sum_{r=1}^{m} |p_{rk}(t)|\, q_{ri}(t)\Big).$$

Thus the sequences $\{x_k^{(d)}\}$ are locally equibounded in (a, b) for sufficiently large k and $d = 0, 1, \ldots, n-1$. Let I be an arbitrary compact interval such that $I \subset (a, b)$ and $x_0 \in I$. From Helly's theorem it follows that a subsequence $\{x_{v_k}^{(d)}\}$ of $\{x_k^{(d)}\}$ is convergent to a function of finite variation in I for $d = 0, 1, \ldots, n-1$, respectively. Hence we infer (see [8]) that there exists a subsequence $\{x_{u_k}^{(d)}\}$ of $\{x_k^{(d)}\}$ distributionally convergent to a function $x^{(d)}$ of finite variation in (a, b). In view of [1] and [3] (p. 642) we conclude that there exists a distributional limit

$$\lim_{u\to\infty}(d)\, g_{u_k}\big(t, x_{u_k}(t), x'_{u_k}(t), \ldots, x_{u_k}^{(n-1)}(t)\big) = g\big(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\big).$$

Hence the function $x(t)$ is a solution of equation $(*)$.

We shall prove that $x^{(d)}(t_0) = \varkappa_d$. In fact, from the almost uniform convergence of the sequences $\{x_{u_k}^{(\lambda)}\}$ for $\lambda = 0, 1, \ldots, n-2$ we have

$x^{(\lambda)}(t_0) = \varkappa_\lambda$. Moreover, by (2.0), Hypothesis H_2, we obtain

$$(2.2)\qquad x_{u_k}^{(n-1)}(t) = \int_{t_0}^{t} g_{u_k}\big(s, x_{u_k}(s), x'_{u_k}(s), \dots, x_{u_k}^{(n-1)}(s)\big)\,ds + \varkappa_{n-1}$$

$$= \int_{t_0}^{t} f_0\big(s, x_{u_k}(s), x'_{u_k}(s), \dots, x_{u_k}^{(n-1)}(s)\big)\,ds +$$

$$+\Big[\sum_{r=1}^{m} P_{ru_k}(s)\,g_{ru_k}(s)\Big]\Big|_{t_0}^{t} - \int_{t_0}^{t}\Big[\sum_{r=1}^{m} P_{ru_k}(s)\cdot\frac{dg_{ru_k}}{ds}\Big]\,ds + \varkappa_{n-1},$$

where

$$P'_r = p_r, \qquad P_{ru_k}(t) = (P_r * \delta_{u_k})(t)$$
$$\text{and}\qquad g_{ru_k}(t) = f_r\big(t, x_{u_k}(t), x'_{u_k}(t), \dots, x_{u_k}^{(n-2)}(t)\big).$$

From (2.2), Definition 5, Theorem 1 and [3] (p. 642) we infer that $x^{(n-1)}(t_0) = \varkappa_{n-1}$. This ends the proof of Theorem 2.

Similarly one may prove the following

THEOREM 3. *Let Hypothesis* H_1 *be fulfilled. Then equation* $(*)$ *has at least one solution in the class* $V_{(a,b)}^{(n-1)}$.

THEOREM 4. *Let Hypothesis* H_3 *be fulfilled. Then problem* $(**)$ *has exactly one solution in the class* $V_{(a,b)}^{(n-1)}$.

Proof. Suppose that $\tilde{x}_1$ and $\tilde{x}_2$ are solutions of problem $(**)$ such that $\tilde{x}_1 \neq \tilde{x}_2$ and $\tilde{x}_1, \tilde{x}_2 \in V_{(a,b)}^{(n-1)}$. We denote by $\{\tilde{x}_{1k}\}$, $\{\tilde{x}_{2k}\}$ arbitrary regular sequences of $\tilde{x}_1$ and $\tilde{x}_2$ respectively. We consider the sequences $\{Y_{ik}^{(d)}\}$ defined as follows:

$$(4.0)\quad Y_{ik}^{(d)}(t) = \Big[\int_{t_0}^{t}\frac{(t-s)^{n-1}}{(n-1)!}\,g_k\big(s, \tilde{x}_{ik}(s), \tilde{x}'_{ik}(s), \dots, \tilde{x}_{ik}^{(n-1)}(s)\big)\,ds + W_{n-1}(t)\Big]^{(d)}$$

for $d = 0, 1, \dots, n-1$ and $i = 1, 2$.

From Helly's theorem it follows that a subsequence $\{Y_{iu_k}^{(d)}\}$ of $\{Y_{ik}^{(d)}\}$ is convergent to a function $Y_i^{(d)}$ of finite variation in (a, b) for $i = 1, 2$, $d = 0, 1, \dots, n-1$ and $u \to \infty$. Moreover, $Y_i^{(d)}(t_0) = \varkappa_d$. Hence by (4.0) we get

$$(4.1)\qquad Y_i(t) = \tilde{x}_i(t) \quad \text{for } i = 1, 2.$$

We put

$$(4.2)\quad Z_{u_k}(t) = Y_{1u_k}(t) - Y_{2u_k}(t), \qquad E_{du_k}(t) = \big||\tilde{x}_{1u_k}^{(d)}(t) - \tilde{x}_{2u_k}^{(d)}(t)| - |Z_{u_k}^{(d)}(t)|\big|.$$

In view of (4.0), (4.1), (4.2) and Hypothesis H_3 we have

$$(4.3)\qquad |\overline{Y}_{u_k}(t)| \leqslant \Big|\int_{t_0}^{t}|\overline{B}_{u_k}(s)|\,ds\Big|\exp\Big|\int_{t_0}^{t}\|A_{u_k}(s)\|\,ds\Big|,$$

where

$$|\overline{Y}_{u_k}(t)| = \sum_{d=0}^{n-1} |Z_{u_k}^{(d)}(t)|,$$

$$|\bar{B}_{u_k}(t)| = \sum_{i=0}^{n-2} \Big(\sum_{r=1}^{m} |p_{ru_k}(t)|\, q_{ri}(t)\Big)\, E_{iu_k}(t) + \sum_{i=0}^{n-1} q_{0i}(t)\, E_{iu_k}(t).$$

Hence $\lim_{u\to\infty} (d)Z_{u_k} = 0$. By Theorem 3 and (4.1) the proof of our assertion is ended.

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