ANNALES
POLONICI MATHEMATICI
XXVII (1973)

On the existence and uniqueness of periodic solutions for difference equations of second order

by ZDZISŁAW DENKOWSKI (Kraków)

1. In paper [4] A. Lasota and Z. Opial have given following two theorems concerning the existence and uniqueness of periodic solutions for the differential equation

$$y''+P(t,y,y')y = Q(t,y,y'). \tag{1.1}$$

THEOREM 1.1. *Suppose that the functions $P, Q\colon \mathbf{R}^3 \to \mathbf{R}$ are periodic with respect to the first variable (with common period ω) and satisfy the Carathéodory conditions and the inequalities*

$$p_1(t) \leqslant P(t,y,z), \quad |P(t,y,z)| \leqslant p_2(t), \quad 0 \leqslant t \leqslant \omega, \quad y, z \in \mathbf{R},$$

where the functions $p_1, p_2\colon [0,\omega] \to \mathbf{R}$ are integrable and such that

$$p_1(t) \not\equiv 0, \quad \int_0^\omega p_1(t)\,dt \geqslant 0, \quad \omega \int_0^\omega p_2(t)\,dt \leqslant 16.$$

Suppose, moreover, that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^\omega \sup_{|y|+|z|\leqslant n} |Q(t,y,z)|\,dt = 0.$$

Then equation (1.1) *has at least one ω-periodic solution.*

THEOREM 1.2. *If an ω-periodic function $p\colon \mathbf{R} \to \mathbf{R}$ ($p \not\equiv 0$) is integrable over the interval $[0,\omega]$ and satisfies the inequalities*

$$\int_0^\omega p(t)\,dt \geqslant 0, \quad \omega \int_0^\omega |p(t)|\,dt \leqslant 16,$$

then the equation

$$y''+p(t)y = q(t) \tag{1.2}$$

has exactly one ω-periodic solution for each ω-periodic function $q\colon \mathbf{R} \to \mathbf{R}$ integrable over the interval $[0,\omega]$.

The aim of this paper is to give analogues theorems for the difference equations

$$\nabla\Delta v_i+\tilde{P}(i, v_i, \Delta v_i)v_i = \tilde{Q}(i, v_i, \Delta v_i), \quad i\in \boldsymbol{Z} \tag{1.3}$$

and

$$\nabla\Delta v_i+\tilde{p}_i v_i = \tilde{q}_i, \quad i\in \boldsymbol{Z} \tag{1.4}$$

(where $\tilde{P}, \tilde{Q}$: $\boldsymbol{Z}\times\boldsymbol{R}^2\to\boldsymbol{R}$, $\tilde{p}_i, \tilde{q}_i\in\boldsymbol{R}$ for $i\in\boldsymbol{Z}$), and to consider the possibility of approximating the solution (if it is unique) of differential equation (1.2) by the periodic solutions of appropriately defined difference equations of the form (1.4).

This last problem will be solved in Section 4. In the proof of the approximation theorem we shall use two lemmas given in Section 2 and Lemma 1 of paper [2]. The existence and uniqueness theorems for the solutions of difference equations are contained in Section 3. The proofs of these theorems are based on Theorem 4.1 and Theorem 4.2 of paper [1], which are discrete analogues suitable theorems for the continuous case given in [3].

Throughout the paper we will utilize notions and notations which are explained in detail in paper [1], [2].

Furthermore, we will assume in the sequel that the coefficients of the difference equations (1.3) and (1.4) satisfy, respectively, the periodic conditions

$$\tilde{P}(i+n, w) = \tilde{P}(i, w), \quad \tilde{Q}(i+n, w) = \tilde{Q}(i, w), \quad i\in\boldsymbol{Z}, w\in\boldsymbol{R}^2, \tag{1.5}$$

$$\tilde{p}_{i+n} = \tilde{p}_i, \quad \tilde{q}_{i+n} = \tilde{q}_i, \quad i\in\boldsymbol{Z}. \tag{1.6}$$

By a periodic solution of the difference equation (1.3) or (1.4) (with periodic coefficients) we will mean any vector $v\in\boldsymbol{R}^{\boldsymbol{Z}}$ whose coordinates fulfil equations (1.3) and (1.4), respectively, and the following conditions of periodicity:

$$v_{i+n} = v_i, \quad i\in\boldsymbol{Z}. \tag{1.7}$$

Finally, the author wishes to express his heartfelt thanks to A. Lasota for suggesting this paper and for valuable ideas.

2. We start with two lemmas. The first one is a discrete analogue of Gronwall's well-known inequality and the second one gives an a priori estimation of the minimum of the periodic solution of equation (1.4)

LEMMA 2.1. *If the vector $v\in\boldsymbol{R}^{n+1}$ satisfies the inequality*

$$|v_i|\leqslant C+\sum_{j=0}^{i-1}a_j|v_j|, \quad i = 0, \ldots, n, \tag{2.1}$$

with non-negative coefficients a_i, *then the estimate*

$$(2.2)\qquad |v_i| \leqslant C\cdot\exp\Big(\sum_{j=0}^{i-1} a_j\Big), \quad i = 0, \dots, n$$

holds true. (*We set* $\sum_{i=\nu}^{\mu} = 0$ *if* $\mu < \nu$.)

Proof. Notice that it suffices to consider the case of $a_i > 0$ ($i = 0, \dots, n$) because in the remaining cases we can repeat our reasoning for $a_i' = a_i + \varepsilon$ ($\varepsilon > 0$, $i = 0, \dots, n$) and then pass to the limit with $\varepsilon \to 0$. Setting for $i = 0, \dots, n$

$$w_i = C + \sum_{j=0}^{i-1} a_j |v_j|$$

we have

$$\Delta w_i - a_i w_i \leqslant 0.$$

Multiplying both sides by $\exp\Big(-\sum_{j=0}^{i} a_j\Big)$ and using the elementary inequality

$$e^t < \frac{e^t - 1}{t} \quad \text{for } t < 0,$$

we obtain

$$\Delta\Big(w_i\cdot\exp\Big(-\sum_{j=0}^{i-1} a_j\Big)\Big) \leqslant 0.$$

Since $w_0 = C$, the last inequality implies

$$w_i\cdot\exp\Big(-\sum_{j=0}^{i-1} a_j\Big) \leqslant C, \quad i = 0, \dots, n,$$

which immediately gives the required inequality (2.2).

LEMMA 2.2. *If a vector* $v \in \mathbf{R}^{\mathbf{Z}}$ *is a periodic solution of difference equation* (1.4) *satisfying the conditions*

$$(2.3)\qquad v_i \neq 0, \quad i \in \mathbf{Z}$$

and if the coefficients $\tilde{p}_i$ ($i \in \mathbf{Z}$) *fulfil the additional inequality*

$$(2.4)\qquad \sum_{i=1}^{n} \tilde{p}_i > 0,$$

then the estimation

$$(2.5)\qquad \min_{i \in \mathbf{Z}} |v_i| \leqslant \frac{\sum_{i=1}^{n} |\tilde{q}_i|}{\sum_{i=1}^{n} \tilde{p}_i}$$

is valid.

Proof. It is evident that the coordinates of v satisfy the difference equation

$$\frac{\nabla\Delta v_i}{v_i}+\tilde{p}_i=\frac{\tilde{q}_i}{v_i}, \quad i=1,\ldots,n. \tag{2.6}$$

Summing these equations, we obtain

$$\sum_{i=1}^{n}(\nabla\Delta v_i)\frac{1}{v_i}+\sum_{i=1}^{n}\tilde{p}_i=\sum_{i=1}^{n}\frac{\tilde{q}_i}{v_i}.$$

Hence, applying the summation by parts formula (see [1], [2]), we get, by the periodicity of v, the inequality

$$\sum_{i=1}^{n}\tilde{p}_i\leqslant\frac{1}{\min\limits_{1\leqslant i\leqslant n}|v_i|}\cdot\sum_{i=1}^{n}|\tilde{q}_i|,$$

which completes the proof.

3. We now sate two theorems which are discrete analogues of Theorem 1.1 and Theorem 1.2.

THEOREM 3.1. *Suppose that the functions $\tilde{P},\tilde{Q}$ in the difference equation* (1.3) *are continuous, periodic and satisfy the conditions*

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{n}\sup_{|w|\leqslant k}|\tilde{Q}(i,w)|=0 \quad (w\in\boldsymbol{R}^2) \tag{3.1}$$

and

$$p_{i1}\leqslant\tilde{P}(i,w), \quad |\tilde{P}(i,w)|\leqslant p_{i2} \quad (i\in\boldsymbol{Z},\ w\in\boldsymbol{R}^2), \tag{3.2}$$

where the sequences $\{p_{i1}\}_{i\in\boldsymbol{Z}}$, $\{p_{i2}\}_{i\in\boldsymbol{Z}}$ of real numbers fulfil the inequalities

$$\sum_{i=1}^{n}p_{i1}\geqslant 0, \quad n\sum_{i=1}^{n}p_{i2}<16. \tag{3.3}$$

Moreover, suppose that there is an index $j\in\boldsymbol{Z}$ such that $p_{j1}\neq 0$. Then the difference equation (1.3) *has at least one periodic solution.*

Proof. First of all notice that the existence problem of the periodic solution of equation (1.3) is equivalent to the existence problem of the solution of the difference equation

$$\nabla\Delta v_i+\tilde{P}(i,v_i,\Delta v_i)v_i=\tilde{Q}(i,v_i,\Delta v_i), \quad i=1,\ldots,n, \tag{3.4}$$

satisfying the boundary condition

$$v_0=v_n, \quad v_1=v_{n+1}. \tag{3.5}$$

We can write problem (3.4), (3.5) in the vectorial form (see [1])

$$\Delta_s u_i = f(i, u_i), \quad i = 1, \ldots, n, \tag{3.6}$$

$$Lu = 0, \tag{3.7}$$

where

$$u = (u^1, u^2) \in (\boldsymbol{R}^2)^{n+2}, \quad u_i^1 = v_i, \quad u_i^2 = \Delta v_i \quad (i = 0, \ldots, n), \quad u_{n+1}^2 = 0.$$

The multi-index $s = (s_1, s_2)$ $(s_i \in \{-1, 1\})$ is of the form $s = (1, -1)$, and therefore

$$\Delta_s u_i = (\Delta u_i^1, \nabla u_i^2) \quad (i = 1, \ldots, n), \quad \Delta_s u_0 = (\Delta u_0^1, 0), \quad \Delta_s u_{n+1} = (0, 0);$$

the map f: $\{0, \ldots, n+1\} \times \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is given by

$$f(i, u_i) = \begin{cases} (u_i^2, 0), & i = 0, \\ \big(u_i^2, -\tilde{P}(i, u_i)u_i^1 + \tilde{Q}(i, u_i)\big), & i = 1, \ldots, n, \\ (0, 0), & i = n+1. \end{cases}$$

Finally, the linear operator L: $(\boldsymbol{R}^2)^{n+2} \to \boldsymbol{R}^2$ is defined as follows:

$$Lu = (u_0^1 - u_n^1, u_1^1 - u_{n+1}^1). \tag{3.8}$$

Side by side with equation (3.6) we consider the following equation with a multi-valued right-hand side (the contingent equation):

$$\Delta_s u_i \in \boldsymbol{F}(i, u_i), \quad i = 1, \ldots, n, \tag{3.9}$$

where the map $\boldsymbol{F}$: $\{0, \ldots, n+1\} \times \boldsymbol{R}^2 \to \mathrm{cf}(\boldsymbol{R}^2)$ is given by

$$\boldsymbol{F}(i, u_i) = \begin{cases} (u_i^2, 0), & & i = 0, \\ \{(u_i^2, z) : p_{i1} u_i^1 \leqslant z \leqslant p_{i2} u_i^1\} & \text{if } u_i^1 \geqslant 0, & i = 1, \ldots, n, \\ \qquad\qquad\qquad\quad \geqslant \quad \geqslant & \quad\;\; u_i^1 < 0, & \\ (0, 0), & & i = n+1. \end{cases}$$

Here $\mathrm{cf}(\boldsymbol{R}^2)$ denotes the set of all convex and closed subsets of $\boldsymbol{R}^2$.

It is easy to observe that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{n} \sup_{|w| \leqslant k} \delta\big(f(i, w), \boldsymbol{F}(i, w)\big) = 0, \tag{3.10}$$

where $\delta(x, B)$ denotes the distance from the point x to the set B.

According to the generalized Fredholm theorem (see [3], [1]) in order to prove the existence of solutions of problem (3.6), (3.7) it is sufficient to show the uniqueness of the solution of problem (3.9), (3.7).

Suppose u satisfies (3.9) and (3.7). Setting

$$v_i = u_i^1 \quad (i = 0, \ldots, n+1), \quad v_{i+n} = v_i, \quad i \in \boldsymbol{Z},$$

we confirm that, under the notation of paper [2], the vector v is a solution of the difference equation

$$\nabla_\tau \Delta_\tau v_i + p_i v_i = 0, \quad i \in \boldsymbol{Z}, \tag{3.11}$$

where

$$p_i = \begin{cases} \dfrac{-\nabla \Delta v_i}{h^2 v_i} & \text{if } v_i \neq 0, \\ 0 & \text{if } v_i = 0 \end{cases}$$

and

$$\tau = \{t_i\}_{i \in \boldsymbol{Z}}, \quad t_i = i \cdot h$$

(h denotes an arbitrary positive real number). Moreover, we have

$$\sum_{i=1}^{n} p_{i1} \leqslant h^2 \sum_{i=1}^{n} p_i, \quad h^2 \sum_{i=1}^{n} |p_i| \leqslant \sum_{i=1}^{n} p_{i2}$$

and consequently by assumption (3.3) and the assumption concerning p_{j1} we obtain

$$\sum_{i=1}^{n} p_i \geqslant 0, \quad h \sum_{i=1}^{n} |p_i| < \frac{16}{nh}, \quad p_i \not\equiv 0.$$

These inequalities imply (see Corollary 4.1 in [2]) that $v = 0$ (i.e. $v_i = 0$ for $i \in \boldsymbol{Z}$), which completes the proof of Theorem 3.1.

In a similar way we can prove the following

THEOREM 3.2. *If the coefficients $\tilde{p}_i$, $\tilde{q}_i$ ($i \in \boldsymbol{Z}$) of difference equation* (1.4) *are periodic and satisfy the inequalities*

$$\sum_{i=1}^{n} \tilde{p}_i \geqslant 0, \quad n \sum_{i=1}^{n} |\tilde{p}_i| < 16, \quad \tilde{p}_i \not\equiv 0, \tag{3.12}$$

then equation (1.4) *has exactly one periodic solution.*

Proof. This theorem is an immediate consequence of Corollary 4.1 of paper [2].

4. We shall now consider the problem of approximating the periodic solution of differential equation (1.2) by periodic solutions of appropriately defined difference equations of the form (1.4).

The existence of ω-periodic solution of differential equation (1.2) is equivalent to the boundary value problem

$$y'' + p(t)y = q(t), \quad y(0) = y(\omega), \quad y'(0) = y'(\omega). \tag{4.1}$$

Consider the sequence ($n = 1, 2, \ldots$) of the boundary-value problems of the form:

$$\nabla\Delta v_i^n + h_n^2 p_i^n v_i^n = h_n^2 q_i^n, \quad i = 1, \ldots, n, \tag{4.2}$$

$$v_0^n = v_n^n, \quad v_1^n = v_{n+1}^n, \tag{4.3}$$

where

$$h_n = \omega/n, \quad t_i^n = i\cdot h_n, \quad p_i^n = p(t_i^n), \quad q_i^n = q(t_i^n) \quad (i \in \boldsymbol{Z}). \tag{4.4}$$

Let us put

$$\tilde{p}_i^n = h_n^2 p_i^n, \quad \tilde{q}_i^n = h_n^2 q_i^n. \tag{4.5}$$

We are now in a position to prove.

THEOREM 4.1. *If the continuous functions $p, q: \boldsymbol{R} \to \boldsymbol{R}$ are periodic (with ω-period) and the function p ($p \not\equiv 0$) satisfies the inequalities*

$$\int_0^\omega p(t)\,dt > 0, \quad \omega\int_0^\omega |p(t)|\,dt < 16, \tag{4.6}$$

then

1° *for sufficiently large n there is exactly one solution $v^n \in \boldsymbol{R}^{\boldsymbol{Z}}$ of problem* (4.2), (4.3),

2° $\lim\limits_{n\to\infty} |v_i^n - y_i^n| = 0 \quad (y_i^n = y(t_i^n))$,

where y denotes the unique periodic solution of the differential equation (1.2).

The convergence in condition 2° is uniform with respect to i.

Proof. Note that the functions p, q satisfy the assumptions of Theorem 1.2; so there is exactly one periodic solution of differential equation (1.2).

It is also easy to see that the coefficients $\tilde{p}_i^n, \tilde{q}_i^n$ fulfil (for fixed n) the periodicity condition (1.6) and, for sufficiently large n, inequalities (3.12).

Indeed, the integral inequalities (4.6) imply (for sufficiently large n) the following inequalities for approximation sums of suitable integrals:

$$h_n\sum_{i=1}^n p_i^n > 0, \quad nh_n^2\sum_{i=1}^n |p_i^n| < 16, \tag{4.7}$$

which imply in turn inequalities (3.12). Thus, for such n, problem (4.2), (4.3) has by Theorem 3.2 exactly one solution and the proof of 1° is completed. Now we pass to the proof of 2°.

By the mean-value theorem from (1.2) we have

$$\nabla\Delta y_i^n = -\tilde{p}_i^n y_i^n + \tilde{q}_i^n + h_n^2\delta_i^n, \quad i = 1, \ldots, n, \tag{4.8}$$

where (for $i = 1, \ldots, n$)

$$\delta_i^n = \big(p(\xi_i^n)y(\xi_i^n) - p(t_i^n)y(t_i^n)\big) + \big(q(\xi_i^n) - q(t_i^n)\big)$$

and

$$\xi_i^n \in (t_{i-1}^n, t_i^n).$$

It is easily seen that

$$\lim_{n\to\infty} \delta_i^n = 0 \tag{4.9}$$

and

$$\delta_{i+n}^n = \delta_i^n \quad (i \in \boldsymbol{Z}). \tag{4.10}$$

Subtracting equation (4.8) from (4.2) and setting

$$\tilde{z}_i^n = v_i^n - y_i^n \quad (i = 1, \ldots, n), \qquad \tilde{z}_{i+n}^n = \tilde{z}_i^n \quad (i \in \boldsymbol{Z}), \tag{4.11}$$

we obtain the equation

$$\nabla \Delta \tilde{z}_i^n + \tilde{p}_i^n \tilde{z}_i^n = -h_n^2 \delta_i^n, \quad i \in \boldsymbol{Z}, \tag{4.12}$$

which in view of Theorem 3.2 has exactly one periodic solution. Setting

$$z_i^n = \begin{cases} \tilde{z}_i^n - \min\limits_{1\leqslant j\leqslant n} |\tilde{z}_j^n| & \text{if } \min\limits_{1\leqslant j\leqslant n} |\tilde{z}_j^n| = \tilde{z}_k \quad \text{for some } k, \\ \tilde{z}_i^n + \min\limits_{1\leqslant j\leqslant n} |\tilde{z}_j^n| & \text{if } \min\limits_{1\leqslant j\leqslant n} |\tilde{z}_j^n| = -\tilde{z}_k \quad \text{for some } k, \end{cases} \tag{4.13}$$

and changing the numbering, if necessary, we obtain the difference equation

$$\nabla \Delta z_i^n + \tilde{p}_i^n z_i^n = h_n^2 r_i^n, \quad i \in \boldsymbol{Z} \tag{4.14}$$

(where $r_i^n = \pm p_i^n \min\limits_{1\leqslant j\leqslant n} |\tilde{z}_j^n| - \delta_i^n$) such that for its (unique) periodic solution we have

$$z_0^n = 0, \quad z_1^n \neq 0. \tag{4.15}$$

It easily follows by Lemma 2.2 that

$$|r_i^n| \leqslant \max_{[0,\omega]} |p(t)| \frac{h_n \sum\limits_{i=1}^{n} |\delta_i^n|}{h_n \sum\limits_{i=1}^{n} p_i^n} + |\delta_i^n|$$

and in consequence, by (4.9), we get

$$\lim_{n\to\infty} r_i^n = 0. \tag{4.16}$$

We now consider a piece-linear function $\varphi_n\colon \boldsymbol{R} \to \boldsymbol{R}$ which has as its graph on plane $\boldsymbol{R}^2$ the polygonal line with points (t_i^n, z_i^n) as vertices.

It is easy to see (compare the proof of Theorem 1 in [2]) that φ_n has at least one zero in the interval (t_0, t_n). Denote by t'_{k_n} the smallest zero of φ_n in (t_0, t_n) and suppose that

$$t'_{k_n} \in (t^n_{k_n-1}, t^n_{k_n})$$

(by (4.15) it has to be $k_n > 1$). We extend the net $\tau_n = \{t^n_i\}_{i\in Z}$ (with t^n_i given by (4.4)) adding t'_{k_n}, so that the extended net is of the form

$$\tau' = \{t'_i\}_{i\in Z},$$

where

$$t'_i = \begin{cases} t^n_i & i \leqslant k_n - 1, \\ t'_{k_n} & i = k_n, \\ t^n_{i-1} & i \geqslant k_n + 1. \end{cases}$$

Setting

$$\tilde{\tilde{p}}_i = \begin{cases} p^n_i & i = 1, \ldots, k_n - 1, \\ 0 & i = k_n, \\ p^n_{i-1} & i = k_n + 1, \ldots, n, \end{cases}$$

and

$$\tilde{r}_i = \begin{cases} r^n_i & i = 1, \ldots, k_n - 1, \\ 0 & i = k_n, \\ r^n_{i-1} & i = k_n + 1, \ldots, n, \end{cases}$$

and

$$\tilde{\tilde{z}} = (z^n_0, \ldots, z^n_{k_n-1}, 0, z^n_{k_n}, \ldots, z^n_n),$$

we easily confirm that the vector

$$\tilde{\tilde{z}}' = (z^n_0, \ldots, z^n_{k_n-1}, 0) \in \boldsymbol{R}^{k_n+1}$$

(considered as the vector of $\boldsymbol{R}^{\boldsymbol{Z}}$ with the remaining coordinates equal to zero) is a solution of the difference equation

$$\nabla_{\tau'} \Delta_{\tau'} \tilde{\tilde{z}}_\tau + \tilde{\tilde{p}}_i \tilde{\tilde{z}}_i - \tilde{r}_i = 0, \quad i = 1, \ldots, k_n - 1,$$

atisfying by (4.15) the condition

$$\tilde{\tilde{z}}'_0 = 0, \quad \tilde{\tilde{z}}'_{k_n} = 0.$$

Similarly, the vector

$$\tilde{\tilde{z}}'' = (0, z^n_{k_n}, \ldots, z^n_n) \in \boldsymbol{R}^{n-k_n+2}$$

is a solution of the difference equation

$$\nabla_{\tau'} \Delta_{\tau'} \tilde{\tilde{z}}_i + \tilde{\tilde{p}}_i \tilde{\tilde{z}}_i - r_i = 0, \quad i = k_n + 1, \ldots, n,$$

satisfying, by the periodicity of z^n and (4.15), the condition

$$\tilde{\tilde{z}}''_{k_n} = 0, \quad \tilde{\tilde{z}}''_{n+1} = z^n_n = 0.$$

Hence, by Lemma 1 of paper [2] (formula (3.3) in the case of vector $\tilde{\tilde{z}}'$ and formula (3.4) in the case of vector $\tilde{\tilde{z}}''$), we obtain

$$(4.17) \qquad \tilde{\tilde{z}}_i = h_n \sum_{j=1}^{k_n-1} \Gamma_{ij}^{k_n}(\tilde{\tilde{p}}_j \tilde{\tilde{z}}_j - \tilde{r}_j), \quad i = 1, \ldots, k_n-1,$$

$$(4.18) \qquad \tilde{\tilde{z}}_{n+1-i} = h_n \sum_{j=k_n+1}^{n} \Gamma^{n+1-k_n}_{n+1-j,n+1-i}(\tilde{\tilde{p}}_{n+1-j}\tilde{\tilde{z}}_{n+1-j} - \tilde{r}_{n+1-j}), \quad i = 1, \ldots, n-k_n.$$

Since

$$\Gamma_{ij}^{k_n} \leqslant \tfrac{1}{4}(t'_{k_n} - t'_0), \qquad \Gamma^{n+1-k_n}_{n+1-j,n+1-i} \leqslant \tfrac{1}{4}(t'_{n+1} - t'_{k_n})$$

(see (3.6) in [2]), we easily get from (4.17) and (4.18) the following estimations:

$$(4.19) \qquad \max_{1\leqslant i\leqslant k_n-1} |z_i^n| \leqslant h_n \frac{t'_{k_n} - t_0}{4}\Big(\max_{1\leqslant i\leqslant k_n-1} |z_i^n| \sum_{j=1}^{k_n-1} |p_j^n| + \sum_{j=1}^{k_n-1} |r_j^n|\Big),$$

$$(4.20) \qquad \max_{k_n\leqslant i\leqslant n-1} |z_i^n| \leqslant h_n \frac{t_n - t'_{k_n}}{4}\Big(\max_{k_n\leqslant i\leqslant n-1} |z_i^n| \sum_{j=k_n}^{n-1} |p_j^n| + \sum_{j=k_n}^{n-1} |r_j^n|\Big).$$

Let us observe now that the second inequality in (4.6) implies that there is a positive number δ such that

$$\omega \int_0^\omega |p(t)|\, dt < 16 - \delta,$$

and in consequence for sufficiently large n we have

$$n h_n^2 \sum_{n=1}^{n} |p_i^n| < 16 - \delta/2.$$

Thus, for such n, at least one of the inequalities

$$(4.21) \qquad k_n h_n^2 \sum_{i=1}^{k_n} |p_i^n| < 4 - \delta/8, \quad (n-k_n) h_n^2 \sum_{i=k_n+1}^{n} |p_i^n| < 4 - \delta/8$$

is fulfilled. Without any loss of generality we may assume that the first one is true for each sufficiently large n.

Under this assumption, using the obvious inequality

$$t'_{k_n} - t_0 < k_n \cdot h_n,$$

we obtain from (4.19)

$$\max_{1 \leqslant i \leqslant k_n - 1} |z_i^n| \leqslant \tfrac{1}{4} k_n h_n^2 \sum_{j=1}^{k_n} |r_j^n| \Big(1 - \tfrac{1}{4} k_n h_n^2 \sum_{j=1}^{k_n} |p_j^n|\Big)^{-1}. \tag{4.22}$$

In order to estimate $|z_i^n|$ for $i = k_n, \ldots, n$ we shall use Lemma 2.1. To this and observe that from (4.14) it follows that the vector

$$(z_0^n, \ldots, z_n^n) \in \boldsymbol{R}^{n+1}$$

(as the vector of $\boldsymbol{R}^{\boldsymbol{Z}}$ with the remaining coordinates equal to zero) is a solution of the difference equation

$$\nabla_\tau \Delta_\tau z_i^n + p_i^n z_i^n = r_i^n, \quad i = 1, \ldots, n-1.$$

Therefore its coordinates are given (see formula (2.5) in [2]) by

$$z_i^n = z_s^n + (i-s) h_n \Delta_\tau z_s + h_n^2 \sum_{j=s+1}^{i-1} (-p_j^n z_j^n + r_j^n)(i-j), \tag{4.23}$$

where s denotes a fixed index of the set $\{0, \ldots, n-1\}$, $i = s, \ldots, n-1$. Assume that in (4.15) we have

$$z_1^n > 0$$

(the case of $z_1^n < 0$ is similar) and that for some index s_0 of $\{1, \ldots, k_n - 1\}$

$$z_{s_0}^n = \max_{1 \leqslant i \leqslant k_n - 1} |z_i^n|.$$

Then we have

$$\Delta_\tau z_{s_0}^n \leqslant 0,$$

and we may write, by (4.23) (with $s = s_0$) and (4.22), the following estimation:

$$\begin{aligned} &|z_i^n| \\ &\leqslant \tfrac{1}{4} k_n h_n^2 \sum_{j=1}^{k_n} |r_j^n| \Big(1 - \tfrac{1}{4} k_n h_n^2 \sum_{j=1}^{k_n} |p_j^n|\Big)^{-1} + n h_n^2 \sum_{j=1}^{i-1} |r_j^n| + n h_n^2 \sum_{j=1}^{i-1} |p_j^n| \, |z_j^n| \end{aligned} \tag{4.24}$$

for $i \in \{k_n, \ldots, n\}$ such that $z_i^n > 0$.

However, for $i \in \{1, \ldots, n\}$ such that $z_i^n < 0$ we obtain by (4.23) (with $s = 0$) and (4.15) (with $z_1 > 0$) the estimation

$$|z_i^n| \leqslant nh_n^2 \sum_{j=1}^{i-1} |r_j^n| + nh_n^2 \sum_{j=1}^{i-1} |p_j^n|\,|z_j^n|. \tag{4.25}$$

Thus, setting

$$C_n = \tfrac{1}{4} nh_n^2 \sum_{j=1}^{n} |r_j^n| \Big(1 - \tfrac{1}{4} k_n h_n^2 \cdot \sum_{j=1}^{k_n} |p_j^n|\Big)^{-1} + nh_n^2 \sum_{j=1}^{n} |r_j^n|,$$

we have the inequality

$$|z_i^n| \leqslant C_n + \sum_{j=0}^{i-1} nh_n^2 |p_j^n|\,|z_j^n|, \qquad i = 1, \ldots, n,$$

which immediately implies, by Lemma 2.1 (a discrete analogue of Gronwall's inequality), the following estimation:

$$|z_i^n| \leqslant C_n \exp\Big(nh_n^2 \sum_{j=0}^{i-1} |p_j^n|\Big), \qquad i = 1, \ldots, n. \tag{4.26}$$

But from (4.4), (4.16) and from the assumption that the first of the inequalities of (4.21) is valid for sufficiently large n, we obtain

$$\lim_{n\to\infty} C_n = 0,$$

$$\lim_{n\to\infty} \exp\Big(nh_n^2 \sum_{j=0}^{i-1} |p_j^n|\Big) \leqslant \exp\Big(\omega \int_0^{\omega} |p(t)|\,dt\Big),$$

so by (4.26) we confirm that

$$\lim_{n\to\infty} |z_i^n| = 0, \tag{4.27}$$

and this convergence is uniform with respect to i.

Finally, from Lemma 2.2 it follows that

$$\min_{1\leqslant i\leqslant n} |\tilde{z}_i^n| \leqslant \Big(h_n^2 \sum_{j=1}^{n} |\delta_i^n|\Big)\Big(\sum_{j=1}^{n} \tilde{p}_i^n\Big)^{-1}$$

for $\tilde{z}$ given by (4.11).

This implies by (4.9), (4.4) and (4.6) that

$$\lim_{n\to\infty} \big(\min_{1\leqslant i\leqslant n} |\tilde{z}_i^n|\big) \leqslant \Big(\lim_{n\to\infty} \frac{\omega}{n} \sum_{j=1}^{n} |\delta_i^n|\Big) \int_0^{\omega} p(t)\,dt = 0.$$

Hence, by (4.13) and (4.27), we immediately obtain part 2° of the theorem. Thus the proof of Theorem 4.1 is completed.

References

[1] Z. Denkowski, *Linear problems for systems of difference equations*, Ann. Polon. Math. 24 (1970), p. 85-94.

[2] — *On a criterion of uniqueness for periodic solutions of the linear second order difference equations*, ibidem 27 (1972), p. 41-42.

[3] A. Lasota, *Une généralisation du premier théorème de Fredholm et ses application à la théorie des équations différentielles ordninaires*, ibidem 18 (1966), p. 65-77.

[4] — and Z. Opial, *Sur les solutions périodiques des équations différentielles ordinaires*, ibidem 16 (1964), p. 69-94.

Reçu par la Rédaction le 6. 10. 1971