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On Lipschitzian solutions of a functional equation

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Abstract. In this paper there is proved a theorem on the existence of the unique solution φ of the first order functional equation $\varphi(x) = h(x, \varphi[f(x)])$, fulfilling a Lipschitz condition in a neighbourhood of the fixed point of the function f. Under some additional conditions the properties of monotonicity and convexity of the solution φ are investigated.

In the present paper we shall deal with the questions of the existence and uniqueness of solutions of the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{1}$$

where φ denotes the unknown function and is assumed to fulfil the Lipschitz condition. Applying the results obtained we shall prove a theorem on the existence and uniqueness of convex solutions of equation (1).

The problem of the existence of Lipschitzian solutions of equation (1) was investigated in [6] under stronger assumptions. Convex solutions of equation (1) were treated in [2], [4] and [5] by means of other methods.

1. Let I be an interval. We assume that

(i) f is defined and continuous in I; there exists a $\xi \in I$ such that

$$0 < \frac{f(x)-\xi}{x-\xi} < 1, \quad x \in I, \ x \neq \xi.$$

Remark 1. Hypothesis (i) implies that $f(\xi) = \xi$, $f(I_1) \subset I_1$ for every interval $I_1 \subset I$, such that $\xi \in I_1$, and $\lim_{n\to\infty} f^n(x) = \xi$ for every $x \in I$ (cf. [3], p. 20). Here f^n denotes the nth iteration of the function f.

Concerning the function h we assume:

(ii) h is continuous in a domain $\Omega \subset R^2$; there exists an η such that

$$(\xi, \eta) \in \Omega, \quad h(\xi, \eta) = \eta; \tag{2}$$

(iii) for every $x \in I$ the set $\Omega_x = \{y: (x, y) \in \Omega\}$ is a non-empty open interval and

$$h(f(x), \Omega_{f(x)}) \subset \Omega_x.$$

We shall prove the following.

THEOREM 1. *If hypotheses* (i)–(iii) *are fulfilled and there exist positive numbers* a, β, s, k, l *such that*

$$|f(x)-f(\bar{x})| \leqslant s|x-\bar{x}|, \qquad x, \bar{x} \in I \cap \langle \xi - a, \xi + a\rangle, \tag{4}$$

$$|h(x, y)-h(\bar{x}, \bar{y})| \leqslant k|x-\bar{x}|+l|y-\bar{y}|, \tag{5}$$
$$(x, y), (\bar{x}, \bar{y}) \in \Omega \cap (\langle \xi - a, \xi + a\rangle \times \langle \eta - \beta, \eta + \beta\rangle),$$

and

$$ls < 1, \tag{6}$$

then equation (1) *has in* I *exactly one solution* φ *fulfilling a Lipschitz condition in a neighbourhood of* $x = \xi$ *and such that* $\varphi(\xi) = \eta$.

Proof. Without loss of generality we may assume that ξ is the left endpoint of I (cf. [3], p. 48) and $\xi = \eta = 0$. First we shall prove the existence of the desired solution.

Put

$$v = \frac{k}{1-ls}, \tag{7}$$

and choose a number c

$$0 < c \leqslant \min(a, \beta/v) \tag{8}$$

in such a manner that the following inclusion holds:

$$D = \{(x, y):\ 0 \leqslant x \leqslant c,\ |y| \leqslant vx\} \subset \Omega. \tag{9}$$

It is easily seen that $D \subset \langle 0, a\rangle \times \langle -\beta, \beta\rangle$.

Let us define F as the space of functions γ which are continuous in $\langle 0, c\rangle$, with the norm $\|\gamma\| = \sup\limits_{\langle 0,c\rangle} |\gamma(x)|$. Evidently, F is a Banach vector space.

We denote by A the set of functions $\varphi \in F$ which fulfil the following conditions:

$$\varphi(0) = 0, \tag{10}$$

$$|\varphi(x)-\varphi(\bar{x})| \leqslant v|x-\bar{x}|, \qquad x, \bar{x} \in \langle 0, c\rangle. \tag{11}$$

Setting in (11) $\bar{x} = 0$ and taking into accout (10) and (8) we obtain

$$|\varphi(x)| \leqslant v|x| \leqslant v \cdot \frac{\beta}{v} = \beta, \qquad x \in \langle 0, c\rangle. \tag{12}$$

Hence and by (11) A is a set of functions equibounded and equicontinuous in $\langle 0, c\rangle$. By Arzela's theorem A is a compact subset of F. Evidently, A is also convex.

Now we shall prove that the transformation $\psi = T(\varphi)$ defined by the formula

$$\psi(x) = h\big(x, \varphi[f(x)]\big) \tag{13}$$

maps A into itself. Let $\varphi \in A$ and let ψ be given by (13). Evidently, $\psi \in F$. Moreover, according to Remark 1 and relation (2) we obtain $\psi(0) = h\big(0, \varphi[f(0)]\big) = h(0, 0) = 0$, i.e. φ fulfils condition (10). Let $x, \bar{x} \in \langle 0, c\rangle$. Then the points $\big(x, \varphi[f(x)]\big)$ and $\big(\bar{x}, \varphi[f(\bar{x})]\big)$ belong to $D \subset \Omega \cap (\langle 0, a\rangle \times \langle -\beta, \beta\rangle)$. Now, in view of (5), (11), (4) and (7) we have

$$\begin{aligned}|\psi(x)-\psi(\bar{x})| &= \big|h\big(x, \varphi[f(x)]\big) - h\big(\bar{x}, \varphi[f(\bar{x})]\big)\big| \\ &\leqslant k|x-\bar{x}| + l\,|\varphi[f(x)] - \varphi[f(\bar{x})]| \\ &\leqslant k|x-\bar{x}| + lvs\,|x-\bar{x}| = v\,|x-\bar{x}|,\end{aligned}$$

which means that ψ fulfils conditions (11). This completes the proof of the inclusion $T(A) \subset A$.

Transformation (13) is continuous in A. Indeed, suppose that $\varphi_n \in A$, $n = 0, 1, \ldots$, and φ_n tends to φ_0 uniformly in $\langle 0, c\rangle$ (i.e. in the norm of F) and write $\psi_n(x) = h\big(x, \varphi_n[f(x)]\big)$. According to the previous part of the proof, $\psi_n \in A$, $n = 0, 1, \ldots$ Now, the uniform convergence $\psi_n \to \psi_0$ results from the uniform continuity of h in D and from the uniform convergence $\varphi_n \to \varphi_0$. Thus T is continuous in A.

Applying Schauder's principle we obtain the existence of at least one solution $\varphi \in A$ of equation (1). This solution may be uniquely extended onto the whole interval I. The solution thus obtained is continuous in I (cf. [3], p. 70, Theorem 3.2).

Now we shall prove that this solution is unique. Suppose that φ_1, φ_2 are solutions of equation (1) such that

$$\varphi_1(0) = \varphi_2(0) = 0 \tag{14}$$

and

$$|\varphi_i(x) - \varphi_i(\bar{x})| \leqslant \mu\,|x-\bar{x}|, \qquad x, \bar{x} \in \langle 0, d\rangle,\ i = 1, 2. \tag{15}$$

We may assume that $d > 0$ is chosen in such a manner that $|\varphi_i(x)| \leqslant \beta$ for $x \in \langle 0, d\rangle$ and $i = 1, 2$. We have

$$|\varphi_1(x) - \varphi_2(x)| \leqslant l^n\,|\varphi_1[f^n(x)] - \varphi_2[f^n(x)]|, \qquad x \in \langle 0, d\rangle,\ n = 1, 2, \ldots \tag{16}$$

In view of (16), (15), (14), and (4) we obtain for $x \in \langle 0, d\rangle$

$$\begin{aligned}|\varphi_1(x) - \varphi_2(x)| &\leqslant l^n\big(|\varphi_1[f^n(x)]| + |\varphi_2[f^n(x)]|\big) \\ &\leqslant 2l^n\mu\,|f^n(x)| \leqslant 2\mu(ls)^n|x| \leqslant 2\mu d(ls)^n, \qquad n = 1, 2, \ldots\end{aligned}$$

Hence and by (6), letting $n \to \infty$, we obtain

$$\varphi_1(x) = \varphi_2(x), \quad x \in \langle 0, d \rangle.$$

It follows that $\varphi_1(x) = \varphi_2(x)$ for $x \in I$ (cf. [3], p. 70) and this completes the proof of the uniqueness of the obtained solution.

2. In this section we shall prove the following

THEOREM 2. *Let hypotheses* (i)–(iii) *and inequalities* (4)–(6) *be fulfilled. If the function f is increasing in I and h is increasing with respect to each variable in Ω, then the unique solution φ fulfilling the Lipschitz condition in a neighbourhood of the point $x = \xi$ and such that $\varphi(\xi) = \eta$ is increasing in I.*

Proof. We may assume that ξ is the left endpoint of I and $\xi = \eta = 0$. Let $A_1 = \{\varphi \in A: \varphi$ increasing in $\langle 0, c \rangle\}$, where A is the set of functions defined as in the proof of Theorem 1. It is easy to verify that A_1 is a compact and convex subset of F, and that transformation (13) maps A_1 into itself. On account of Schauder's principle, the unique solution φ fulfilling a Lipschitz condition and such that $\varphi(\xi) = \eta$ must be increasing in $\langle 0, c \rangle$. Now we shall prove that φ is increasing in I. To this end we denote by x_0 the supremum of all b such that φ is increasing in $\langle 0, b \rangle$ and suppose that $I \setminus \langle 0, x_0 \rangle$ is non-empty. From (i) we have $f(x_0) < x_0$. If follows from the continuity of f that there exists a $\delta > x_0$ such that $\langle 0, \delta \rangle \subset I$ and for $x \in \langle 0, \delta \rangle$ we have $f(x) < x_0$. Hence for $0 \leqslant x_1 \leqslant x_2 \leqslant \delta$ we obtain

$$\varphi(x_1) = h\big(x_1, \varphi[f(x_1)]\big) \leqslant h\big(x_2, \varphi[f(x_2)]\big) = \varphi(x_2),$$

i.e. φ is increasing in $\langle 0, \delta \rangle$, $\delta > x_0$. This contradiction completes the proof of Theorem 2.

3. Now we shall prove a theorem on the existence and uniqueness of convex solution of equation (1) (cf. [2], [4], [5]).

We denote by $f'_r(\xi)$ the right derivative of f at the point $x = \xi$. If ξ is the right endpoint of I, we assume $f'_r(\xi) \overset{\text{df}}{=} \lim_{x \to \xi-} f'_r(x)$.

THEOREM 3. *Let hypotheses* (i)–(iii) *be fulfilled and let Ω be a convex domain. Suppose that f is increasing and convex in I, h is increasing with respect to each variable and convex in Ω. If h has a total differential at the point (ξ, η) and*

$$f'_r(\xi) \frac{\partial h}{\partial y}(\xi, \eta) < 1, \tag{17}$$

then there exists in I exactly one convex solution φ of equation (1) *such that $\varphi(\xi) = \eta$ and $\varphi'_r(\xi) \neq -\infty$. This solution is increasing in I.*

Proof. We assume that ξ is the left endpoint of I and $\xi = \eta = 0$. It follows from the convexity of f and h and from the existence of a total differential of h at $(0, 0)$, that for every $\varepsilon > 0$ there exist $\alpha > 0$ and $\beta > 0$ such that inequalities (4) and (5) are valid with

$$s = f_r'(0)+\varepsilon, \quad k = \frac{\partial h}{\partial x}(0, 0)+\varepsilon, \quad l = \frac{\partial h}{\partial y}(0, 0)+\varepsilon$$

(cf. [1], p. 18). On account of (17), we can choose an $\varepsilon > 0$ so small tha- for s and l defined above inequality (6) holds. We define the set of func tions

$$A_2 = \{\varphi\colon \varphi \in A_1, \varphi \text{ is convex in } \langle 0, c\rangle\},$$

where A_1 is defined in the proof of Theorem 2. It is easy to verify that A_2 is a compact and convex subset of F and T defined by (13) maps A_2 into itself. In view of Schauder's principle the unique solution of equation (1) belonging to A_1 must be convex in $\langle 0, c\rangle$. Now, by a similar argument as that employed in Theorem 2 we can prove that this solution is convex in I. This completes the proof.

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