ANNALES
POLONICI MATHEMATICI
XV (1964)

Determination of linear differential geometric objects of the first class, with two components, in a two-dimensional space

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§ 1. In the present paper we deal with linear, purely differential geometric objects of the first class, with two components, in a two-dimensional space, i.e., according to the terminology of J. Aczél and S. Gołąb (cf. [2], p. 15), with linear geometric objects of the type [2, 2, 1]. After a change of the coordinate system

$$\begin{aligned} \xi^{1'} &= \xi^{1'}(\xi^1, \xi^2)\,, \\ \xi^{2'} &= \xi^{2'}(\xi^1, \xi^2) \end{aligned} \tag{1}$$

the components ω_1, ω_2 of the object are transformed according to the rule

$$\begin{aligned} \omega_1' &= f_{11}\omega_1 + f_{12}\omega_2 + g_1\,, \\ \omega_2' &= f_{21}\omega_1 + f_{22}\omega_2 + g_2 \end{aligned} \tag{2}$$

where the functions f_{ij} and g_k depend on transformation (1). Since we deal with objects of the first class, the functions f_{ij} and g_k depend only on the first derivatives

$$A_\lambda^{\lambda'} \overset{\text{df}}{=} \frac{\partial \xi^{\lambda'}}{\partial \xi^\lambda}\,, \quad \lambda = 1, 2;\ \lambda' = 1', 2'\,,$$

of the new variables with respect to the old ones (and these derivatives are calculated at a fixed point of the space, namely at the point at which the object is fastened).

In the sequel we shall use the matrix notation

$$\Omega = \begin{Vmatrix} \omega_1 \\ \omega_2 \end{Vmatrix}, \quad \Omega' = \begin{Vmatrix} \omega_1' \\ \omega_2' \end{Vmatrix},$$

$$F = \begin{Vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{Vmatrix}, \quad G = \begin{Vmatrix} g_1 \\ g_2 \end{Vmatrix}, \quad A = \begin{Vmatrix} A_1^{1'} & A_2^{1'} \\ A_1^{2'} & A_2^{2'} \end{Vmatrix}. \tag{3}$$

With the aid of notation (3) the transformation law (2) of the object may be written shortly in the form

(4) $$\Omega' = F(A)\Omega + G(A)\,.$$

Since, for a prescribed system of values $A_\lambda^{\lambda'}$ fulfilling the condition

$$J \stackrel{\mathrm{df}}{=} \det A = A_1^{1'}A_2^{2'} - A_2^{1'}A_1^{2'} \neq 0\,,$$

one can always find a locally invertible transformation for which the values of the derivatives at the given point equal $A_\lambda^{\lambda'}$, in formula (4) A may denote an arbitrary regular matrix.

In the case where $G(A) \equiv 0$ all the objects with transformation formula (4) have been determined in our earlier paper [6]. Now we do not make any assumption about the functions $F(A)$ and $G(A)$ except that they are defined for all regular 2×2 matrices A.

Since the transformation law (4) must have a group property, the functions $F(A)$ and $G(A)$ cannot be quite arbitrary but must satisfy (for arbitrary regular matrices A and B) the system of matrix-functional equations

(5) $$\begin{aligned} F(BA) &= F(B)F(A)\,,\\ G(BA) &= F(B)G(A) + G(B)\,, \end{aligned}$$

and the matrix $F(A)$ must be regular for every regular matrix A. System (5) has been solved in our paper [8] (cf. also [4], [5]). Namely, we have the following

LEMMA 1. *If the functions $F(A)$ and $G(A)$ satisfy system (5) for arbitrary regular matrices A, B and if the matrix $F(A)$ is regular for every regular matrix A, then the functions $F(A)$ and $G(A)$ must have one of the following seven forms*:

(6) $$F(A) = C\begin{Vmatrix}\varphi(J) & 0\\ 0 & \varphi(J)\end{Vmatrix}AC^{-1}\,,\qquad G(A) = [F(A)-E]D\,,$$

(7) $$F(A) = C\begin{Vmatrix}\varphi_1(J) & 0\\ 0 & \varphi_2(J)\end{Vmatrix}C^{-1}\,,\qquad G(A) = [F(A)-E]D\,,$$

(8) $$F(A) = C\begin{Vmatrix}\varphi(J) & \varphi(J)\alpha(J)\\ 0 & \varphi(J)\end{Vmatrix}C^{-1}\,,\qquad G(A) = [F(A)-E]D\,,$$

(9) $$F(A) = C\begin{Vmatrix}\varkappa(J) & -\sigma(J)\\ \sigma(J) & \varkappa(J)\end{Vmatrix}C^{-1}\,,\qquad G(A) = [F(A)-E]D\,,$$

(10) $$F(A) = C\begin{Vmatrix}1 & 0\\ 0 & \psi(J)\end{Vmatrix}C^{-1},\qquad G(A) = \begin{Vmatrix}\ln|\varphi(J)|\\ \lambda[\psi(J)-1]\end{Vmatrix},$$

(11) $$F(A) \equiv E\,,\qquad G(A) = \begin{Vmatrix}\ln|\varphi(J)|\\ \ln|\psi(J)|\end{Vmatrix},$$

(12) $$F(A) = C\begin{Vmatrix}1 & \alpha(J)\\ 0 & 1\end{Vmatrix}C^{-1}\,,\qquad G(A) = \begin{Vmatrix}\ln|\varphi(J)| + \mu\alpha^2(J)\\ 2\mu\alpha(J)\end{Vmatrix}.$$

In formulae (6)-(12) λ and μ are arbitrary constants, $C = \begin{Vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{Vmatrix}$ is an arbitrary regular matrix, $D = \begin{Vmatrix} \delta_1 \\ \delta_2 \end{Vmatrix}$ is an arbitrary constant matrix, $E = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$ is the unit matrix, and J denotes the determinant of the matrix A. $\varphi, \varphi_1, \varphi_2, \psi$ are arbitrary, non-vanishing identically, multiplicative functions, i.e. they satisfy the functional equation

$$\varphi(xy) = \varphi(x)\varphi(y) , \quad xy \neq 0 . \tag{13}$$

$a(x)$ is an arbitrary solution of the functional equation

$$a(xy) = a(x) + a(y) , \quad xy \neq 0 , \tag{14}$$

and we may assume that $a(x) \not\equiv 0$, for otherwise formulae (8) and (12) would coincide with (7) and (11), respectively. Finally, the functions $\varkappa(x)$ and $\sigma(x)$ are an arbitrary solution of the system of functional equations

$$\begin{aligned} \varkappa(xy) &= \varkappa(x)\varkappa(y) - \sigma(x)\sigma(y) , \\ \sigma(xy) &= \varkappa(x)\sigma(y) + \sigma(x)\varkappa(y) , \end{aligned} \qquad xy \neq 0 .$$

Lemma 1 immediately implies the following

THEOREM 1. *Every linear differential geometric object of the first class, with two components, in a two-dimensional space must be of the form* (4) (*with shortened notation* (3)), *where $F(A)$ and $G(A)$ have one of the forms* (6)-(12).

§ 2. Among the objects thus obtained there are, however, many equivalent ones. Thus the next problem is to investigate the equivalence of the objects with transformation formula (4). Now we proceed to solve this problem under the supposition that the functions f_{ij} and g_k (occurring in the transformation formulae of the objects investigated) are measurable.

The notion of the equivalence (or similarity) of geometric objects has been introduced by S. Gołąb [3] (cf. also [2], p. 16):

DEFINITION. Two geometric objects Ω and Σ are called *equivalent* if there exists an invertible function H such that the relation $\Sigma = H(\Omega)$ holds in every coordinate system (is invariant under transformations of the coordinate system).

LEMMA 2. *The relation of the equivalence of objects is reflexive, symmetric and transitive.*

Given n objects $\Omega_1, \ldots, \Omega_n$ (each with an arbitrary number of components), we may unify them into one new object $\Omega = (\Omega_1, \ldots, \Omega_n)$ (cf. [2], p. 13).

LEMMA 3. *If an object Ω_i is equivalent to an object Σ_i $(i = 1, \ldots, n)$, then the object $\Omega = (\Omega_1, \ldots, \Omega_n)$ is equivalent to the object $\Sigma = (\Sigma_1, \ldots, \Sigma_n)$.*

We omit the simple proofs of lemmas 2 and 3. We shall prove, however, the following

LEMMA 4. *If C is a regular matrix, then the object Ω with the transformation formula*

$$\Omega' = CF(A)C^{-1}\Omega + CG(A) \tag{15}$$

is equivalent to an object Σ with the transformation formula

$$\Sigma' = F(A)\Sigma + G(A) . \tag{16}$$

Proof. Let us put $H(\Omega) = C^{-1}\Omega$. The function $H(\Omega)$ is invertible, since the matrix C is regular. Moreover, we have by (15)

$$H(\Omega') = C^{-1}\Omega' = C^{-1}CF(A)C^{-1}\Omega + C^{-1}CG(A) = F(A)C^{-1}\Omega + G(A)$$
$$= F(A)H(\Omega) + G(A) ,$$

which means that $\Sigma = H(\Omega)$ is really a geometric object with the transformation formula (16).

We note also the following two lemmas:

LEMMA 5 (cf. [9], theorem 2). *Every linear geometric object of the first class, with one component, with a measurable transformation rule, is equivalent either to the G-density (ordinary density)*

$$\omega' = J\omega ,$$

or to the W-density (Weyl density)

$$\omega' = |J|\omega ,$$

or to the biscalar

$$\omega' = (\operatorname{sgn} J)\omega .$$

LEMMA 6 (cf. [6], lemma 13). ***Every object with two components, consisting of a pair of densities of a pair of biscalars, of a density and a biscalar, of a density and a scalar, or of a biscalar and a scalar, is equivalent to the object consisting of a pair of W-densities of weight −1, to the object consisting of a pair of G-densities of weight −1, or to the object consisting of a pair of biscalars.***

§ 3. In the case of the homogeneous transformation law, i.e. if $G(A) \equiv 0$, the equivalence of linear geometric objects of type $[2,2,1]$ has been established (under the supposition of the measurability of the functions f_{ij}) in our paper [6] (cf. also [7]). Namely, we have the following

LEMMA 7. *Every linear homogeneous geometric object of type $[2,2,1]$ with a measurable transformation formula is equivalent to one and only one* (¹) *of the following objects:*

(¹) I.e. objects (17), (18), (24), (25) corresponding to different values of the parameter p are not equivalent. Thus we have here not nine objects but indeed an infinite number of non-equivalent objects: five single objects ((19), (20), (21), (22) and (23)) and four one-parameter families of objects ((17), (18), (24) and (25)).

1. *Contravariant vector-W-density of some definite weight* $-p$:

$$\Omega' = |J|^p A\Omega\,, \quad p \neq 0\,. \tag{17}$$

2. *Contravariant vector-G-density of some definite weight* $-p$:

$$\Omega' = (|J|^p \operatorname{sgn} J) A\Omega\,, \quad p \neq 0\,. \tag{18}$$

3. *Contravariant vector*:

$$\Omega' = A\Omega\,. \tag{19}$$

4. *Contravariant G-vector*:

$$\Omega' = (\operatorname{sgn} J) A\Omega\,. \tag{20}$$

5. *Pair of W-densities of weight* -1:

$$\omega_1' = |J|\,\omega_1\,, \quad \omega_2' = |J|\,\omega_2\,. \tag{21}$$

6. *Pair of G-densities of weight* -1:

$$\omega_1' = J\,\omega_1\,, \quad \omega_2' = J\omega_2\,. \tag{22}$$

7. *Pair of biscalars*:

$$\omega_1' = (\operatorname{sgn} J)\,\omega_1\,, \quad \omega_2' = (\operatorname{sgn} J)\,\omega_2\,. \tag{23}$$

8. *One of the family of objects*:

$$\Omega' = \begin{Vmatrix} \cos(p\ln|J|) & -\sin(p\ln|J|) \\ \sin(p\ln|J|) & \cos(p\ln|J|) \end{Vmatrix} \Omega\,, \quad p > 0\,. \tag{24}$$

9. *One of the family of objects*:

$$\Omega' = (\operatorname{sgn} J)\begin{Vmatrix} \cos(p\ln|J|) & -\sin(p\ln|J|) \\ \sin(p\ln|J|) & \cos(p\ln|J|) \end{Vmatrix} \Omega\,, \quad p > 0\,. \tag{25}$$

Now we shall show that (under the supposition of the measurability of the functions f_{ij} and g_k) an object with transformation formula (4) must be equivalent also to one of the above objects. Namely we shall prove the following

THEOREM 2. *Every linear differential geometric object of the first class, with two components, in a two-dimensional space, with a measurable law of transformation, is equivalent to one and only one* [2] *of the objects* (17)-(25).

Proof. In the proof we shall distinguish 4 cases, depending on the form of the functions $F(A)$ and $G(A)$ (formulae (6)-(12); cf. lemma 1).

1. The object Ω has the transformation rule

$$\Omega' = F(A)\Omega + [F(A) - E]D \tag{26}$$

[2] Compare footnote [1].

(formulae (6)-(9)). We put $H(\Omega) = \Omega + D$. The function $H(\Omega)$ is evidently invertible. $\Sigma = H(\Omega)$ is a geometric object with the transformation formula

$$\Sigma' = F(A)\Sigma .$$

In fact, we have by (26)

$$\Sigma' = H(\Omega') = \Omega' + D = F(A)\Omega + F(A)D - D + D = F(A)[\Omega + D]$$
$$= F(A)H(\Omega) = F(A)\Sigma .$$

Thus, according to lemmas 2 and 7, the object Ω is equivalent to exactly one of the objects (17)-(25).

2. Object Ω has a transformation rule expressed by (4) with the functions $F(A)$ and $G(A)$ given by formulae (10). In virtue of lemma 4 the object Ω is equivalent to an object Σ with the transformation formula

$$\Sigma' = \begin{Vmatrix} 1 & 0 \\ 0 & \psi(J) \end{Vmatrix} \Sigma + \begin{Vmatrix} \ln|\varphi(J)| \\ \lambda[\psi(J) - 1] \end{Vmatrix} . \tag{27}$$

Formula (27) can be written in components in the form

$$\sigma_1' = \sigma_1 + \ln|\varphi(J)| , \qquad \sigma_2' = \psi(J)\sigma_2 + \lambda[\psi(J) - 1] .$$

Thus, as we see, Σ is an object consisting of two linear geometric objects with one component. On account of lemmas 5, 3 and 6 the object Σ, and thus also the object Ω, is equivalent to one of the objects (21), (22), (23).

3. The object Ω has transformation formula (4) with the functions $F(A)$ and $G(A)$ given by formulae (11). Having written the transformation rule of the object Ω in the form

$$\omega_1' = \omega_1 + \ln|\varphi(J)| , \qquad \omega_2' = \omega_2 + \ln|\psi(J)| ,$$

we see that Ω consists of two linear geometric objects of the first class with one component. Thus, on account of lemmas 5, 3 and 6 it is equivalent to one of the objects (21), (22), (23).

4. The object Ω has transformation rule (4) with the functions $F(A)$ and $G(A)$ given by formulae (12). According to lemmas 4 and 2 we may assume that the transformation rule of the object Ω has the form

$$\Omega' = \begin{Vmatrix} 1 & a(J) \\ 0 & 1 \end{Vmatrix} \Omega + \begin{Vmatrix} \ln|\varphi(J)| + \mu a^2(J) \\ 2\mu a(J) \end{Vmatrix} . \tag{28}$$

Since we have supposed that the functions occurring in the transformation rule of the object Ω are measurable, we may insert into formula (28) the measurable solutions of equations (13) and (14) (cf. [1], p. 48):

$$\varphi(x) = |x|^{\varrho} \qquad \text{or} \qquad \varphi(x) = (\operatorname{sgn} x)|x|^{\varrho} ,$$
$$a(x) = \pi \ln|x|$$

(ϱ, π — arbitrary constants). Thus formula (28) can be written in the form

$$\begin{aligned} \omega_1' &= \omega_1 + \omega_2\pi\ln|J| + \varrho\ln|J| + \mu\pi^2\ln^2|J| , \\ \omega_2' &= \omega_2 + 2\mu\pi\ln|J| . \end{aligned} \tag{29}$$

By the assumption $a(x) \not\equiv 0$, we have $\pi \neq 0$. But ϱ and μ may as well equal zero. Therefore we must distinguish two further cases:

(a) $\mu = 0$. Then formulae (29) take the form

$$\begin{aligned} \omega_1' &= \omega_1 + (\omega_2\pi + \varrho)\ln|J| , \\ \omega_2' &= \omega_2 . \end{aligned}$$

As we see, ω_2 is a scalar, and consequently only ω_1 is a linear geometric object of the first class with one component. Thus, on account of lemmas 5, 3 and 6, the object Ω is equivalent to one of the objects (21), (22), (23).

(b) $\mu \neq 0$. We put

$$\begin{aligned} h_1(\omega_1, \omega_2) &= \omega_1 - \frac{\varrho}{2\mu\pi}\,\omega_2 - \frac{1}{4\mu}\,\omega_2^2 , \\ h_2(\omega_1, \omega_2) &= \omega_2 . \end{aligned} \tag{30}$$

The function $H(\Omega) = \left\| \begin{matrix} h_1(\omega_1, \omega_2) \\ h_2(\omega_1, \omega_2) \end{matrix} \right\|$ is invertible. In fact, for given h_1 and h_2 we can uniquely determine ω_2 from the second relation of (30) and then (ω_2 being already known) we can find ω_1 from the first relation of (30).

Now we shall verify how the components of the object

$$\Sigma = \left\| \begin{matrix} \sigma_1 \\ \sigma_2 \end{matrix} \right\| = \left\| \begin{matrix} h_1(\omega_1, \omega_2) \\ h_2(\omega_1, \omega_2) \end{matrix} \right\|$$

change with a change of the coordinate system. We have by (30) and (29)

$$\begin{aligned} \sigma_1' = h_1(\omega_1', \omega_2') &= \omega_1' - \frac{\varrho}{2\mu\pi}\,\omega_2' - \frac{1}{4\mu}\,[\omega_2']^2 \\ &= \omega_1 + \omega_2\pi\ln|J| + \varrho\ln|J| + \mu\pi^2\ln^2|J| - \\ &\quad - \frac{\varrho}{2\mu\pi}\,[\omega_2 + 2\mu\pi\ln|J|] - \frac{1}{4\mu}\,[\omega_2 + 2\mu\pi\ln|J|]^2 \\ &= \omega_1 - \frac{\varrho}{2\mu\pi}\,\omega_2 - \frac{1}{4\mu}\,\omega_2^2 = \sigma_1 , \end{aligned}$$

and

$$\sigma_2' = h_2(\omega_1', \omega_2') = \omega_2' = \omega_2 + 2\mu\pi\ln|J| = \sigma_2 + 2\mu\pi\ln|J| .$$

As we see, σ_1 is a scalar and σ_2 is a linear geometric object of the first class with one component. Consequently, on account of lemmas 5, 3 and 6 the object Σ, and thus also the object Ω, are equivalent to one of the objects (21), (22), (23).

Thus we have proved that for each possible form (6)-(12) of the functions $F(A)$ and $G(A)$ (under the supposition of the measurability of these functions) an object Ω with transformation rule (4) is equivalent to one of the objects (17)-(25), (which, as has been proved in [6], are not equivalent to one another (3)). On account of theorem 1, this completes the proof of theorem 2.

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(3) Compare footnote (1).

Reçu par la Rédaction le 16. 5. 1962