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Lyapunov numbers for a countable system of ordinary differential equations

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Abstract. Consider the system of differential equations

$$(1)\qquad \dot{x}_i = g_i(t) + \sum_{\substack{j=1\\ i\neq j}}^{\infty} (g_{ij}(t) + \tilde{g}_{ij}(t))x_j; \qquad x_i(t_0) = x_i^0, \qquad i = 1, 2, \ldots,$$

where $g_i(t)$, $g_{ij}(t)$ and $\tilde{g}_{ij}(t)$ are continuous functions (in general, complex-valued) of the real variable t, for $t_0 \leqslant t < \infty$ and $x^0 = (x_1^0, x_2^0, \ldots) \in C$, the space of convergent sequences. System (1) can be written in the form $\dot{x} = A(t)x$, $x(0) = x^0$. Suppose that for each $t \in [0, \infty)$, $A(t)$ is the infinitesimal generator of a C^0-semigroup, $A(t)$: $D \subset C \to C$ is densely defined, and assume the existence, uniqueness and continuous dependence of the solutions of (1), in $[0, \infty)$. It is proved that for $t > t_0$ and under some conditions on the coefficients $g_i(t)$, $g_{ij}(t)$ and $\tilde{g}_{ij}(t)$, there exists a system of linearly independent solutions, $(x_i(t))_1^{\infty}$, $x_i(t) = (x_{1i}, x_{2i}, \ldots)$ such that $\lim_{t\to\infty} (x_{ik}(t)/x_{kk}(t)) = 0$ for every $i \neq k$. Furthermore, if $\lambda(x_i(t))$ is the Lyapunov number of the solution $x_i(t)$, $i = 1, 2, \ldots$, then $\lambda(x_i(t)) = \overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t \operatorname{Re}(g_i(s))ds$. The results generalize theorems of Szmydt [12] and Perron [11]; the proof relies on a topological method presented in [15].

1. Introduction. The first investigations of countable systems of ordinary differential equations date back to the origins of functional analysis around the last turn of the century.

In course of time one was led to consider such systems in connection with concrete problems in natural science such as branching processes, Hille [5], solution of partial differential equations by Fourier Methods, Lewis [7], Dickey [4], and semi-discretization of Cauchy's problems for parabolic equations, Voigt [13]. In the following we apply a topological method presented in [15] to study the asymptotic behavior and give a formula to calculate the Lyapunov number of a countable linear non-autonomous system

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of ordinary differential equations. An overview of the theory of countable systems of differential equations is given in the books of Deimling [3] and Zautykov and Valleev [14].

2. Preliminaries. We begin by recalling a few definitions and results from [6].

Suppose X is a Banach space, $\mathbf{R}^+ = [0, \infty)$, u: $\mathbf{R} \times X \times \mathbf{R}^+ \to X$ is a given mapping and define $U(\sigma, t)$: $X \to X$ for $\sigma \in \mathbf{R}^+$ by $U(\sigma, t)x = u(\sigma, x, t)$. A *process* on X is a mapping u: $\mathbf{R} \times X \times \mathbf{R}^+ \to X$ satisfying the following properties:

(i) u is continuous,

(ii) $U(\sigma, 0) = I$ (identity),

(iii) $U(\sigma+s, t)U(\sigma, s) = U(\sigma, s+t)$.

A process is said to be an *autonomous process* or a *semidynamical system* if $U(\sigma, t)$ is independent of σ; that is, $T(t) = U(0, t)$, $t \geqslant 0$, and $T(t)x$ is continuous for $(t, x) \in \mathbf{R}^+ \times X$.

DEFINITION. Suppose u is a process on X.

The *trajectory* $\tau^+(\sigma, x)$ through $(\sigma, x) \in \mathbf{R} \times X$ is the set in $\mathbf{R} \times X$ defined by

$$\tau^+(\sigma, x) = \{(\sigma+t, U(\sigma, t)x) |\ t \in \mathbf{R}^+\}.$$

The *orbit* $\gamma^+(\sigma, x)$ through (σ, x) is the set in X defined by

$$\gamma^+(\sigma, x) = \{U(\sigma, t)x |\ t \in \mathbf{R}^+\}.$$

We assume in the following that the integral through each $(\sigma, x) \in \mathbf{R} \times X$ is unique. We define $\tau^{-1}(x) = \{(\sigma, y) \in \mathbf{R} \times X |\ \exists t > 0$ such that $U(\sigma, t)y = x\}$. If $P_0 = (\sigma, x) \in \mathbf{R} \times X$ and $z \in \gamma^+(\sigma, x)$, we define

$$t_z = \inf\{t \geqslant 0 |\ U(\sigma, t)x = z\}, \qquad Q_z = (\sigma + t_z, U(\sigma, t_z)x),$$

$$[P_0, Q_z] = \{(\sigma+t, U(\sigma, t)x) |\ 0 \leqslant t \leqslant t_z\}.$$

Let Ω be an open set of $\mathbf{R} \times X$, ω an open set of Ω, $\omega \neq \emptyset$ and $\partial\omega = \bar{\omega} \cap \overline{(\Omega - \omega)}$ the boundary of ω with respect to Ω. We put

$$S^0 = \{P_0 = (\sigma, x) \in \partial\omega |\ \exists z \in \gamma^+(\sigma, x) \text{ with } (P_0, Q_z) \neq \emptyset \text{ and } (P_0, Q_z) \cap \bar{\omega} = \emptyset\},$$

$$S = \{Q \in \partial\omega |\ \exists P_0 = (\sigma, x) \in \omega \text{ with } Q \in \tau^+(\sigma, x) \text{ and } [P_0, Q) \subset \omega\},$$

$$S^* = S^0 \cap S.$$

The points of S are called *egress points*. The points of S^* are called *strict egress points*. Given a point $P_0 = (\sigma, x) \in \omega$, if the trajectory $\tau^+(\sigma, x)$ of the process is contained in ω for every $t > 0$, we say that the trajectory is asymptotic with respect to ω. If the trajectory $\tau^+(\sigma, x)$ is not asymptotic with respect to ω, there exists $t > 0$ such that $(\sigma+t,\ U(\sigma,\ t)x) \in \partial\omega$. Taking

$$t_{po} = \inf\{t > 0 \mid (\sigma+t,\ U(\sigma,\ t)x) \in \partial\omega\}, \qquad Q = (\sigma+t_{po},\ U(\sigma,\ t_{po})x) = C(P_0),$$

we have

$$[P_0,\ Q) \subset \omega.$$

The point $C(P_0)$ is called after Poincaré the *consequent of* P_0. Define G to be the set of all $P_0 = (\sigma,\ x) \in \omega$ such that $C(P_0)$ exists and belongs to S^*. G is called the *left shadow of* ω. Consider the mapping, called the *consequent operator*,

$$K\colon\ S^* \cup G \to S^*,$$

$K(P_0) = C(P_0)$ if $P_0 \in \omega$ and $K(P_0) = P_0$ if $P_0 \in S^*$.

The following lemma is proved in [10]; see also [15].

LEMMA 1. *The consequent operator* $K\colon\ S^* \cup G \to S^*$ *is continuous.*

We will need the following result of Ważewski [15], [10], [6].

LEMMA 2. *Assume that there exist sets* W, S, Z; W *open,* $S \subset \partial W$ *and* $Z \subset W \cup S$, $Z \neq \emptyset$, *such that*

(i) $S = S^*$,

(ii) $Z \cap S$ *is not a retract of* Z,

(iii) $Z \cap S$ *is a retract of* S.

Then there exists at least one point $P_0 = (\sigma,\ x) \in Z \cap W$ *such that* $C(P_0)$ *does not exist; that is,* $\tau^+(\sigma,\ x) \subset W$.

Examples of processes are described by differential equations in a Banach space X,

$$\dot{x} = Ax + f(t,\ x), \qquad x(0) = x_0.$$

3. Main results. Consider the system of differential equations

$$\dot{x}_i = g_i(t)x_i + \sum_{j=1}^{\infty} (g_{ij}(t) + \tilde{g}_{ij}(t))x_j, \qquad x_i(t_0) = x_i^0, \qquad i = 1, 2, \ldots, \tag{1}$$

where $g_i(t)$, $g_{ij}(t)$ and $\tilde{g}_{ij}(t)$ are continuous functions (in general, complex-valued) of the real variable t for $t_0 < t < \infty$ and $x^0 = (x_1^0,\ x_2^0, \ldots) \in C$, the space of convergent sequences with norm $\|x^0\| = \sup_i(|x_i^0|)$. System (1) can be written in the form

$$\dot{x} = A(t)x, \qquad x(0) = x^0. \tag{2}$$

Suppose that, for each $t \in [0,\ \infty)$, $A(t)$ is the infinitesimal generator of a C^0-semigroup and $A(t)\colon\ D \subset C \to C$ is densely defined in C. We assume the existence and uniqueness of a continuously differentiable solution of (1), as well as continuous dependence of solutions with respect to initial data [3], [9].

Our purpose is to study asymptotic directions of the solutions of (1) and to generalize the results of Perron [8], p. 173, and Szmydt [12], Corollary 1, Remark 2, p. 30.

The principal tool used here is a topological method developed in [15]. Theorems and notations used here can be found in [15], [6], [10].

LEMMA 3. *Consider the differential inequalities*

$$\dot{z}(t) < \sigma_1(t)z - [\gamma(t)+\tilde{\gamma}(t)], \qquad \dot{w}(t) > -\sigma_2(t)w + [\gamma(t)+\tilde{\gamma}(t)]. \tag{3}$$

Let us suppose that the functions $\gamma(t)$, $\tilde{\gamma}(t)$, $\sigma_i(t)$ $(i = 1, 2)$ *satisfy the following conditions:* $\gamma(t)$, $\tilde{\gamma}(t)$, $\sigma_i(t)$ $(i = 1, 2)$ *are continuous for* $T \leqslant t < \infty$, $\gamma(t) \geqslant 0$, $\tilde{\gamma}(t) \geqslant 0$, $\sigma_i(t) \geqslant 0$, $\sigma_i(t) > 0$ *if* $\gamma(t) \not\equiv 0$, $i = 1, 2$, $T < t < \infty$ *and*

$$\begin{gathered} \int^{\infty} \sigma_i(t)dt = \infty, \qquad \lim \frac{\gamma(t)}{\sigma_i(t)} = 0, \qquad i = 1, 2, \\ \int^{\infty} \tilde{\gamma}(t)dt < \infty. \end{gathered} \tag{4}$$

Then inequalities (3) *admit solutions*

$$z = \varphi(t), \qquad w = \psi(t),$$

so that

$$\begin{gathered} \varphi(t) > 0, \qquad \psi(t) > 0 \quad \textit{for } T < t, \\ \lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} \psi(t) = 0. \end{gathered} \tag{5}$$

Proof. Each solution $z = \varphi(t)$, $w = \psi(t)$ of the system

$$\dot{z} = \sigma_1(t)z - \tilde{\gamma}(t) - \delta_1(t), \qquad \dot{w} = \sigma_2(t)w + \gamma(t) + \delta_2(t), \tag{6}$$

where $\delta_i(t) = \gamma(t) + \tau(t)\sigma_i(t)$ and $\tau(t)$ is any continuous positive function which tends to zero when $t \to \infty$, satisfies the system of differential inequalities (3).

We show that the functions

$$\varphi(t) = \exp\int_T^t \sigma_1(v)dv \int_t^{\infty} (\delta_1(y)+\tilde{\gamma}(y))\{\exp(-\int_T^y \sigma_1(u)du)\}dy, \tag{7}$$

$$\psi(t) = \exp\Big(-\int_T^t \sigma_2(v)dv \int_T^t (\delta_2(y)+\tilde{\gamma}(y))\{\exp\int_T^y \sigma_2(u)du\}dy\Big), \tag{8}$$

which are solutions of (6), satisfy (5). In fact,

$$\begin{aligned} \exp\int_T^t \sigma_1(v)dv \int_t^{\infty} \tilde{\gamma}(y)\{\exp(-\int_t^y \sigma_1(v)dv)\}dy &= \int_t^{\infty} \tilde{\gamma}(y)\exp(-\int_t^y \sigma_1(v)dv)dy \\ &\leqslant \int_t^{\infty} \tilde{\gamma}(y)dy \to 0 \quad \text{as } t \to \infty. \end{aligned}$$

From (4) and the definition of $\delta_i(t)$ $(i = 1, 2)$ we have

$$\lim_{t\to\infty}\frac{\delta_i(t)}{\sigma_i(t)} = 0. \tag{9}$$

Since $\delta_i(t) > 0$ and $\sigma_i(t) > 0$, we have for t large enough

$$\delta_i(t) < \sigma_i(t). \tag{10}$$

The right side of (7) is well-defined because the limit

$$\lim_{t\to\infty}\int_T^t \delta_1(y)\{\exp(-\int_T^y \sigma_1(u)du)\}dy \tag{11}$$

is finite. This follows from the existence of the limit of

$$\int_T^t \sigma_1(y)\exp(-\int_T^y \sigma_1(u)du)dy = -\exp(-\int_T^t \sigma_1(u)du)+1.$$

Since $\varphi(t)$ given by (7) is positive, we have only to show that $\lim_{t\to\infty}\varphi(t) = 0$. By (4), (9) and l'Hospital rule,

$$\lim_{t\to\infty}\varphi_1(t) = \lim_{t\to\infty}\frac{\int_t^\infty \sigma_1(y)\exp(-\int_T^y \sigma_1(u)du)dy}{\exp(-\int_T^t \sigma_1(v)dv)} = \lim_{t\to\infty}\frac{\delta_1(t)}{\sigma_1(t)} = 0.$$

Thus $\lim_{t\to\infty}\varphi(t) = 0$. The proof that

$$\lim_{t\to\infty}\exp-\int_T^t \sigma_2(v)dv\int_T^t\delta_2(y)\{\exp(\int_T^y \sigma_2(u)du)\}dy = 0$$

is quite similar. To prove that

$$\lim_{t\to\infty}\exp-\int_T^t \sigma_2(v)dv\int_T^t\tilde{\gamma}(y)\{\exp\int_T^y \sigma_2(u)du\}dy = 0,$$

define

$$F_t(y) = \tilde{\gamma}(y)\exp\left[-\int_y^t \sigma_2(u)du\right] \quad \text{if } T\leqslant t < \infty,$$

$$F_t(y) = 0 \quad \text{if } t = \infty.$$

We have $\int^\infty F_t(y)dy \leqslant \int^\infty \tilde{\gamma}(y)dy$ for every t, $T\leqslant t$, and by the Lebesgue dominated convergence theorem,

$$\lim_{t\to\infty}\int_T^t F_t(y)dy = \int_T^t \lim_{t\to\infty}F_t(y)dy = 0.$$

Then $\lim \psi(t) = 0$ and the lemma is proved.

THEOREM 1. *Let system* (1) *satisfy the hypothesis*

$$\mathrm{Re}(g_i - g_{i+1}) \geq 0, \quad \mathrm{Re}(g_i - g_{i+1}) > 0 \quad \text{if } g_{ij} \neq 0$$

for some i, j;

$$\sum_{i,j} |g_{ij}| < \infty, \quad \sum_{i \neq j} \{|\mathrm{Re}(\tilde{g}_{ii} - \tilde{g}_{jj}) + |\tilde{g}_{ij}|\} < \infty,$$

$$\int^{\infty} \mathrm{Re}(g_i(t) - g_{i+1}(t))dt = \infty, \quad i = 1, 2 \ldots,$$

$$\lim_{t \to \infty} \frac{\sum_{i,j} |g_{ij}(t)|}{\mathrm{Re}(g_k(t) - g_{k+1}(t))} = 0, \quad k = 1, 2, \ldots, \tag{12}$$

$$\int^{\infty} \sum_{i \neq j} \{|\mathrm{Re}(\tilde{g}_{ii}(t) - \tilde{g}_{jj}(t))| + |\tilde{g}_{ij}(t)|\} dt < \infty.$$

Then there exists a system of linearly independent solutions

$$(x_1(t), x_2(t), \ldots) = \begin{bmatrix} x_{11}(t), \ldots, x_{1n}(t), \ldots \\ x_{21}(t), \ldots, x_{2n}(t), \ldots \\ x_{n1}(t), \ldots, x_{nn}(t), \ldots \end{bmatrix}$$

such that

$$\lim_{t \to \infty} \frac{x_{ik}(t)}{x_{kk}(t)} = 0 \quad \text{for every } i \neq k. \tag{13}$$

Proof. For every fixed integer p we set

$$w_p = \{P = (t, x) |\ |x_i|^2 - |x_p|^2 \psi^2(t) < 0,\ |x_j|^2 - |x_p|^2 \psi^2(t) < 0,$$
$$i = 1, 2, \ldots, p-1,\ j = p+1, \ldots,\ t > t_0\},$$

where $\varphi(t)$, $\psi(t)$ and t_0 will be conveniently chosen so that, for every $t \geq t_0$, $\varphi(t) > 0$, $\psi(t) > 0$, φ and ψ are differentiable, $\lim_{t \to \infty} \varphi(t) = \lim_{t \to \infty} \psi(t) = 0$. Let

$$H_i(P) = |x_i|^2 - |x_p|^2 \varphi^2(t), \quad i = 1, 2, \ldots, p-1,$$
$$H_j(P) = |x_j|^2 - |x_p|^2 \psi^2(t), \quad j = p+1, \ldots,$$
$$H_p(P) = t_0 - t.$$

It follows that

$$w_p = \{P |\ H_k(P) < 0,\ k = 1, 2, \ldots,\ k \neq p,\ t > t_0\}.$$

Set $M = \{Q = (t, x) |\ x = 0\}$. Assuming that P is fixed, define for any $q \geq 1$:

$$\Gamma_q = \{P | H_q(P) = 0,\ H_i(P) \leq 0 \text{ for all } q \geq 1\}, \quad \tilde{\Gamma}_q = \Gamma_q \backslash M.$$

For each p, w_p is homeomorphic to a wedge.

The sets Γ_i, $i = 1, 2, \ldots$, are called the *faces of* w_p.

We show that the points of $\tilde{\Gamma}_q$ for $q < p$ are strict egress points and the points of $\tilde{\Gamma}_r$ for $r > p$ are ingress points. The origin is not an egress point. An easy computation shows that DH_q, the derivative of H_q along the solutions of (1) on the faces $\tilde{\Gamma}_q$, is positive, since

$$\begin{aligned}\tfrac{1}{2}[DH_q(P)]_{p\in\tilde{\Gamma}_q} \geqslant & -|x_p|^2\psi(t)\dot{\psi}(t)+\mathrm{Re}(g_q-g_p+\tilde{g}_{qq}-\tilde{g}_{pp})|x_p|^2\varphi^2(t)\\ & -|x_p|^2\varphi^2(t)\left(\sum_{k=1}^{\infty}|g_{qk}||x_k|/|x_q|+\sum_{k=1}^{\infty}|g_{pk}||x_k|/|x_p|\right)\\ & -|x_p|^2\varphi^2(t)\left(\sum_{\substack{k=1\\k\neq q}}^{\infty}|\tilde{g}_{qk}|(|x_k|/|x_q|)+\sum_{\substack{k=1\\k\neq p}}^{\infty}|\tilde{g}_{pk}|(|x_k|/|x_p|)\right).\end{aligned}$$

Since $|x_k| \leqslant |x_p|\varphi(t)$ if $k < p$ and $|x_k| \leqslant |x_p|\psi(t)$ if $k > p$, if follows that $|x_k|/|x_p| \leqslant \varphi(t)$ or $|x_k|/|x_p| \leqslant \psi(t)$.

As we want to have $\varphi(t) > 0$ and $\lim_{t\to\infty}\varphi(t) = 0$, $\psi(t) > 0$ and $\lim_{t\to\infty}\psi(t) = 0$, we take t_0 sufficiently large so that $\varphi(t) < 1$ and $\psi(t) < 1$. Then

$$\begin{aligned}\tfrac{1}{2}[DH_q(P)]_{p\in\tilde{\Gamma}_q} > & \;\varphi(t)|x_p|^2[\varphi(t)\mathrm{Re}(g_q-g_p+\tilde{g}_{qq}-\tilde{g}_{pp})\\ & -\dot{\varphi}(t)-\sum_{k=1}^{\infty}(|g_{pk}|+|g_{qk}|)-\sum_{k=1}^{\infty}|\tilde{g}_{pk}|-\sum_{\substack{k=1\\k\neq p}}^{\infty}|\tilde{g}_{qk}|]\\ \geqslant & \;\varphi(t)|x_p|^2[\varphi(t)\mathrm{Re}(g_q-g_p)-\dot{\varphi}(t)-\gamma(t)-\tilde{\gamma}(t)],\end{aligned}$$

where

$$\gamma(t) = \sum_{i,j}|g_{ij}| \quad\text{and}\quad \tilde{\gamma}(t) = \sum_{i\neq j}\{\mathrm{Re}(\tilde{g}_{ii}-\tilde{g}_{jj})+|\tilde{g}_{ij}|\}.$$

In order to have $[DH_q(P)]_{p\in\tilde{\Gamma}_q} > 0$, $q = 1, 2, \ldots, p-1$, it is sufficient to choose $\varphi(t)$ in such a way that

$$-\dot{\varphi}(t)+\sigma_1(t)\varphi(t)-[\gamma(t)+\tilde{\gamma}(t)] > 0,$$

where $\sigma_1(t) = \mathrm{Re}(g_q(t)-g_p(t)) > 0$, or

$$\dot{\varphi}(t) < \sigma_1(t)\varphi(t)-[\gamma(t)+\tilde{\gamma}(t)]. \tag{14}$$

For the faces $\tilde{\Gamma}_r$ we have

$$\begin{aligned}\tfrac{1}{2}[DH_r(P)]_{p\in\tilde{\Gamma}_r} \leqslant & |x_p|^2\psi(t)[-\dot{\psi}(t)-\mathrm{Re}(g_q-g_r+\tilde{g}_{pp}-\tilde{g}_{rr})\psi(t)\\ & +\left(\sum_{k=1}^{\infty}|g_{rk}|\frac{|x_k|\,|x_r|}{|x_r|\,|x_p|}+\sum_{k=1}^{\infty}g_{pk}\frac{|x_k|\,|x_r|}{|x_p|\,|x_p|}\right)+\left(\sum_{\substack{k=1\\k\neq r}}^{\infty}\tilde{g}_{rk}\frac{|x_k|\,|x_r|}{|x_r|\,|x_p|}+\sum_{\substack{k=1\\k\neq p}}^{\infty}\tilde{g}_{pk}\frac{|x_k|\,|x_r|}{|x_p|\,|x_p|}\right)]\\ \leqslant & |x_p|^2\psi(t)[-\dot{\psi}(t)+\mathrm{Re}(g_r-g_p+\tilde{g}_{rr}-\tilde{g}_{pp})\psi(t)+\sum_{i,j}|g_{ij}|+\sum_{\substack{i,j\\i\neq j}}|\tilde{g}_{ij}|],\end{aligned}$$

because $|x_k|/|x_p| \leqslant \varphi(t)$ if $k < p$ and $|x_k|/|x_p| \leqslant \psi(t)$ if $k > p$, with $\varphi(t) < 1$ and $\psi(t) < 1$ for $t > t_0$.

Hence

$$\tfrac{1}{2}[DH_r(P)]_{p\in\tilde{\Gamma}_r} \leqslant |x_p|^2\psi(t)(-\dot{\psi}(t)+\sigma_2(t)\psi(t)+(\gamma(t)+\tilde{\gamma}(t))),$$

where $\sigma_2(t) = \mathrm{Re}(g_r - g_p) > 0$.

In order to have $\frac{1}{2}[DH_r(P)]_{p\in\tilde{\Gamma}_r} < 0$ for $r \neq p$, we have to choose $\psi(t)$ in such a way that

$$\dot{\psi}(t) > \sigma_2(t)\psi(t)+[\gamma(t)+\tilde{\gamma}(t)]. \tag{15}$$

In view of (12), the system of differential inequalities given by (14) and (15) satisfies Lemma 3 and hence possesses differentiable solutions $\varphi(t)$ and $\psi(t)$ such that

$$\lim_{t\to\infty}\varphi(t) = \lim_{t\to\infty}\psi(t) = 0, \quad \varphi(t) > 0 \quad \text{and} \quad \psi(t) > 0.$$

Then for $i < p$ the points of $\tilde{\Gamma}_i$ are strict egress; the points of $M_i = M = \{(t, x)|\ x = 0,\ t \geqslant 0\}$ are not egress points.

For $j > p$ the points of $\tilde{\Gamma}_j$ are ingress points. Since $[DH_p(P)]_{p\in\Gamma_p} = -1$, the points of Γ_p are ingress. The equality $S = S^*$ holds; i.e., we have

$$S = S^* = \bigcup_{i<p}\tilde{\Gamma}_i - \bigcup_{j\geqslant p}\tilde{\Gamma}_j$$

and condition (i) of Lemma 2 is satisfied.

Select a $\tau > t_0$ and $x_* \in C$ such that $x_{*p} \neq 0$, $|x_{*p}| < \psi(\tau)|x_{*p}|$ for $j > p$ and define

$$Z_p = \{P = (t, x)|\ t = \tau,\ x_p = x_{*p'},\ |x_i| \leqslant |x_p|\varphi(t) \text{ for } i < p\}.$$

Then Z_p is a ball B^{2p-2} in $\boldsymbol{R}^{2p-2}$ with respect to the maximum norm. We have also

$$Z_p \cap S = Z_p \cap (\bigcup_{i<p}\tilde{\Gamma}_i \backslash \bigcup_{j\geqslant p}\Gamma_j) = \bigcup_{i<p}\bigcap_{j\geqslant p} Z_p \cap (\tilde{\Gamma}_i \backslash \Gamma_j) = \bigcup_{i<p} Z_p \cap \tilde{\Gamma}_i,$$

because $Z_p \cap \Gamma_j = \emptyset$. Thus

$$Z_p \cap S = \{P \in Z_p|\ \exists i < p,\ |x_i| = |x_p|\ \ \varphi(t)\},$$

which means that $Z_p \cap S$ is the boundary of B^{2p-2}, i.e., a $(2p-3)$-dimensional sphere. $Z_p \cap S$ is a retract of S; the retraction $r\colon S \to Z_p \cap S$ is given by

$$r(P) = P^* \quad \text{with } t^* = \tau,\ x_p^* = x_p^0,\ x_i^* = \frac{\varphi(t)|x_p^0|}{\varphi(t)|x_p|}x_i,\ i < p,$$

and $x_j^* = x_j^0$ if $j > p$, so that condition (iii) of Lemma 2 is satisfied. If $p = 1$, all points of the faces $\tilde{\Gamma}_r$ are ingress points.

From Lemma 2 we conclude that there exists at least one point

$P_0 = (\tau, x_0) \in Z_p \cap \omega_p$ such that the trajectory of system (1) through P_0 stays in ω_p. This means that the solution $x_p(t) = (x_{1p}(t), \ldots, x_{np}(t), x_{n+1p}, \ldots)$ through P_0 satisfies

$$|x_{ip}|/|x_{pp}| < \max\{\varphi(t), \psi(t)\} \quad \text{for } i \neq p,\ t \geq \tau \geq t_0.$$

Letting $p = 1, 2, \ldots$, we find a countable set of solutions $(x_1(t), x_2(t), \ldots)$ with the required property. We now show that these solutions can be taken linearly independent. By choosing Z_p with sufficiently large τ and $x^0_{pp} = 1$, the absolute values of the coordinates x_{ip}, $i \neq p$, of the points of Z_p can be made arbitrarily small.

We then have

$$(x_1(\tau), x_2(\tau), \ldots) = \begin{bmatrix} \varepsilon_{11}, \ldots, \varepsilon_{1n}, \ldots \\ \varepsilon_{21}, \varepsilon_{22}, \ldots, \varepsilon_{2n} \\ \varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn} \end{bmatrix},$$

where $\varepsilon_{ii} = 1$ and the ε_{ij}'s are arbitrarily small positive numbers for $i \neq j$.

Let us now consider

$$\dot{x}_i = \sum_{j=1}^{\infty} (g_{ij}(t) + \tilde{g}_{ij}(t))x_j, \quad i = 1, 2, \ldots \tag{16}$$

If we set $g_{ii} + \tilde{g}_{ii} = g_i$, system (16) becomes

$$\dot{x}_i = g_i(t)x_i + \sum_{\substack{j=1 \\ i \neq j}}^{\infty} (g_{ij}(t) + \tilde{g}_{ij}(t)x_j), \quad i = 1, \ldots \tag{17}$$

The following theorem generalizes a theorem of Perron [10]; see also [8], p. 173.

The Lyapunov number of a solution $x(t)$ of equation (17) is defined by (see [2], p. 117)

$$\lambda(x(t)) = \overline{\lim_{t \to \infty}} \frac{\log\|x(t)\|}{t}. \tag{18}$$

THEOREM 2. *Assume the hypotheses of Theorem* 1 *with respect to system* (17). *If* $\sum_{j=1}^{\infty} |g_{ij}(t)|$ *is bounded in* $[0, \infty)$, $i, j = 1, 2, \ldots$, *then there exists a system of linearly independent solutions of* (17), $(x_1(t), x_2(t), \ldots)$, *such that*

$$\lambda(x_i(t)) = \overline{\lim_{t \to \infty}} \frac{1}{t} \int_0^t \mathrm{Re}(g_i(s))ds,$$

where $\lambda(x_i(t))$ *is the Lyapunov number of the solution* $x_i(t) = (x_{1i}(t), x_{2i}(t), \ldots)$.

Proof. By Theorem 1, system (17) has a system of linearly independent solutions which satisfy (13). Multiplying the i-th component of (17) by $1/x_{ii}$, integrating and taking the norm we get, if $|x_{ii}(t_0)| = b_0$,

$$|b_0|\exp\left[\int_0^t \operatorname{Re}(g_i(s))ds - \int_{t_0}^t \sum_{\substack{j=1\\ i\neq j}}^{\infty} |g_{ij}(s)| - \frac{|x_{ji}(s)|}{|x_{ii}(s)|}ds - \int_{t_0}^t \sum_{\substack{j=1\\ i\neq j}}^{\infty} |\tilde{g}_{ij}(s)\frac{|x_{ji}(s)|}{|x_{ii}(s)|}ds\right]$$

$$\leqslant |x_{ii}(t)| \leqslant |b_0|\exp\left[\int_{t_0}^t \operatorname{Re}(g_i(s))ds\right.$$

$$\left.+\int_{t_0}^t \sum_{\substack{j=1\\ i\neq j}}^{\infty} |g_{ij}(s)|\frac{|x_{ji}(s)|}{|x_{ii}(s)|}ds + \int_{t_0}^t \sum_{\substack{j=1\\ i\neq j}}^{\infty} |\tilde{g}_{ij}(s)|\frac{|x_{ji}(s)|}{|x_{ii}(s)|}ds\right].$$

Since the sums $\sum_{i\neq j} |g_{ij}(t)|$ are bounded by a constant K for $t \geqslant t_0$, we can take for every $\varepsilon > 0$ a number t_0 sufficiently large to have the inequality

$$\frac{|x_{ji}(t)|}{|x_{ii}(t)|} < \frac{\varepsilon}{K}$$

satisfied for every pair i, j with $i \neq j$. Then

$$\int_{t_0}^t \sum_{\substack{j=1\\ j\neq i}}^{\infty} |g_{ij}(s)|\frac{|x_{ji}(s)|}{|x_{ii}(s)|}ds \leqslant \int_{t_0}^t K\frac{\varepsilon}{K}ds \leqslant \varepsilon t, \qquad t \geqslant t_0.$$

We may take $t_0 \geqslant 1$ so large that

$$\int_{t_0}^t \sum_{\substack{j=1\\ j\neq i}}^{\infty} |\tilde{g}_{ij}(s)|ds \leqslant k \qquad \text{for } t \geqslant t_0.$$

We have

$$\int_{t_0}^t \sum_{\substack{j=1\\ j\neq 1}}^{\infty} |\tilde{g}_{ij}(s)|\left|\frac{x_{ji}(s)}{x_{ii}(s)}\right|ds \leqslant \varepsilon t, \qquad t \geqslant t_0 > 1.$$

Then

$$|b_0|\exp\left(\int_{t_0}^t \operatorname{Re}(g_{ii}(s))ds - 2\varepsilon t\right) \leqslant |x_{ii}(t)| \leqslant |b_0|\exp\left(\int_{t_0}^t \operatorname{Re}(g_i(s))ds + 2\varepsilon t\right).$$

By Theorem 1, $|x_{ji}(t)| \leqslant |x_{ii}(t)|$ for t sufficiently large, and we have

$$|b_0|\exp\left(\int_{t_0}^t \operatorname{Re}(g_i(s))ds - 2\varepsilon t\right) \leqslant \|x_i(t)\| \leqslant \sup_j \{|x_{ji}(t)|\}$$

$$\leqslant |b_0|\exp\left(\int_{t_0}^t \operatorname{Re}(g_{ii}(s))ds + 2\varepsilon t\right).$$

Using the well-known properties of Lyapunov numbers

$$\lambda_i = \lambda(x_i) = \lambda\big(\exp\int_{t_0}^{t} \mathrm{Re}(g_i(s))ds\big),$$

we have by definition (18)

$$\lambda_i = \lambda(x_i) = \overline{\lim}\frac{1}{t}\int_0^t \mathrm{Re}(g_i(s))ds.$$

EXAMPLE. Let $\{a_i\} \in C$ be a sequence with $\lim_{i\to\infty} a_i = a_\infty \neq 0$, $\alpha^i = (0, 0, \ldots, a_i, 0, \ldots)$. Define $T(t)\alpha^i = \{e^{\lambda_i}\alpha^i\}$, $-\infty < \mathrm{Re}\lambda_i \leqslant \omega < \infty$. $T(t)$ is a strongly continuous semigroup with infinitesimal generator A given by $A\alpha^i = \{\lambda_i \alpha^i\}$. $T(t)$ is compact if and only if $\underline{\lim}\,\mathrm{Re}\lambda_i = -\infty$; [1], p. 189. Take $\mathrm{Re}\lambda_i > \mathrm{Re}\lambda_{i+1}$, $i = 1, 2, \ldots$, and consider the system

$$\dot{x}_i = \lambda_i x_i + \sum_{i=1}^{\infty} g_{ij}(t)x_j, \qquad x_i(t_0) = x_i^0,$$

with $\lim_{t\to\infty} \sum_{j=1}^{\infty} |g_{ij}(t)| = 0$, $\sum_{i\neq j} |g_{ij}| < \infty$. By Theorem 2, there exists a system of linearly independent solutions $(x_1(t), \ldots)$ such that $\lambda(x_i(t)) = \mathrm{Re}(\lambda_i)$, $i = 1, 2, \ldots$

References

[1] Balakrishnan, *Applied Functional Analysis*, Springer-Verlag, New York 1976.

[2] Yu. L. Daleckiĭ and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Spaces*, Translations of Mathematical Monographs 43, AMS, Providence 1974.

[3] K. Deimling, *Ordinary Differential Equations in Banach spaces*, Lectures Notes 596, Springer-Verlag, 1977.

[4] R. W. Dickey, *Infinite systems of nonlinear oscillations equation related to the string*, Proc. Amer. Math. Soc. 23 (1969), 459–468.

[4a] —, *Infinite systems of nonlinear oscillations equations*, J. Diff. Eqs. 8 (1970), 16–26.

[5] E. Hille, *Pathology of infinite systems of linear first order differential equations with constant coefficients*, Ann. Mat. Pura Appl. 55 (1961), 133–148.

[6] A. F. Izé, *On a fixed point index method for the analysis of the asymptotic behavior and boundary problems of infinite dimensional dynamical systems and processes*, J. of Differential Equations, Vol. 52, 2, April 1984.

[7] R. C. Lewis, *Infinite systems of ordinary differential equations with applications to certain second order partial differential equations*, Trans. Amer. Math. Soc. 34 (1933), 792–823.

[8] V. V. Nemitski and V. V. Stepanov, *Qualitative theory of differential equations*, Princeton Mathematical Series.

[9] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences 44, Springer-Verlag.

[10] A. Pelczar, *A local version of the generalized retract theorem of Ważewski with applications in the theory of Partial Differential Equations of first order*, Ann. Polon. Math. 36 (1979), 11–28.

[11] O. Perron, *Über lineare Differentialgleichungen bei denen die unabhängig variable reel ist, I, II*, J. Reine Angew. Math. 142 (1913), 254–270 and 143 (1913), 25–50 (8,17).
[12] Z. Szmydt, *Sur l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Math. Polon. 24(2) (1951), 17–34.
[13] A. Voigt, *Line method approximations to the Cauchy problem for nonlinear parabolic differential equations*, Number Math. 23 (1974), 23–36.
[14] O. A. Zautykov and K. G. Valeev, *Infinite systems of differential equations*, Izdat. „Nauka Kazakh. S.S.R," Alma Ata 1974.
[15] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. 20 (1947), 279–313.

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