ANNALES
POLONICI MATHEMATICI
LI (1990)

On the structure of Mellin distributions

by GRZEGORZ ŁYSIK (Warszawa)

Zdzisław Opial in memoriam

Abstract. The notions of a Mellin order and transformational Mellin order of Mellin distributions are investigated. A structure theorem for Mellin distributions is given. The Mellin convolution is studied.

Throughout the paper, we use the notation, definitions and results on the Mellin transformation stated in paper [2].

1. Transformational Mellin order of a Mellin distribution. In this section, we give definitions of the Mellin order and transformational Mellin order of a Mellin distribution and study relations between them.

For the definition that a distribution $U \in \mathfrak{M}'_{(\omega)}$ is of Mellin order $\leqslant m$, $m \in N_0$, see [2].

DEFINITION. Let $U \in \mathfrak{M}'_{(\omega)}$, $m \in N$. We say that *U is of Mellin order* $\leqslant -m$ and write $\mathrm{MO}(U) \leqslant -m$ if distributions $V_{jl} = \tilde{D}_j^l U$, $j = 1, \ldots, n$, $l = 0, 1, \ldots, m$ are of Mellin order $\leqslant 0$.

For a given distribution $U \in \mathfrak{M}'_{(\omega)}$, denote by K the set $\{k \in \boldsymbol{Z}: \mathrm{MO}(U) \leqslant k\}$.

Let

$$m = \begin{cases} -\infty & \text{if } K = \boldsymbol{Z}, \\ +\infty & \text{if } K = \emptyset, \\ \min K & \text{otherwise.} \end{cases}$$

Then we say that U is *of Mellin order* m and write $\mathrm{MO}(U) = m$.

LEMMA 1. *Let* $\omega \in (\boldsymbol{R} \cup \{-\infty\})^n$. *If* f *is a measurable function on* J *such that*

$$\int_J |x^a f(x)| dx < \infty \quad \textit{for every } a > \omega,$$

then $U_f \in \mathfrak{M}'_{(-\omega-1)}$ *and* $\mathrm{MO}(U_f) \leqslant 0$, *where*

$$U_f[\varphi] = \int_J f(x)\varphi(x)dx \quad \textit{for } \varphi \in \mathfrak{M}_{(-\omega-1)}.$$

Proof. Let $\varphi \in \mathfrak{M}_{(-\omega-1)}$. Then there exists $a < -\omega-1$ such that $\varphi \in \mathfrak{M}_a$. We derive

$$|U_f[\varphi]| \leqslant \int_J |x^{-a-1} f(x)|\,dx \cdot \varrho_{a,0}(\varphi) \leqslant C_a \cdot \varrho_{a,0}(\varphi) \quad \text{since } -a-1 > \omega.$$

LEMMA 2. *Let $m \in \mathbf{N}$ and let f be m times differentiable on J. Write*

$$f_{jp}(x) = \left(x_j \frac{\partial}{\partial x_j}\right)^p f(x) \quad \textit{for } x \in J,\ j = 1, \ldots, n,\ p = 0, \ldots, m.$$

Let

$$\int_J |x^a f_{jp}(x)|\,dx < \infty \quad \textit{for every } a > \omega,\ j = 1, \ldots, n,\ p = 0, \ldots, m.$$

Then

$$\left.\frac{\partial^p f(x)}{\partial x_j^p}\right|_{x_j = r_j} = 0 \quad \textit{for} \quad j = 1, \ldots, n,\ p = 0, \ldots, m-1 \tag{1}$$

if and only if

$$\tilde{D}_j^p U_f = U_{f_{jp}} \quad \textit{for} \quad j = 1, \ldots, n,\ p = 1, \ldots, m.$$

The proof goes along the same lines as that of Lemma 3 from [2].

From Lemmas 1 and 2 we get

PROPOSITION 1. *Let f be the same as in Lemma 2. Then $U_f \in \mathfrak{M}'_{(-\omega-1)}$ and if condition* (1) *holds then* $\mathrm{MO}(U_f) \leqslant -m$.

DEFINITION. Let $U \in \mathfrak{M}'_{(\omega)}(J)$, $s \in \mathbf{R}$. We say that U is *of transformational Mellin order* $\leqslant s$ and write $\mathrm{TMO}(U) \leqslant s$ if for every $a < \omega$ there exists a constant $C = C(a)$ such that

$$|(MU)(z)| \leqslant C\langle z\rangle^s r^{-\mathrm{Re}\,z} \quad \text{for} \quad \mathrm{Re}\,z \leqslant a, \tag{2}$$

where $\langle z\rangle^2 = 1 + |z_1|^2 + \ldots + |z_n|^2$.

LEMMA 3. *Let $U \in \mathfrak{M}'_{(\omega)}$, $s \in \mathbf{R}$, $\alpha \in \mathbf{N}_0^n$. If* $\mathrm{TMO}(U) \leqslant s$ *then* $\mathrm{TMO}(\tilde{D}^\alpha U) \leqslant s + |\alpha|$.

LEMMA 4. *Let $U \in \mathfrak{M}'_{(\omega)}$, $s \in \mathbf{R}$, $m \in N$. If* $\mathrm{TMO}(\tilde{D}_j^m U) \leqslant s+m$ *for $j = 1, \ldots, n$, then* $\mathrm{TMO}(U) \leqslant s$.

Proof. Let $a < \omega$. For $j = 1, \ldots, n$ we have

$$|M(\tilde{D}_j^m U)(z)| = |z_j^m||MU(z)| \leqslant C\langle z\rangle^{s+m} r^{-\mathrm{Re}\,z} \quad \text{for } \mathrm{Re}\,z \leqslant a.$$

So

$$\sum_{j=1}^n |z_j|^m |MU(z)| \leqslant Cn\langle z\rangle^{s+m} r^{-\mathrm{Re}\,z} \quad \text{for} \quad \mathrm{Re}\,z \leqslant a.$$

Since $MU(z)$ is bounded in bounded subsets of $\{\mathrm{Re}\, z < a\}$ and $\langle z\rangle^m / \sum_{j=1}^{m} |z_j|^m$ is bounded in $C^n \backslash B(0, 1)$, we get the assertion.

The proof of the following lemma is easy but tedious and therefore will be omitted.

LEMMA 5. *Let* $a \in \mathbf{R}^n$, $\alpha \in \mathbf{N}_0^n$. *The following spaces of linear operators coincide*:

$$\operatorname{span}\{x^{a+\beta}D^\beta\}_{\beta\leqslant\alpha}, \qquad \operatorname{span}\{x^\beta D^\beta(x^a)\}_{\beta\leqslant\alpha},$$
$$\operatorname{span}\{(xD)^\beta(x^a)\}_{\beta\leqslant\alpha}, \qquad \operatorname{span}\{x^a(Dx)^\beta\}_{\beta\leqslant\alpha}.$$

For $a \in \mathbf{R}^n$ and $\omega \in (\mathbf{R} \cup \{+\infty\})^n$, we write $X_a = \operatorname{span}\{x^{-z-1}\}_{\mathrm{Re}\, z \leqslant a}$, $X_{(\omega)} = \bigcup_{a<\omega} X_a$. Observe that $X_a \subset \mathfrak{M}_a$, $X_{(\omega)} \subset \mathfrak{M}_{(\omega)}$.

PROPOSITION 2. $X_{(\omega)}$ *is a dense subset of* $\mathfrak{M}_{(\omega)}$.

Proof. We can assume that $\omega < \infty$. Let $\varphi \in \mathfrak{M}_{(\omega)}$. Thus $\varphi \in \mathfrak{M}_b$ for certain $b < \omega$. Let $b < a < \omega$. Clearly, $\varphi \in \mathfrak{M}_a$. It is sufficient to show that for every $\varepsilon > 0$, $m \in \mathbf{N}$ there exists $\eta \in X_a$ such that

$$\sum_{|\alpha|\leqslant m} \varrho_{a,\alpha}(\varphi - \eta) < \varepsilon. \tag{3}$$

Let $\varepsilon > 0$, $m \in \mathbf{N}$. Since $C_0^\infty(J)$ is dense in $\mathfrak{M}_{(\omega)}$, we have

$$\sum_{|\alpha|\leqslant m} \varrho_{a,\alpha}(\varphi - \psi) < \varepsilon/2 \quad \text{for a certain function } \psi \in C_0^\infty(J). \tag{4}$$

By Lemma 5 we can find constants $C_\beta(a, \alpha)$, $\beta \leqslant \alpha$ such that

$$x^{a+\alpha+1}D^\alpha f = \sum_{\beta\leqslant\alpha} C_\beta(a, \alpha) x^\beta D^\beta(x^{a+1}f) \quad \text{for every } f \in C^\infty(\mathbf{R}_+^n). \tag{5}$$

Let $C = C(a, m) = \sum_{|\alpha|\leqslant m} \sum_{\beta\leqslant\alpha} |C_\beta(a,\alpha)| r^{|\beta|}$.

Since $x^{a+1}\psi(x)$ is a smooth function in $\bar{J}$, by the Weierstrass theorem we can find a polynomial $w(x) = \sum_{\gamma\in \mathbf{N}_0^n} b_\gamma x^\gamma$, $b_\gamma \in \mathbf{C}$, such that

$$\sup_{|\alpha|\leqslant m} \sup_{x\in J} \left|D^\alpha\big(x^{a+1}\psi(x) - w(x)\big)\right| < \varepsilon/2C. \tag{6}$$

Let $\eta(x) = x^{-a-1}w(x) = \sum_\gamma b_\gamma x^{\gamma-a-1} \in X_a$. Using (5) for the function $f = \psi - \eta$ and (6), we get

$$\sum_{|\alpha|\leqslant m} \varrho_{a,\alpha}(\psi - \eta) < \varepsilon/2. \tag{7}$$

From (4) and (7) we get (3) by the triangle inequality.

THEOREM 1. *Let* $U \in \mathfrak{M}'_{(\omega)}$, $m \in \mathbf{Z}$. *If* $\mathrm{MO}(U) \leqslant m$ *then* $\mathrm{TMO}(U) \leqslant m$.

Proof. It follows from Lemmas 3 and 4 that it is sufficient to prove the theorem for $m \in N_0$. So we assume that $m \in N_0$.

Suppose $\mathrm{MO}(U) \leqslant m$. Let $a < \omega$. Using (1) from [2] for the function $\varphi(x) = x^{-z-1}$, $\mathrm{Re}\, z \leqslant a$, we derive

$$|(MU)(z)| \leqslant C \cdot \sum_{|\alpha| \leqslant m} \varrho_{a,\alpha}(x^{-z-1}) \leqslant C \cdot r^{a} \cdot \sum_{|\alpha| \leqslant m} |W_\alpha(z)| r^{-\mathrm{Re}\, z}, \tag{8}$$

where $W_\alpha(z) = \prod_{j=1}^{n} \prod_{k=1}^{\alpha_j} (-z_j - k)$.

Since $|W_\alpha(z)| \leqslant C(\alpha)\langle z \rangle^{|\alpha|} \leqslant C(\alpha)\langle z \rangle^{m}$ for $|\alpha| \leqslant m$, we obtain estimation (2).

For $\omega \in (\boldsymbol{R} \cup \{-\infty\})^n$, let us write

$$\mathfrak{M}_{[\omega]} = \bigcap_{a > \omega} \mathfrak{M}_a,$$

$$\mathring{\mathfrak{M}}_{[\omega]} = \{\varphi \in \mathfrak{M}_{[\omega]} \text{ satisfying (1) for } j = 1, \ldots, n,\ p = 0, 1, \ldots\}.$$

We can reformulate Theorem 2 from [2] in the following manner.

THEOREM 2. *Let* $\omega \in (\boldsymbol{R} \cup \{-\infty\})^n$. *Then*

$$f \in \mathring{\mathfrak{M}}_{[\omega]} \quad \textit{iff} \quad U_f \in \mathfrak{M}'_{(-\omega-1)} \textit{ is of transformational Mellin order } -\infty.$$

Remark 1. Note that the if part already follows from Proposition 1.

Let us denote by $L_{(\omega)}$, $\omega \in \boldsymbol{R}^n$, the set of operators $P(\tilde{D})$, where P is a polynomial such that $P(z) \neq 0$ for $\mathrm{Re}\, z < \omega$.

PROPOSITION 3. *Let* $P \in L_{(\omega)}$ *be of the form* $P(\tilde{D}) = P_1(\tilde{D}_1) \cdot \ldots \cdot P_n(\tilde{D}_n)$. *Let* $k = \min_{i=1,\ldots,n} \deg P_i$, $s \in \boldsymbol{R}$, $f \in \mathfrak{M}'_{(\omega)}$.

If $\mathrm{TMO}(f) \leqslant s$ *then there exists* $U \in \mathfrak{M}'_{(\omega)}$ *such that* $\mathrm{TMO}(U) \leqslant s - k$ *and* $P(\tilde{D})U = f$.

Proof. Let

$$G(z) = \frac{Mf(z)}{P(z)} \in \mathcal{O}(\mathrm{Re}\, z < \omega).$$

Since for $\mathrm{Re}\, z \leqslant a < \omega$ we have the estimation $\langle z \rangle^k \leqslant C_a |P(z)|$, the rest of the proof follows from Theorem 1 [2].

For a general $P \in L_{(\omega)}$ we have only the following

PROPOSITION 4. *Let* $P \in L_{(\omega)}$, $s \in \boldsymbol{R}$, $f \in \mathfrak{M}'_{(\omega)}$. *If* $\mathrm{TMO}(f) \leqslant s$ *then there exists* $U \in \mathfrak{M}'_{(\omega)}$ *such that* $\mathrm{TMO}(U) \leqslant s$ *and* $P(D)U = f$.

Proof. It is sufficient to note that $1/P(z)$ is a bounded holomorphic function for $\mathrm{Re}\, z \leqslant a$ for every $a < \omega$ and use Theorem 1 [2].

2. Structure theorem for Mellin distributions. In this section we give a characterization of Mellin distributions from $\mathfrak{M}'_{(\omega)}$.

LEMMA 6. *Let* $\psi \in C_0^1(J)$, $J = (0, r]^n$. *Then*

$$\sup_{x \in J} |x^1 \psi(x)| \leqslant \sum_{\gamma \in \{0,1\}^n} \|\tau^\gamma D^\gamma \psi(\tau)\|_{L^1(J)}. \tag{9}$$

Proof. For $n = 1$ we have the formula

$$x\psi(x) = \int_0^x \tau\psi'(\tau)d\tau + \int_0^x \psi(\tau)d\tau.$$

Hence

$$|x\psi(x)| \leqslant \int_0^r |\tau\psi'(\tau)|d\tau + \int_0^r |\psi(\tau)|d\tau \quad \text{for } x \in J,$$

and, taking supremum, we get (9).

For $n > 1$, the proof goes by induction with respect to n.

PROPOSITION 5. *Let* $a \in \mathbf{R}^n$, $U \in \mathfrak{M}'_a$. *Then there exist* $m \in \mathbf{N}_0$ *and bounded functions* h_α *for* $|\alpha| \leqslant m$ *such that*

$$U = \sum_{|\alpha| \leqslant m} \tilde{D}^\alpha (x^a h_\alpha) \quad \text{in } \mathfrak{M}'_{(a)}. \tag{10}$$

Proof. Let $U \in \mathfrak{M}'_a$. Then there exist constants $C > 0$, $p \in \mathbf{N}_0$ such that

$$|U[\varphi]| \leqslant C \max_{|\alpha| \leqslant p} \varrho_{a,\alpha}(\varphi) \quad \text{for } \varphi \in C_0^\infty(J). \tag{11}$$

Take any $\varphi \in C_0^\infty(J)$. Let $\psi(x) = x^{a+\alpha} D^\alpha \varphi(x)$. By Lemma 5 we get $\psi(x) = \sum_{\beta \leqslant \alpha} C_\alpha(\beta)(xD)^\beta (x^a \varphi(x))$. Thus applying Lemma 6 to the function ψ and using Lemma 5 we derive

$$\begin{aligned} \varrho_{a,\alpha}(\varphi) &= \sup_{x \in J} |x^1 \psi(x)| \\ &\leqslant \sum_{\gamma \in \{0,1\}^n} \Big\|\tau^\gamma D^\gamma \sum_{\beta \leqslant \alpha} C_\alpha(\beta)(\tau D)^\beta \tau^a \varphi(\tau)\Big\|_{L^1(J)} \\ &\leqslant \sum_{\beta \leqslant \alpha + 1} C_\alpha(\beta) \Big\|(\tau D)^\beta (\tau^a \varphi(\tau))\Big\|_{L^1(J)}. \end{aligned}$$

Hence, by (11) we get

$$|U[\varphi]| \leqslant C_2 \max_{|\alpha| \leqslant p+n} \Big\|(\tau D)^\alpha (\tau^a \varphi(\tau))\Big\|_{L^1(J)} \quad \text{for } \varphi \in C_0^\infty(J). \tag{12}$$

Let $m = p+n$, $N = \operatorname{card}\{\alpha\colon |\alpha| \leqslant m\} = \binom{m+n}{n}$. We define [1] the operator

$$C_0^\infty(J) \ni \varphi \to A\varphi = \{(\tau D)^\alpha (\tau^a \varphi)\}_{|\alpha| \leqslant m}.$$

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[1] This part of the proof is similar to the proof of the structure theorem for distributions from S'. See e.g. [3].

Note that $A\colon C_0^\infty \to AC_0^\infty$ is one to one and $A(C_0^\infty(J)) \subset (L^1(J))^N$. We also define the functional

$$A(C_0^\infty(J)) \ni \psi \to T[\psi] = U[A^{-1}\psi].$$

Due to (12), T is continuous on $A(C_0^\infty)$ in the norm $(L^1)^N$. By the Hahn–Banach theorem there exists a continuous extension $\tilde{T}$ of T to the whole $(L^1)^N$. So by Riesz theorem there exist functions $\tilde{h}_\alpha \in L^\infty(J)$ for $|\alpha| \leqslant m$ such that

$$\tilde{T}[\{\sigma_\alpha\}_{|\alpha| \leqslant m}] = \sum_{|\alpha| \leqslant m} \tilde{h}_\alpha[\sigma_\alpha] \quad \text{for } \sigma_\alpha \in L^1.$$

Hence, for $\varphi \in C_0^\infty(J)$

$$U[\varphi] = T[A\varphi] = \tilde{T}[\{(\tau D)^\alpha(\tau^a \varphi)\}_{|\alpha| \leqslant m}] = \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} x^a (Dx)^\alpha \tilde{h}_\alpha[\varphi],$$

and on the basis of Lemma 5 this is equal to

$$\sum_{|\alpha| \leqslant m} (xD)^\alpha (x^a h_\alpha)[\varphi],$$

where for every α, $|\alpha| \leqslant m$, the function h_α is a linear combination of $\tilde{h}_\beta$, $|\beta| \leqslant m$. Since $C_0^\infty(J)$ is dense in $\mathfrak{M}_{(a)}$, we get (10) with $h_\alpha \in L^\infty(J)$.

Since $\mathfrak{M}'_{(\omega)} = \bigcap_{a<\omega} \mathfrak{M}'_a$ and functions $x^a h$ with h bounded belong to $\mathfrak{M}'_{(a)}$, we obtain

THEOREM 3. *Let $\omega \in (\boldsymbol{R} \cup \{+\infty\})^n$. In order that U belongs to $\mathfrak{M}'_{(\omega)}(J)$ it is necessary and sufficient that for every $a < \omega$ there exist $m \in \boldsymbol{N}_0$ and functions $h_\alpha \in L^\infty(J)$ for $|\alpha| \leqslant m$ such that* (10) *holds in $\mathfrak{M}'_{(a)}$.*

Remark 2. If $U \in \mathfrak{M}'_{(\omega)}$ is of Mellin order $\leqslant k < \infty$, then the m in (10) can be taken independently of a, namely $m = k+n$.

3. Example. The aim of this section is to construct a distribution $U \in \mathfrak{M}'_{(0)}$ of Mellin order $+\infty$. We shall restrict ourselves to dimension one (the passage to higher dimensions is simple).

The proof of the following lemma is simple and we omit it.

LEMMA 7. *Let $s \in \boldsymbol{R}$, $k \in \boldsymbol{N}_0$. Then*

$$\left(x\frac{d}{dx}\right)^k (\sin x^s) = \sum_{j=0}^{k} B_{k,j} \cdot s^k \cdot x^{js} \cdot \sin\left(x^s + \tfrac{1}{2} j\pi\right),$$

where the coefficients $B_{k,j}$ are defined by the relations

$$\begin{aligned} &B_{k,k} = 1 && \textit{for} \quad k = 0, 1, \ldots, \\ &B_{k,0} = 0, \quad B_{k,1} = 1 && \textit{for} \quad k = 1, 2, \ldots, \\ &B_{k+1,j} = jB_{k,j} + B_{k,j-1} && \textit{for} \quad k = 2, 3, \ldots, \; j = 2, \ldots, k \end{aligned}$$

and satisfy the inequality

$$\sum_{j=0}^{k} B_{k,j} \leqslant k!.$$

COROLLARY 1. *Let* $s < 0$, $k \in N_0$. *Let*

$$g_{k,s}(x) = s^{-k} \cdot x^{-ks} \cdot \left(x\frac{d}{dx}\right)^k (\sin x^s) \quad \text{for } x \in (0, 1].$$

Then

$$\sup_{x\in(0,1]} |g_{k,s}(x)| \leqslant k!.$$

EXAMPLE 1. Let $a_k = -1/k!$ for $k \in N$. Let

$$U(x) = \sum_{k=1}^{\infty} \left(x\frac{d}{dx}\right)^k (\sin x^{a_k}) \quad \text{for } x \in (0, 1]. \tag{13}$$

Then $U \in \mathfrak{M}'_{(0)}$ is of Mellin order $+\infty$.

Proof. First of all we shall prove that $U \in \mathfrak{M}'_{(0)}$. To this end we take any $\varepsilon > 0$. Let $i \in N$ be such that $i! \geqslant 1/\varepsilon$. We observe that

$$U_1(x) = \sum_{k=1}^{i} \left(x\frac{d}{dx}\right)^k (\sin x^{a_k}) \in \mathfrak{M}'_{(0)}.$$

Applying Lemma 7 for $s = a_k$, we get

$$U(x) - U_1(x) = x^{-\varepsilon} \cdot \sum_{k=i+1}^{\infty} a_k^k \cdot x^{\varepsilon + a_{k-1}} \cdot g_{k,a_k}(x)$$

and $\sup_{x\in(0,1]} |g_{k,a_k}(x)| \leqslant k!$.

Since $\varepsilon + a_{k-1} \geqslant 0$ for $k \geqslant i+1$, the numerator is bounded. Hence, $U - U_1 \in \mathfrak{M}'_{(-\varepsilon)}$ and $U \in \mathfrak{M}'_{(-\varepsilon)}$. Since $\varepsilon > 0$ was arbitrary, we have $U \in \mathfrak{M}'_{(0)}$(²).

Let us suppose to the contrary that U is of finite Mellin order $m \in N_0$. Let $\varepsilon = 1/(m+3)!$. By Theorem 4 and Remark 2 there exist bounded functions $h_{k,\varepsilon}$, $k = 0, \ldots, m+1$, such that

$$U = \sum_{k=0}^{m+1} \left(x\frac{d}{dx}\right)^k (x^{-\varepsilon} h_{k,\varepsilon}) \quad \text{in } \mathfrak{M}'_{(-\varepsilon)}. \tag{14}$$

Let us write

$$f_{k,j}(x) = x^{ja_k} \sin(x^{a_k} + \tfrac{1}{2}j\pi), \qquad k \in N,\ j \in N_0,$$

$$F_{k,l}(x) = a_k^{k-l} \sum_{j=0}^{k-l} B_{k-l,j} f_{k,j}(x), \qquad k \in N,\ l = 0, \ldots, k.$$

(²) Actually, one can prove that $U \in C^\infty((0, 1])$.

By Lemma 7, $F_{k,l}(x) = \left(x\frac{d}{dx}\right)^{k-l}(\sin x^{a_k})$. The functions $f_{k,j}$, $k \in N$, $j \in N_0$ are linearly independent, so since $B_{k-l,j} \neq 0$ for $k \neq l$, the functions $F_{k,l}$, $k \in N$, $l = 0, \ldots, k$, are also linearly independent. We observe that $x^{\varepsilon} \times \sum_{k=m+4}^{\infty} F_{k,0}(x)$ is a bounded function and that the functions $x^{\varepsilon} F_{k,l}(x)$, $k = m+2$, $m+3$, $l = 0, 1, \ldots, m+1$ are unbounded. So the decomposition (14) is impossible with bounded $h_{k,\varepsilon}$, which proves that U is of Mellin order $+\infty$.

Remark 3. For U as in Example 1, the degree of the polynomial P from Theorem 1 in [2] indeed depends on $b < 0$.

4. The Mellin convolution. In this section we study the Mellin convolution which can readily be analysed by means of the Mellin transformation. The Mellin convolution in dimension $n = 1$ was studied in [1] (Section 11.11) for distributions of finite Mellin order.

Throughout this section we use the following notation.

For $x, y \in \boldsymbol{R}_+^n$, we write $xy = (x_1 y_1, \ldots, x_n y_n)$, $J^1 = (0, r^1]$, $J^2 = (0, r^2]$, $J = (0, r^1 r^2]$, $\omega^1, \omega^2 \in (\boldsymbol{R} \cup \{+\infty\})^n$.

We shall show in Theorem 4 that the following definition is correct.

DEFINITION. For $i = 1, 2$ let $U^i \in \mathfrak{M}'_{(\omega^i)}(J^i)$. Let $\omega = \min(\omega^1, \omega^2)$. Then the formal definition

$$U[\varphi] = U_x^1[U_y^2[\varphi(xy)]] \quad \text{for } \varphi \in \mathfrak{M}_{(\omega)}(J) \tag{15}$$

defines correctly $U \in \mathfrak{M}'_{(\omega)}(J)$. We denote this functional U by $U^1 *_m U^2$ and call it the *Mellin convolution* of U^1 and U^2.

THEOREM 4. *Under the assumptions of the above definition, the Mellin convolution $U^1 *_m U^2$ exists and belongs to $\mathfrak{M}'_{(\omega)}(J)$.*

Proof. Without loss of generality we may assume that $\omega^i < \infty$, $i = 1, 2$. (If $U^1 *_m U^2$ exists then $\operatorname{supp} U^1 *_m U^2 \subset J$.) Take $\varphi \in \mathfrak{M}_{(\omega)}(J)$. It is sufficient to show that the function $\psi(x) = U_y^2[\varphi(xy)]$, $x > 0$ belongs to $M_{(\omega)}(J^1)$. There exist $a < \omega$ and bounded functions $\varphi_\alpha(s)$, $\alpha \in N_0^n$ such that

$$s^{\alpha+1}\varphi^{(\alpha)}(s) = s^{-a}\varphi_\alpha(s). \tag{16}$$

Let $a < b < \omega$. We shall show that $\psi \in \mathfrak{M}_b(J^1) \subset \mathfrak{M}_{(\omega)}(J^1)$. Since $U^2 \in \mathfrak{M}'_{(\omega)}(J^2)$, there exist constants $C > 0$, $m \in N$ such that

$$|U^2[\varphi]| \leqslant C \cdot \sum_{|\beta| \leqslant m} \sup_y |y^{b+\beta+1}\varphi^{(\beta)}(y)| \quad \text{for } \varphi \in \mathfrak{M}_b(J^2). \tag{17}$$

For α, β, $\delta \in N_0^n$, $\delta \leqslant \beta$ let us denote

$$C_\alpha(\beta, \delta) = \prod_{i=1}^{n} \binom{\beta_i}{\delta_i} \alpha_i(\alpha_i - 1) \dots (\alpha_i - \beta_i + \delta_i + 1),$$

$$C_\alpha(\beta) = \sup_{\delta \leqslant \beta} C_\alpha(\beta, \delta), \qquad K_\alpha(\beta) = \sum_{\delta \leqslant \beta} \sup_s |\varphi_{\alpha+\delta}(s)|.$$

For $x > 0$, applying the Leibniz formula, we derive

$$\begin{aligned} |x^{b+\alpha+1}\psi^{(\alpha)}(x)| &\overset{(17)}{\leqslant} C \sum_{|\beta| \leqslant m} \sup_{y \in J^2} \left| y^{b+\beta+1} D_y^\beta \left(x^{b+1}(xy)^\alpha \cdot \varphi^{(\alpha)}(xy)\right)\right| \\ &\overset{(16)}{=} C \sum_{|\beta| \leqslant m} \sup_{y \in J^2} \left| \sum_{\delta \leqslant \beta} C_\alpha(\beta, \delta)(xy)^{b-a} \cdot \varphi_{\alpha+\delta}(xy)\right| \\ &\leqslant C \sum_{|\beta| \leqslant m} C_\alpha(\beta) x^{b-a} \cdot \sup_{y \in J^2} \left| y^{b-a} \cdot \sum_{\delta \leqslant \beta} \varphi_{\alpha+\delta}(xy)\right| \\ &\leqslant C \sum_{|\beta| \leqslant m} C_\alpha(\beta) \cdot K_\alpha(\beta) \cdot (r^2)^{b-a} \cdot x^{b-a}. \end{aligned}$$

So for every $\alpha \in N_0^n$, $\varrho_{b,\alpha}(\psi) < \infty$. Thus $\psi \in \mathfrak{M}_b(J^1)$ and for $\varphi_j \to 0$ in $\mathfrak{M}_{(\omega)}(J)$, we have $\psi_j \to 0$ in $\mathfrak{M}_b(J^1)$.

PROPOSITION 6. *Let* $U \in \mathfrak{M}'_{(\omega^1)}(J^1)$, V_j, $V \in \mathfrak{M}'_{(\omega^2)}(J^2)$ *for* $j \in N_0$, $\omega = \min(\omega^1, \omega^2)$. *If* $V_j \to V$ *in* $\mathfrak{M}'_{(\omega^2)}(J^2)$ *then* $U *_m V_j \to U *_m V$ *in* $\mathfrak{M}'_{(\omega)}(J)$.

THEOREM 5. *Under the assumption of Theorem* 4

$$M_r(U^1 *_m U^2)(z) = (M_{r^1}U^1)(z) \cdot (M_{r^2}U^2)(z) \quad \text{for } \operatorname{Re} z < \omega.$$

Proof. It is sufficient to put $\varphi(s) = s^{-z-1}$ in (15).

COROLLARY 2. *The Mellin convolution is commutative and associative.*

COROLLARY 3.

$$\tilde{D}^\alpha(U^1 *_m U^2) = (\tilde{D}^\alpha U^1) *_m U^2 \qquad \text{for } \alpha \in N_0^n,$$

$$x^s(U^1 *_m U^2) = (x^s U^1) *_m (x^s U^2) \qquad \text{for } s \in \mathbf{R}^n.$$

EXAMPLE 2. If $U \in \mathfrak{M}'_{(\omega)}(J)$, $\alpha \in N_0^n$ then $U *_m \delta_1^{(\alpha)} = D^\alpha(x^\alpha U)$.

By Theorem 5 we obtain

PROPOSITION 7. *Under the assumptions of Theorem* 4 *if* $\mathrm{TMO}(U^i) \leqslant s^i$, $i = 1, 2$ *then* $\mathrm{TMO}(U^1 *_m U^2) \leqslant s^1 + s^2$.

From Proposition 7 and Theorem 2 we get

COROLLARY 4. *If* $U \in \mathfrak{M}'_{(\omega)}(J^1)$, $\varphi \in \mathring{\mathfrak{M}}_{[-\omega-1]}(J^2)$ *then* $U *_m \varphi \in \mathring{\mathfrak{M}}_{[-\omega-1]}(J)$.

EXAMPLE 3. Let $\varphi_j \in C_0^\infty((0, \mathbf{1}))$ for $j \in N$ be such that

$$\varphi_j \geqslant 0, \quad \operatorname{supp} \varphi_j \subset [1 - 2^{-j}, 1]^n, \quad \int_{(0,1)} \varphi_j(x) dx = 1.$$

Then for every $\omega \in (\boldsymbol{R} \cup \{+\infty\})^n$, $\varphi_j \to \delta_1$ in $\mathfrak{M}'_{(\omega)}((0, 1])$.

PROPOSITION 8. $\mathring{\mathfrak{M}}_{[-\omega-1]}(J)$ *is a dense subset of* $\mathfrak{M}'_{(\omega)}(J)$.

Proof. Let $\varphi_j \in C_0^\infty((0,1))$ be a sequence such that $\varphi_j \to \delta_1$ in $\mathfrak{M}'_{(\omega)}((0, 1])$. Let $U \in \mathfrak{M}'_{(\omega)}(J)$. Let us write $U_j = U *_m \varphi_j$. Since $\varphi_j \in \mathring{\mathfrak{M}}_{[-\omega-1]}((0, 1])$, it follows from Corollary 4 that $U_j \in \mathring{\mathfrak{M}}_{[-\omega-1]}(J)$. By Proposition 6 we get $U_j \to U *_m \delta_1 = U$ in $\mathfrak{M}'_{(\omega)}(J)$.

The author would like to thank Professor Z. Szmydt for her helpful advice during preparation of this paper.

References

[1] O. P. Misra, J. L. Lavoine, *Transform Analysis of Generalized Functions*, North-Holland, 1986.

[2] Z. Szmydt, *The Paley–Wiener Theorem for the Mellin Transformation*, Ann. Polon. Math., this volume, 315–326.

[3] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, 1967.

[4] B. Ziemian, *Taylor Formula for Distributions in Several Dimensions*, Bull. Pol. Acad. Math. 34(1986), 277–286.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, WARSAW

Reçu par la Rédaction le 19.05.1988