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Young Bae Jun
Madad Khan
Florentin Smarandache
Seok-Zun Song*

LENGTH NEUTROSOPHIC SUBALGEBRAS OF BCK/BCI-ALGEBRAS[1]

Abstract

Given $i, j, k \in \{1, 2, 3, 4\}$, the notion of (i, j, k)-length neutrosophic subalgebras in BCK/BCI-algebras is introduced, and their properties are investigated. Characterizations of length neutrosophic subalgebras are discussed by using level sets of interval neutrosophic sets. Conditions for level sets of interval neutrosophic sets to be subalgebras are provided.

Keywords: Interval neutrosophic set, interval neutrosophic length, length neutrosophic subalgebra.

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*Corresponding author.

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1. Introduction

The intuitionistic fuzzy set, which has been introduced by Atanassov [1], consider both truth-membership and falsity membership. The neutrosophic set developed by Smarandache [6, 7, 8] is a formal framework which generalizes the concept of the classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set, interval valued intuitionistic fuzzy set and paraconsistent set etc. Neutrosophic set theory is applied to various part, including algebra, topology, control theory, decision making problems, medicines and in many real life problems. Wang et al. [9, 11] presented the concept of interval neutrosophic sets, which is more precise and more flexible than the single-valued neutrosophic set. An interval-valued neutrosophic set is a generalization of the concept of single-valued neutrosophic set, in which three membership (t, i, f) functions are independent, and their values belong to the unit interval $[0, 1]$. The interval neutrosophic set can represent uncertain, imprecise, incomplete and inconsistent information which exists in real world. Jun et al. [4] discussed interval neutrosophic sets in BCK/BCI-algebras. They introduced the notion of $(T(i, j), I(k, l), F(m, n))$-interval neutrosophic subalgebras in BCK/BCI-algebras for $i, j, k, l, m, n \in \{1, 2, 3, 4\}$, and investigated several properties and relations. They also introduced the notion of interval neutrosophic length of an interval neutrosophic set, and investigated related properties.

In this paper, we introduce the notion of (i, j, k)-length neutrosophic subalgebras in BCK/BCI-algebras for $i, j, k \in \{1, 2, 3, 4\}$, and investigate several properties. We consider relations of (i, j, k)-length neutrosophic subalgebras, and discuss characterizations of (i, j, k)-length neutrosophic subalgebras. Using subalgebras of a BCK-algebra, we construct (i, j, k)-length neutrosophic subalgebras for $i, j, k \in \{1, 4\}$. We consider conditions for level sets of interval neutrosophic set to be subalgebras of a BCK/BCI-algebra.

2. Preliminaries

By a *BCI-algebra* we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0,$

(II) $(x * (x * y)) * y = 0,$

(III) $x * x = 0$,

(IV) $x * y = y * x = 0 \ \Rightarrow \ x = y$

for all $x, y, z \in X$. If a BCI-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK-algebra*.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [2] and [5] for further information regarding BCK/BCI-algebras.

By a *fuzzy structure* over a nonempty set X we mean an ordered pair (X, ρ) of X and a fuzzy set ρ on X.

Definition 2.1 ([3]). For any $(X, *, 0) \in \mathcal{B}(X)$, a fuzzy structure (X, μ) over $(X, *, 0)$ is called a

- *fuzzy subalgebra* of $(X, *, 0)$ with type 1 (briefly, 1-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X) \left(\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \right), \tag{2.1}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 2 (briefly, 2-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X) \left(\mu(x * y) \leq \min\{\mu(x), \mu(y)\} \right), \tag{2.2}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 3 (briefly, 3-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X) \left(\mu(x * y) \geq \max\{\mu(x), \mu(y)\} \right), \tag{2.3}$$

- *fuzzy subalgebra* of $(X, *, 0)$ with type 4 (briefly, 4-*fuzzy subalgebra* of $(X, *, 0)$) if

$$(\forall x, y \in X) \left(\mu(x * y) \leq \max\{\mu(x), \mu(y)\} \right). \tag{2.4}$$

Let X be a non-empty set. A neutrosophic set (NS) in X (see [7]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function.

An interval neutrosophic set (INS) A in X is characterized by truth-membership function T_A, indeterminacy membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0,1]$ (see [11, 10]).

In what follows, let $(X, *, 0) \in \mathcal{B}(X)$ and $\mathcal{P}^*([0,1])$ be the family of all subintervals of $[0,1]$ unless otherwise specified.

Definition 2.2 ([11, 10]). An *interval neutrosophic set* in a nonempty set X is a structure of the form:

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}$$

where

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1])$$

which is called *interval truth-membership function*,

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1])$$

which is called *interval indeterminacy-membership function*, and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1])$$

which is called *interval falsity-membership function.*

For the sake of simplicity, we will use the notation $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ for the interval neutrosophic set

$$\mathcal{I} := \{\langle x, \mathcal{I}[T](x), \mathcal{I}[I](x), \mathcal{I}[F](x)\rangle \mid x \in X\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we consider the following functions (see [4]):

$$\mathcal{I}[T]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[T](x)\}$$
$$\mathcal{I}[I]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[I](x)\}$$
$$\mathcal{I}[F]_{\inf} : X \to [0,1],\ x \mapsto \inf\{\mathcal{I}[F](x)\}$$

and

$$\mathcal{I}[T]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[T](x)\}$$
$$\mathcal{I}[I]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[I](x)\}$$
$$\mathcal{I}[F]_{\sup} : X \to [0,1],\ x \mapsto \sup\{\mathcal{I}[F](x)\}.$$

DEFINITION 2.3 ([4]). Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X, we define the *interval neutrosophic length* of $\mathcal{I}$ as an ordered triple $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ where

$$\mathcal{I}[T]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x),$$

$$\mathcal{I}[I]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x),$$

and

$$\mathcal{I}[F]_\ell : X \to [0,1],\ x \mapsto \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x),$$

which are called *interval neutrosophic T-length*, *interval neutrosophic I-length* and *interval neutrosophic F-length* of $\mathcal{I}$, respectively.

3. Length neutrosophic subalgebras

DEFINITION 3.1. Given $i, j, k \in \{1, 2, 3, 4\}$, an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in X is called an *(i, j, k)-length neutrosophic subalgebra* of $(X, *, 0)$ if the interval neutrosophic T-length of $\mathcal{I}$ is an i-fuzzy subalgebra of $(X, *, 0)$, the interval neutrosophic I-length of $\mathcal{I}$ is a j-fuzzy subalgebra of $(X, *, 0)$, and the interval neutrosophic F-length of $\mathcal{I}$ is a k-fuzzy subalgebra of $(X, *, 0)$.

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1 (see [5]).
Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ where $\mathcal{I}[T]$, $\mathcal{I}[I]$ and $\mathcal{I}[F]$ are given as follows:

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]), \quad x \mapsto \begin{cases} [0.1, 0.8) & \text{if } x = 0, \\ (0.3, 0.7] & \text{if } x = 1, \\ [0.0, 0.6] & \text{if } x = 2, \\ [0.4, 0.8] & \text{if } x = 3, \\ [0.2, 0.5] & \text{if } x = 4, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]), \quad x \mapsto \begin{cases} [0.2, 0.8) & \text{if } x = 0, \\ (0.4, 0.8] & \text{if } x = 1, \\ [0.1, 0.6] & \text{if } x = 2, \\ [0.6, 0.9] & \text{if } x = 3, \\ [0.3, 0.5] & \text{if } x = 4, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]), \quad x \mapsto \begin{cases} [0.1, 0.4) & \text{if } x = 0, \\ (0.4, 0.8] & \text{if } x = 1, \\ [0.1, 0.5] & \text{if } x = 2, \\ [0.2, 0.7) & \text{if } x = 3, \\ [0.3, 0.9] & \text{if } x = 4. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 2.
It is routine to verify that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,4)$-length neutrosophic subalgebra of $(X, *, 0)$.

Table 2. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.7	0.6	0.3
1	0.4	0.4	0.4
2	0.6	0.5	0.4
3	0.4	0.3	0.5
4	0.3	0.2	0.6

Proposition 3.3. Given an (i, j, k)-length neutrosophic subalgebra $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ of $(X, *, 0)$, we have the following assertions.

(1) If $i, j, k \in \{1, 3\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \geq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \geq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \geq \mathcal{I}[F]_\ell(x)). \tag{3.1}$$

(2) If $i, j, k \in \{2, 4\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \leq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \leq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \leq \mathcal{I}[F]_\ell(x)). \tag{3.2}$$

(3) If $i, j \in \{1, 3\}$ and $k \in \{2, 4\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \geq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \geq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \leq \mathcal{I}[F]_\ell(x)). \tag{3.3}$$

(4) If $i, j \in \{2, 4\}$ and $k \in \{1, 3\}$, then

$$(\forall x \in X)(\mathcal{I}[T]_\ell(0) \leq \mathcal{I}[T]_\ell(x), \mathcal{I}[I]_\ell(0) \leq \mathcal{I}[I]_\ell(x), \mathcal{I}[F]_\ell(0) \geq \mathcal{I}[F]_\ell(x)). \tag{3.4}$$

Proof: Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$. If $(i, j, k) = (1, 3, 1)$, then

$$\mathcal{I}[T]_\ell(0) = \mathcal{I}[T]_\ell(x * x) \geq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(x)\} = \mathcal{I}[T]_\ell(x)$$

$$\mathcal{I}[I]_\ell(0) = \mathcal{I}[I]_\ell(x * x) \geq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(x)\} = \mathcal{I}[I]_\ell(x)$$

$$\mathcal{I}[F]_\ell(0) = \mathcal{I}[F]_\ell(x * x) \geq \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(x)\} = \mathcal{I}[F]_\ell(x)$$

for all $x \in X$. Similarly, we can verify that (3.1) is true for other cases of (i, j, k). Using the similar way to the proof of (1), we can prove that (2), (3) and (4) hold. □

THEOREM 3.4. *Given a subalgebra S of $(X, *, 0)$ and $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathcal{P}^*([0,1])$, let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by*

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} A_2 & \text{if } x \in S, \\ A_1 & \text{otherwise,} \end{cases} \tag{3.5}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} B_2 & \text{if } x \in S, \\ B_1 & \text{otherwise,} \end{cases} \tag{3.6}$$

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} C_2 & \text{if } x \in S, \\ C_1 & \text{otherwise.} \end{cases} \tag{3.7}$$

(1) *If $A_1 \subsetneq A_2$, $B_1 \subsetneq B_2$ and $C_1 \subsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,1)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(2) *If $A_1 \supsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \supsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4,4,4)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(3) *If $A_1 \subsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \subsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,4,1)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(4) *If $A_1 \supsetneq A_2$, $B_1 \subsetneq B_2$ and $C_1 \supsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4,1,4)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(5) *If $A_1 \subsetneq A_2$, $B_1 \subsetneq B_2$ and $C_1 \supsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,4)$-length neutrosophic subalgebra of $(X, *, 0)$.*

(6) *If $A_1 \supsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \subsetneq C_2$, then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4,4,1)$-length neutrosophic subalgebra of $(X, *, 0)$.*

PROOF: We will prove (3) only, and others can be obtained by the similar way. Assume that $A_1 \subsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \subsetneq C_2$. If $x \in S$, then $\mathcal{I}[T](x) = A_2$, $\mathcal{I}[I](x) = B_2$ and $\mathcal{I}[F](x) = C_2$. Hence

$$\begin{aligned}\mathcal{I}[T]_\ell(x) &= \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \sup\{A_2\} - \inf\{A_2\},\\ \mathcal{I}[I]_\ell(x) &= \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \sup\{B_2\} - \inf\{B_2\},\\ \mathcal{I}[F]_\ell(x) &= \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \sup\{C_2\} - \inf\{C_2\}.\end{aligned}$$

If $x \notin S$, then $\mathcal{I}[T](x) = A_1$, $\mathcal{I}[I](x) = B_1$ and $\mathcal{I}[F](x) = C_1$, and so

$$\begin{aligned}\mathcal{I}[T]_\ell(x) &= \mathcal{I}[T]_{\sup}(x) - \mathcal{I}[T]_{\inf}(x) = \sup\{A_1\} - \inf\{A_1\},\\ \mathcal{I}[I]_\ell(x) &= \mathcal{I}[I]_{\sup}(x) - \mathcal{I}[I]_{\inf}(x) = \sup\{B_1\} - \inf\{B_1\},\\ \mathcal{I}[F]_\ell(x) &= \mathcal{I}[F]_{\sup}(x) - \mathcal{I}[F]_{\inf}(x) = \sup\{C_1\} - \inf\{C_1\}.\end{aligned}$$

Since $A_1 \subsetneq A_2$, $B_1 \supsetneq B_2$ and $C_1 \subsetneq C_2$, we have

$$\begin{aligned}\sup\{A_2\} - \inf\{A_2\} &\geq \sup\{A_1\} - \inf\{A_1\},\\ \sup\{B_2\} - \inf\{B_2\} &\leq \sup\{B_1\} - \inf\{B_1\},\\ \sup\{C_2\} - \inf\{C_2\} &\geq \sup\{C_1\} - \inf\{C_1\}.\end{aligned}$$

Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ and so

$$\begin{aligned}\mathcal{I}[T]_\ell(x * y) &= \sup\{A_2\} - \inf\{A_2\} = \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},\\ \mathcal{I}[I]_\ell(x * y) &= \sup\{B_2\} - \inf\{B_2\} = \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},\\ \mathcal{I}[F]_\ell(x * y) &= \sup\{C_2\} - \inf\{C_2\} = \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}.\end{aligned}$$

If $x, y \notin S$, then

$$\begin{aligned}\mathcal{I}[T]_\ell(x * y) &\geq \sup\{A_1\} - \inf\{A_1\} = \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},\\ \mathcal{I}[I]_\ell(x * y) &\leq \sup\{B_1\} - \inf\{B_1\} = \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},\\ \mathcal{I}[F]_\ell(x * y) &\geq \sup\{C_1\} - \inf\{C_1\} = \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}.\end{aligned}$$

Assume that $x \in S$ and $y \notin S$ (or, $x \notin S$ and $y \in S$). Then

$$\begin{aligned}\mathcal{I}[T]_\ell(x * y) &\geq \sup\{A_1\} - \inf\{A_1\} = \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\},\\ \mathcal{I}[I]_\ell(x * y) &\leq \sup\{B_1\} - \inf\{B_1\} = \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\},\\ \mathcal{I}[F]_\ell(x * y) &\geq \sup\{C_1\} - \inf\{C_1\} = \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}.\end{aligned}$$

Therefore $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$. □

Remark 3.5. We have the following relations.

(1) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{1, 3\}$ is a $(1, 1, 1)$-length neutrosophic subalgebra of $(X, *, 0)$.

(2) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{2, 4\}$ is a $(4, 4, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

(3) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j \in \{1, 3\}$ and $k \in \{2, 4\}$ is a $(1, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

(4) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j \in \{2, 4\}$ and $k \in \{1, 3\}$ is a $(4, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$.

(5) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{2, 4\}$ and $j \in \{1, 3\}$ is a $(4, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$.

(6) Every (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{1, 3\}$ and $j \in \{2, 4\}$ is a $(1, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$.

The following example shows that the converse in Remark 3.5 is not true in general. We consider the cases (5) and (6) only in Remark 3.5.

Example 3.6. Consider the BCK-algebra $(X, *, 0)$ in Example 3.2. Given a subalgebra $S = \{0, 1, 2\}$ of $(X, *, 0)$, let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.7) & \text{if } x \in S, \\ (0.1, 0.8] & \text{otherwise,} \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.9) & \text{if } x \in S, \\ (0.3, 0.7] & \text{otherwise,} \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.4, 0.5) & \text{if } x \in S, \\ (0.3, 0.6] & \text{otherwise.} \end{cases}$$

Then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4, 1, 4)$-length neutrosophic subalgebra of $(X, *, 0)$ by Theorem 3.4(4). Since

$$\mathcal{I}[I]_\ell(2) = \mathcal{I}[I]_{\sup}(2) - \mathcal{I}[I]_{\inf}(2) = 0.9 - 0.2 = 0.7$$

and

$$\mathcal{I}[I]_\ell(3 * 2) = \mathcal{I}[I]_\ell(3) = \mathcal{I}[I]_{\sup}(3) - \mathcal{I}[I]_{\inf}(3) = 0.7 - 0.3 = 0.4,$$

we have $\mathcal{I}[I]_\ell(3 * 2) = 0.4 < 0.7 = \max\{\mathcal{I}[I]_\ell(3), \mathcal{I}[I]_\ell(2)\}$. Hence $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not an $(i, 3, k)$-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{2, 4\}$. Given a subalgebra $S = \{0, 1, 2, 3\}$ of $(X, *, 0)$, let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.7) & \text{if } x \in S, \\ (0.3, 0.5] & \text{otherwise,} \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.4, 0.6) & \text{if } x \in S, \\ (0.3, 0.8] & \text{otherwise,} \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.2, 0.8) & \text{if } x \in S, \\ (0.3, 0.6] & \text{otherwise.} \end{cases}$$

Then $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1, 4, 1)$-length neutrosophic subalgebra of $(X, *, 0)$ by Theorem 3.4(3). But it is not an $(i, 2, k)$-length neutrosophic subalgebra of $(X, *, 0)$ for $i, k \in \{1, 3\}$ since

$$\mathcal{I}[I]_\ell(4 * 2) = \mathcal{I}[I]_\ell(4) = 0.5 > 0.2 = \min\{\mathcal{I}[I]_\ell(4), \mathcal{I}[I]_\ell(2)\}.$$

Given an interval neutrosophic set $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ in $(X, *, 0)$, we consider the following level sets:

$$\begin{aligned} U_\ell(\mathcal{I}[T]; \alpha_T) &:= \{x \in X \mid \mathcal{I}[T]_\ell(x) \geq \alpha_T\}, \\ U_\ell(\mathcal{I}[I]; \alpha_I) &:= \{x \in X \mid \mathcal{I}[I]_\ell(x) \geq \alpha_I\}, \\ U_\ell(\mathcal{I}[F]; \alpha_F) &:= \{x \in X \mid \mathcal{I}[F]_\ell(x) \geq \alpha_F\}, \end{aligned}$$

and

$$\begin{aligned} L_\ell(\mathcal{I}[T]; \beta_T) &:= \{x \in X \mid \mathcal{I}[T]_\ell(x) \leq \beta_T\}, \\ L_\ell(\mathcal{I}[I]; \beta_I) &:= \{x \in X \mid \mathcal{I}[I]_\ell(x) \leq \beta_I\}, \\ L_\ell(\mathcal{I}[F]; \beta_F) &:= \{x \in X \mid \mathcal{I}[F]_\ell(x) \leq \beta_F\}. \end{aligned}$$

THEOREM 3.7. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$ *and for any* α_T, α_I, $\alpha_F \in [0,1]$, *the following assertions are equivalent.*

(1) $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(1,1,1)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

(2) $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ *and* $U_\ell(\mathcal{I}[F]; \alpha_F)$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty.*

PROOF: Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,1)$-length neutrosophic subalgebra of $(X, *, 0)$ and let α_T, α_I, $\alpha_F \in [0,1]$ be such that $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ and $U_\ell(\mathcal{I}[F]; \alpha_F)$ are nonempty. If $x, y \in U_\ell(\mathcal{I}[T]; \alpha_T)$, then $\mathcal{I}[T]_\ell(x) \geq \alpha_T$ and $\mathcal{I}[T]_\ell(y) \geq \alpha_T$. Hence

$$\mathcal{I}[T]_\ell(x * y) \geq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\} \geq \alpha_T,$$

that is, $x * y \in U_\ell(\mathcal{I}[T]; \alpha_T)$. Similarly, we can see that if $x, y \in U_\ell(\mathcal{I}[I]; \alpha_I)$, then $x * y \in U_\ell(\mathcal{I}[I]; \alpha_I)$, and if $x, y \in U_\ell(\mathcal{I}[F]; \alpha_F)$, then $x * y \in U_\ell(\mathcal{I}[F]; \alpha_F)$. Therefore $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ and $U_\ell(\mathcal{I}[F]; \alpha_F)$ are subalgebras of $(X, *, 0)$.

Conversely, suppose that (2) is valid. If there exist $a, b \in X$ such that

$$\mathcal{I}[T]_\ell(a * b) < \min\{\mathcal{I}[T]_\ell(a), \mathcal{I}[T]_\ell(b)\},$$

then $a, b \in U_\ell(\mathcal{I}[T]; \alpha_T)$ by taking $\alpha_T = \min\{\mathcal{I}[T]_\ell(a), \mathcal{I}[T]_\ell(b)\}$, and so $a * b \in U_\ell(\mathcal{I}[T]; \alpha_T)$. It follows that $\mathcal{I}[T]_\ell(a * b) \geq \alpha_T$, a contradiction. Hence

$$\mathcal{I}[T]_\ell(x * y) \geq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\}$$

for all $x, y \in X$. Similarly, we can check that

$$\mathcal{I}[I]_\ell(x * y) \geq \min\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\}$$

and

$$\mathcal{I}[F]_\ell(x * y) \geq \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}$$

for all $x, y \in X$. Thus $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(1,1,1)$-length neutrosophic subalgebra of $(X, *, 0)$. □

Corollary 3.8. If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is an (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{1, 3\}$, then $U_\ell(\mathcal{I}[T]; \alpha_T)$, $U_\ell(\mathcal{I}[I]; \alpha_I)$ and $U_\ell(\mathcal{I}[F]; \alpha_F)$ are subalgebras of $(X, *, 0)$ whenever they are nonempty for all α_T, α_I, $\alpha_F \in [0, 1]$.

The following example shows that the converse of Corollary 3.8 is not true.

Example 3.9. Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with the binary operation $*$ which is given in Table 3 (see [5]).

Table 3. Cayley table for the binary operation "$*$"

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.3, 0.9) & \text{if } x = 0, \\ (0.5, 0.7] & \text{if } x = 1, \\ [0.1, 0.6] & \text{if } x = 2, \\ [0.4, 0.7] & \text{if } x = a, \\ (0.3, 0.5] & \text{if } x = b, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.2, 0.9) & \text{if } x = 0, \\ (0.1, 0.8] & \text{if } x = 1, \\ [0.5, 0.9] & \text{if } x = 2, \\ [0.4, 0.7] & \text{if } x = a, \\ (0.4, 0.7] & \text{if } x = b, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.1, 0.6) & \text{if } x = 0, \\ (0.6, 0.9) & \text{if } x = 1, \\ (0.4, 0.8] & \text{if } x = 2, \\ [0.5, 0.7] & \text{if } x = a, \\ (0.5, 0.7] & \text{if } x = b. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 4.

Table 4. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.6	0.7	0.5
1	0.2	0.7	0.3
2	0.5	0.4	0.4
a	0.3	0.3	0.2
b	0.2	0.3	0.2

Hence we have

$$U_\ell(\mathcal{I}[T]; \alpha_T) = \begin{cases} \emptyset & \text{if } \alpha_T \in (0.6, 1], \\ \{0\} & \text{if } \alpha_T \in (0.5, 0.6], \\ \{0, 2\} & \text{if } \alpha_T \in (0.3, 0.5], \\ \{0, 2, a\} & \text{if } \alpha_T \in (0.2, 0.3], \\ X & \text{if } \alpha_T \in [0, 0.2], \end{cases}$$

$$U_\ell(\mathcal{I}[I]; \alpha_I) = \begin{cases} \emptyset & \text{if } \alpha_I \in (0.7, 1], \\ \{0, 1\} & \text{if } \alpha_I \in (0.4, 0.7], \\ \{0, 1, 2\} & \text{if } \alpha_I \in (0.3, 0.4], \\ X & \text{if } \alpha_I \in [0, 0.3], \end{cases}$$

and

$$U_\ell(\mathcal{I}[F];\alpha_F) = \begin{cases} \emptyset & \text{if } \alpha_F \in (0.5, 1], \\ \{0\} & \text{if } \alpha_F \in (0.4, 0.5], \\ \{0, 2\} & \text{if } \alpha_F \in (0.3, 0.4], \\ \{0, 1, 2\} & \text{if } \alpha_F \in (0.2, 0.3], \\ X & \text{if } \alpha_F \in [0, 0.2], \end{cases}$$

and so $U_\ell(\mathcal{I}[T];\alpha_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X, *, 0)$ for all α_T, α_I, $\alpha_F \in [0, 1]$ such that $U_\ell(\mathcal{I}[T];\alpha_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are nonempty. But $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not an (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{1, 3\}$ with $(i, j, k) \neq (1, 1, 1)$ since

$$\mathcal{I}[T]_\ell(b * 2) = \mathcal{I}[T]_\ell(b) = 0.2 \ngeq 0.5 = \max\{\mathcal{I}[T]_\ell(b), \mathcal{I}[T]_\ell(2)\},$$

$$\mathcal{I}[I]_\ell(a * 1) = \mathcal{I}[I]_\ell(a) = 0.3 \ngeq 0.7 = \max\{\mathcal{I}[I]_\ell(a), \mathcal{I}[I]_\ell(1)\},$$

and/or

$$\mathcal{I}[F]_\ell(b * 1) = \mathcal{I}[F]_\ell(a) = 0.2 \ngeq 0.3 = \max\{\mathcal{I}[F]_\ell(b), \mathcal{I}[F]_\ell(1)\}.$$

THEOREM 3.10. *Given an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *in* $(X, *, 0)$ *and for any* β_T, β_I, $\beta_F \in [0, 1]$*, the following assertions are equivalent.*

(1) $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(4, 4, 4)$*-length neutrosophic subalgebra of* $(X, *, 0)$.

(2) $L_\ell(\mathcal{I}[T];\beta_T)$*,* $L_\ell(\mathcal{I}[I];\beta_I)$ *and* $L_\ell(\mathcal{I}[F];\beta_F)$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty.*

PROOF: Suppose that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4, 4, 4)$-length neutrosophic subalgebra of $(X, *, 0)$ and let β_T, β_I, $\beta_F \in [0, 1]$ be such that $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are nonempty. For any $x, y \in X$, if $x, y \in L_\ell(\mathcal{I}[T];\beta_T)$, then $\mathcal{I}[T]_\ell(x) \leq \beta_T$ and $\mathcal{I}[T]_\ell(y) \leq \beta_T$. It follows that

$$\mathcal{I}[T]_\ell(x * y) \leq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\} \leq \beta_T$$

and so that $x * y \in L_\ell(\mathcal{I}[T];\beta_T)$. Similarly, if $x, y \in L_\ell(\mathcal{I}[I];\beta_I)$, then $x * y \in L_\ell(\mathcal{I}[I];\beta_I)$, and if $x, y \in L_\ell(\mathcal{I}[F];\beta_F)$, then $x * y \in L_\ell(\mathcal{I}[F];\beta_F)$.

Therefore (2) is valid.

Conversely, assume that $L_\ell(\mathcal{I}[T]; \beta_T)$, $L_\ell(\mathcal{I}[I]; \beta_I)$ and $L_\ell(\mathcal{I}[F]; \beta_F)$ are subalgebras of $(X, *, 0)$ whenever they are nonempty for all β_T, β_I, $\beta_F \in [0, 1]$. If there are $a, b \in X$ such that

$$\mathcal{I}[F]_\ell(a * b) > \max\{\mathcal{I}[F]_\ell(a), \mathcal{I}[F]_\ell(b)\},$$

then $a, b \in L_\ell(\mathcal{I}[F]; \beta_F)$ by taking $\beta_F = \max\{\mathcal{I}[F]_\ell(a), \mathcal{I}[F]_\ell(b)\}$. Thus $a * b \in L_\ell(\mathcal{I}[F]; \beta_F)$, which implies that $\mathcal{I}[F]_\ell(a * b) \leq \beta_F$. This is a contradiction, and so

$$\mathcal{I}[F]_\ell(x * y) \leq \max\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\}$$

for all $x, y \in X$. Similarly, we get

$$\mathcal{I}[T]_\ell(x * y) \leq \max\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\}$$

and

$$\mathcal{I}[I]_\ell(x * y) \leq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\}$$

for all $x, y \in X$. Consequently, $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(4, 4, 4)$-length neutrosophic subalgebra of $(X, *, 0)$. □

Corollary 3.11. If $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is an (i, j, k)-length neutrosophic subalgebra of $(X, *, 0)$ for $i, j, k \in \{2, 4\}$, then $L_\ell(\mathcal{I}[T]; \beta_T)$, $L_\ell(\mathcal{I}[I]; \beta_I)$ and $L_\ell(\mathcal{I}[F]; \beta_F)$ are subalgebras of $(X, *, 0)$ whenever they are nonempty for all β_T, β_I, $\beta_F \in [0, 1]$.

The following example shows that the converse of Corollary 3.11 is not true.

Example 3.12. Consider the BCI-algebra $X = \{0, 1, 2, a, b\}$ in Example 3.9 and let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0, 1]),\ x \mapsto \begin{cases} [0.5, 0.7) & \text{if } x = 0, \\ (0.2, 0.6] & \text{if } x = 1, \\ [0.3, 0.6] & \text{if } x = 2, \\ [0.1, 0.7] & \text{if } x = a, \\ (0.2, 0.8] & \text{if } x = b, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.66, 0.99) & \text{if } x = 0, \\ (0.15, 0.59] & \text{if } x = 1, \\ [0.22, 0.88) & \text{if } x = 2, \\ (0.35, 0.90] & \text{if } x = a, \\ (0.20, 0.75) & \text{if } x = b, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.75, 0.90) & \text{if } x = 0, \\ (0.45, 0.90) & \text{if } x = 1, \\ (0.25, 0.50] & \text{if } x = 2, \\ [0.50, 0.85] & \text{if } x = a, \\ (0.15, 0.60] & \text{if } x = b. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 5.

Table 5. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.2	0.33	0.15
1	0.4	0.44	0.45
2	0.3	0.66	0.25
a	0.6	0.55	0.35
b	0.6	0.55	0.45

Hence we have

$$L_\ell(\mathcal{I}[T]; \beta_T) = \begin{cases} \emptyset & \text{if } \beta_T \in [0, 0.2), \\ \{0\} & \text{if } \beta_T \in [0.2, 0.3), \\ \{0, 2\} & \text{if } \beta_T \in [0.3, 0.4), \\ \{0, 1, 2\} & \text{if } \beta_T \in [0.4, 0.6), \\ X & \text{if } \beta_T \in [0.6, 1], \end{cases}$$

$$L_\ell(\mathcal{I}[I]; \beta_I) = \begin{cases} \emptyset & \text{if } \beta_I \in [0, 0.33), \\ \{0\} & \text{if } \beta_I \in [0.33, 0.44), \\ \{0, 1\} & \text{if } \beta_I \in [0.44, 0.55), \\ \{0, 1, a, b\} & \text{if } \beta_I \in [0.55, 0.66), \\ X & \text{if } \beta_I \in [0.66, 1], \end{cases}$$

and

$$L_\ell(\mathcal{I}[F];\beta_F)=\begin{cases}\emptyset & \text{if } \beta_F\in[0,0.15),\\ \{0\} & \text{if } \beta_F\in[0.15,0.25),\\ \{0,2\} & \text{if } \beta_F\in[0.25,0.35),\\ \{0,2,a\} & \text{if } \beta_F\in[0.35,0.45),\\ X & \text{if } \beta_F\in[0.45,1],\end{cases}$$

which are subalgebras of $(X,*,0)$ for all β_T, β_I, $\beta_F \in [0,1]$ such that $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are nonempty. But $\mathcal{I}:=(\mathcal{I}[T]$, $\mathcal{I}[I]$, $\mathcal{I}[F])$ is not an (i,j,k)-length neutrosophic subalgebra of $(X,*,0)$ for $i,j,k\in\{2,4\}$ with $(i,j,k)\neq(4,4,4)$ since

$$\mathcal{I}[T]_\ell(a*1)=0.6\not\leq 0.4=\min\{\mathcal{I}[T]_\ell(a),\mathcal{I}[T]_\ell(1)\},$$

$$\mathcal{I}[I]_\ell(a*0)=0.55\not\leq 0.33=\min\{\mathcal{I}[I]_\ell(a),\mathcal{I}[I]_\ell(0)\},$$

and/or

$$\mathcal{I}[F]_\ell(2*a)=0.35\not\leq 0.25=\min\{\mathcal{I}[F]_\ell(2),\mathcal{I}[F]_\ell(a)\}.$$

Using the similar way to the proofs of Theorems 3.7 and 3.10, we have the following theorem.

THEOREM 3.13. *Given an (i,j,k)-length neutrosophic subalgebra $\mathcal{I}:=(\mathcal{I}[T]$, $\mathcal{I}[I]$, $\mathcal{I}[F])$ of $(X,*,0)$ for $i,j,k\in\{1,2,3,4\}$, the following assertions are valid.*

(1) *If $i,j\in\{1,3\}$ and $k\in\{2,4\}$, then $U_\ell(\mathcal{I}[T];\alpha_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $L_\ell(\mathcal{I}[F];\beta_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(2) *If $i,k\in\{1,3\}$ and $j\in\{2,4\}$, then $U_\ell(\mathcal{I}[T];\alpha_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(3) *If $i\in\{2,4\}$ and $j,k\in\{1,3\}$, then $L_\ell(\mathcal{I}[T];\beta_T)$, $U_\ell(\mathcal{I}[I];\alpha_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(4) *If $i,j\in\{2,4\}$ and $k\in\{1,3\}$, then $L_\ell(\mathcal{I}[T];\beta_T)$, $L_\ell(\mathcal{I}[I];\beta_I)$ and $U_\ell(\mathcal{I}[F];\alpha_F)$ are subalgebras of $(X,*,0)$ whenever they are nonempty.*

(5) *If* $i, k \in \{2, 4\}$ *and* $j \in \{1, 3\}$*, then* $L_\ell(\mathcal{I}[T]; \beta_T)$*,* $U_\ell(\mathcal{I}[I]; \alpha_I)$ *and* $L_\ell(\mathcal{I}[F]; \beta_F)$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty.*

(6) *If* $i \in \{1, 3\}$ *and* $j, k \in \{2, 4\}$*, then* $U_\ell(\mathcal{I}[T]; \alpha_T)$*,* $L_\ell(\mathcal{I}[I]; \beta_I)$ *and* $L_\ell(\mathcal{I}[F]; \beta_F)$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty.*

THEOREM 3.14. *If an interval neutrosophic set* $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ *is a* $(2, 3, 2)$*-length neutrosophic subalgebra of* $(X, *, 0)$*, then* $U_\ell(\mathcal{I}[T]; \alpha_T)^c$*,* $L_\ell(\mathcal{I}[I]; \beta_I)^c$ *and* $U_\ell(\mathcal{I}[F]; \alpha_F)^c$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty for all* α_T*,* β_I*,* $\alpha_F \in [0, 1]$*.*

PROOF: Assume that $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is a $(2, 3, 2)$-length neutrosophic subalgebra of $(X, *, 0)$. Let α_T, β_I, $\alpha_F \in [0, 1]$ be such that $U_\ell(\mathcal{I}[T]; \alpha_T)^c$, $L_\ell(\mathcal{I}[I]; \beta_I)^c$ and $U_\ell(\mathcal{I}[F]; \alpha_F)^c$ are nonempty. If $x, y \in U_\ell(\mathcal{I}[T]; \alpha_T)^c$, then $\mathcal{I}[T]_\ell(x) < \alpha_T$ and $\mathcal{I}[T]_\ell(y) < \alpha_T$. Hence

$$\mathcal{I}[T]_\ell(x * y) \leq \min\{\mathcal{I}[T]_\ell(x), \mathcal{I}[T]_\ell(y)\} < \alpha_T,$$

and so $x * y \in U_\ell(\mathcal{I}[T]; \alpha_T)^c$. If $x, y \in L_\ell(\mathcal{I}[I]; \beta_I)^c$, then $\mathcal{I}[I]_\ell(x) > \beta_I$ and $\mathcal{I}[I]_\ell(y) > \beta_I$. Thus

$$\mathcal{I}[I]_\ell(x * y) \geq \max\{\mathcal{I}[I]_\ell(x), \mathcal{I}[I]_\ell(y)\} > \beta_I,$$

which implies that $x * y \in L_\ell(\mathcal{I}[I]; \beta_I)^c$. Let $x, y \in U_\ell(\mathcal{I}[F]; \alpha_F)^c$. Then $\mathcal{I}[F]_\ell(x) < \alpha_F$ and $\mathcal{I}[F]_\ell(y) < \alpha_F$. Hence

$$\mathcal{I}[F]_\ell(x * y) \leq \min\{\mathcal{I}[F]_\ell(x), \mathcal{I}[F]_\ell(y)\} < \alpha_F,$$

and so $x * y \in U_\ell(\mathcal{I}[F]; \alpha_F)^c$. Therefore $U_\ell(\mathcal{I}[T]; \alpha_T)^c$, $L_\ell(\mathcal{I}[I]; \beta_I)^c$ and $U_\ell(\mathcal{I}[F]; \alpha_F)^c$ are subalgebras of $(X, *, 0)$ for all α_T, β_I, $\alpha_F \in [0, 1]$. □

The converse of Theorem 3.14 is not true in general as seen in the following example.

Example 3.15. Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the binary operation $*$ which is given in Table 6 (see [5]).
Let $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ be an interval neutrosophic set in $(X, *, 0)$ given by

Table 6. Cayley table for the binary operation "$*$"

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

$$\mathcal{I}[T] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.50, 0.75) & \text{if } x = 0, \\ (0.25, 0.70] & \text{if } x = 1, \\ [0.10, 0.65] & \text{if } x = a, \\ [0.05, 0.70) & \text{if } x = b, \\ (0.10, 0.75] & \text{if } x = c, \end{cases}$$

$$\mathcal{I}[I] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.05, 0.80] & \text{if } x = 0, \\ (0.10, 0.80) & \text{if } x = 1, \\ [0.26, 0.89] & \text{if } x = a, \\ (0.16, 0.79) & \text{if } x = b, \\ (0.07, 0.75] & \text{if } x = c, \end{cases}$$

and

$$\mathcal{I}[F] : X \to \mathcal{P}^*([0,1]),\ x \mapsto \begin{cases} [0.23, 0.67) & \text{if } x = 0, \\ (0.03, 0.58] & \text{if } x = 1, \\ (0.18, 0.73) & \text{if } x = a, \\ [0.14, 0.80] & \text{if } x = b, \\ (0.07, 0.73] & \text{if } x = c. \end{cases}$$

Then the interval neutrosophic length $\mathcal{I}_\ell := (\mathcal{I}[T]_\ell, \mathcal{I}[I]_\ell, \mathcal{I}[F]_\ell)$ of $\mathcal{I}$ is given by Table 7.
Then

$$U_\ell(\mathcal{I}[T]; \alpha_T)^c = \begin{cases} \emptyset & \text{if } \alpha_T \in [0, 0.25], \\ \{0\} & \text{if } \alpha_T \in (0.25, 0.45], \\ \{0, 1\} & \text{if } \alpha_T \in (0.45, 0.55], \\ \{0, 1, a\} & \text{if } \alpha_T \in (0.55, 0.65], \\ X & \text{if } \alpha_T \in (0.65, 1], \end{cases}$$

Table 7. Interval neutrosophic length of $\mathcal{I}$

X	$\mathcal{I}[T]_\ell$	$\mathcal{I}[I]_\ell$	$\mathcal{I}[F]_\ell$
0	0.25	0.75	0.44
1	0.45	0.70	0.55
a	0.55	0.63	0.55
b	0.65	0.63	0.66
c	0.65	0.68	0.66

$$L_\ell(\mathcal{I}[I];\beta_I)^c = \begin{cases} \emptyset & \text{if } \beta_I \in [0.75, 1], \\ \{0\} & \text{if } \beta_I \in [0.70, 0.75), \\ \{0, 1\} & \text{if } \beta_I \in [0.68, 0.70), \\ \{0, 1, c\} & \text{if } \beta_I \in [0.63, 0.68), \\ X & \text{if } \beta_I \in [0, 0.63), \end{cases}$$

and

$$U_\ell(\mathcal{I}[F];\alpha_F)^c = \begin{cases} \emptyset & \text{if } \alpha_F \in [0, 0.44], \\ \{0\} & \text{if } \alpha_F \in (0.44, 0.55], \\ \{0, 1, a\} & \text{if } \alpha_F \in (0.55, 0.66], \\ X & \text{if } \alpha_F \in (0.66, 1] \end{cases}$$

are subalgebras of $(X, *, 0)$ whenever they are nonempty for all α_T, β_I, $\alpha_F \in [0, 1]$. But $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ is not a $(2, 3, 2)$-length neutrosophic subalgebra of $(X, *, 0)$ since

$$\mathcal{I}[T]_\ell(b * a) = \mathcal{I}[T]_\ell(c) = 0.65 > 0.55 = \min\{\mathcal{I}[T]_\ell(b), \mathcal{I}[T]_\ell(a)\},$$

$$\mathcal{I}[I]_\ell(b * c) = \mathcal{I}[I]_\ell(a) = 0.63 < 0.68 = \max\{\mathcal{I}[I]_\ell(b), \mathcal{I}[I]_\ell(c)\},$$

and/or

$$\mathcal{I}[F]_\ell(b * a) = \mathcal{I}[F]_\ell(c) = 0.66 > 0.55 = \min\{\mathcal{I}[F]_\ell(b), \mathcal{I}[F]_\ell(a)\}.$$

By the similar way to the proof of Theorem 3.14, we have the following theorem.

THEOREM 3.16. *Given an (i, j, k)-length neutrosophic subalgebra $\mathcal{I} := (\mathcal{I}[T], \mathcal{I}[I], \mathcal{I}[F])$ of $(X, *, 0)$, the following assertions are valid.*

(1) *If* $(i, j, k) = (2, 2, 2)$, *then* $U_\ell(\mathcal{I}[T]; \alpha_T)^c$, $U_\ell(\mathcal{I}[I]; \alpha_I)^c$ *and* $U_\ell(\mathcal{I}[F]; \alpha_F)^c$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty for all* α_T, α_I, $\alpha_F \in [0, 1]$.

(2) *If* $(i, j, k) = (2, 2, 3)$, *then* $U_\ell(\mathcal{I}[T]; \alpha_T)^c$, $U_\ell(\mathcal{I}[I]; \alpha_I)^c$ *and* $L_\ell(\mathcal{I}[F]; \beta_F)^c$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty for all* α_T, α_I, $\beta_F \in [0, 1]$.

(3) *If* $(i, j, k) = (2, 3, 3)$, *then* $U_\ell(\mathcal{I}[T]; \alpha_T)^c$, $L_\ell(\mathcal{I}[I]; \beta_I)^c$ *and* $L_\ell(\mathcal{I}[F]; \beta_F)^c$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty for all* α_T, β_I, $\beta_F \in [0, 1]$.

(4) *If* $(i, j, k) = (3, 2, 2)$, *then* $L_\ell(\mathcal{I}[T]; \beta_T)^c$, $U_\ell(\mathcal{I}[I]; \alpha_I)^c$ *and* $U_\ell(\mathcal{I}[F]; \alpha_F)^c$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty for all* β_T, α_I, $\alpha_F \in [0, 1]$.

(5) *If* $(i, j, k) = (3, 2, 3)$, *then* $L_\ell(\mathcal{I}[T]; \beta_T)^c$, $U_\ell(\mathcal{I}[I]; \alpha_I)^c$ *and* $L_\ell(\mathcal{I}[F]; \beta_F)^c$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty for all* β_T, α_I, $\beta_F \in [0, 1]$.

(6) *If* $(i, j, k) = (3, 3, 2)$, *then* $L_\ell(\mathcal{I}[T]; \beta_T)^c$, $L_\ell(\mathcal{I}[I]; \beta_I)^c$ *and* $U_\ell(\mathcal{I}[F]; \alpha_F)^c$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty for all* β_T, β_I, $\alpha_F \in [0, 1]$.

(7) *If* $(i, j, k) = (3, 3, 3)$, *then* $L_\ell(\mathcal{I}[T]; \beta_T)^c$, $L_\ell(\mathcal{I}[I]; \beta_I)^c$ *and* $L_\ell(\mathcal{I}[F]; \beta_F)^c$ *are subalgebras of* $(X, *, 0)$ *whenever they are nonempty for all* β_T, β_I, $\beta_F \in [0, 1]$.

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Young Bae Jun
Gyeongsang National University
Department of Mathematics Education
Jinju 52828, Korea
e-mail: skywine@gmail.com

Madad Khan
COMSATS Institute of Information Technology
Department of Mathematics
Abbottabad, Pakistan
e-mail: madadmath@yahoo.com

Florentin Smarandache
University of New Mexico
Department of Mathematics
New Mexico 87301, USA
e-mail: fsmarandache@gmail.com

Seok-Zun Song
Jeju National University
Department of Mathematics
Jeju 63243, Korea
e-mail: szsong@jejunu.ac.kr