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Paweł Płaczek

ONE-SIDED SEQUENT SYSTEMS FOR NONASSOCIATIVE BILINEAR LOGIC: CUT ELIMINATION AND COMPLEXITY

Abstract

Bilinear Logic of Lambek amounts to Noncommutative MALL of Abrusci. Lambek proves the cut–elimination theorem for a one-sided (in fact, left-sided) sequent system for this logic. Here we prove an analogous result for the nonassociative version of this logic. Like Lambek, we consider a left-sided system, but the result also holds for its right-sided version, by a natural symmetry. The treatment of nonassociative sequent systems involves some subtleties, not appearing in associative logics. We also prove the PTime complexity of the multiplicative fragment of NBL.

Keywords: Substructural logic, Lambek calculus, nonassociative linear logic, sequent system, PTime complexity.

1. Introduction

Multiplicative-Additive Linear Logic (MALL) was introduced by Girard [8]. Noncommutative MALL (where product $\otimes$ is noncommutative) is due to Abrusci [1]. This logic, presented as a one-sided (precisely: left-sided[1]) sequent system was studied by Lambek [10] under the name: Classical Bilinear Logic. Lambek proved the cut–elimination theorem for this system in a syntactic way.

[1] Two-sided systems admit sequents $\Gamma \Rightarrow \Delta$, right-sided $\Rightarrow \Delta$, left-sided $\Gamma \Rightarrow$

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The present paper studies an analogous system for Nonassociative Bilinear Logic (NBL), being a version of Bilinear Logic with nonassociative $\otimes$. Some related logics, restricted to multiplicative connectives and not admitting multiplicative constant (nor the corresponding unit elements in algebraic models), were studied in [5, 3] under the name: Classical Nonassociative Lambek Calculus (CNL). CNL contains one (cyclic) negation $^{\sim}$, satisfying $a^{\sim\sim} = a$ in algebras. Buszkowski [4] considers a weaker logic, called Involutive Nonassociative Lambek Calculus (InNL), with two negations $^{\sim}, ^{-}$, satisfying $a^{-\sim} = a = a^{\sim-}$.

Here we provide a syntactic proof of the cut–elimination theorem for one-sided systems of NBL in the language $\otimes, \oplus, ^{\sim}, ^{-}, \wedge, \vee, 0, 1$ (also $\bot, \top$). Our notation is different from that of [8, 1]. In particular, we write $\oplus$ for coproduct (par), $\vee$ for additive disjunction and 0 for $\bot$, following standards of substructural logics [6]. $\otimes, \oplus, ^{\sim}, ^{-}, 0, 1$ are reffered to as multiplicative connectives and contants, while $\wedge, \vee, \bot, \top$ as additive ones. In algebras, 1 (resp. 0) is interpreted as the unit element for product (resp. coproduct), $\wedge$ (resp. $\vee$) as meet (resp. join) in a lattice and $\bot$ (resp. $\top$) as the least (resp. the greatest) element.

We follow the path presented in [4]. NBL is InNL extended by the multiplicative constants and additive connectives. All the statements and proofs in this paper are similar to these in [4], so we skip the parts that are identical and focus on the differences. The crucial difference is that Buszkowski [4] considers only sequents consisting of at least two formulas, which makes the proofs much simpler. Here we consider all nonempty sequents.

We write a complete proof of the cut–elimination theorem for a left-sided system (a nonassociative version of the system from [10]) without $\bot, \top$ (these constants are added in the subsection 4.1). In subsection 4.2 we obtain an analogous result for a right-sided system, using a natural symmetry of both systems.

InNL is a conservative extension of Nonassociative Lambek Calculus (NL), due to Lambek [9]; see [5, 3]. It can be shown that NBL is a conservative extension of NL with 1 (NL1). These logics have applications in linguistics as type logics for categorial grammars [10, 5, 3] and seem quite natural from the perspective of modal logics, where $\otimes$ can be regarded as a binary possibility operator.

NL and NL1 are usually presented as intuitionistic systems with sequents $\Gamma \Rightarrow A$; in NL Γ must be nonempty. The syntax of the left-sided system for NBL is quite similar to that of NL1 (in a richer language).

The proof of cut elimination for nonassociative logics is roughly similar to those for associative linear logic [1, 8, 9], but the nonassociative framework involves some new subtleties. For instance, the rule (r-shift), expressing the algebraic compatibility condition (see below), must be replaced by weaker rules. In the resulting system (r-shift) and two rules for $^{\sim\sim}$ and $^{--}$ are shown to be admissible (Lemmas 2.2 and 2.4), which is essentially used in the final proof (Theorem 1). Our proof partially follows that from [4] for InNL, but the richer language makes it more complicated.

In our sequent systems, negations appear at variables only (so we consider formulas in negation normal form). Negations of arbitrary formulas are defined in metalanguage. Some systems with negations of formulas in the language were considered in [2] (right-sided) and [6] (two-sided). The system from [2] does not have the subformula property. That from [6] uses sequents $\Gamma \Rightarrow \Delta$. The cut–elimination theorem for this system is proved in [6] by algebraic methods.

Having the cut–elimination theorem, we can prove the decidability of NBL. In the last section we show that the multiplicative fragment of NBL (MNBL) is PTime. The algorithm essentially uses cut elimination. An analogous result for InNL is given in [4].

By atoms in NBL-language we mean two constants: 0 and 1, and $p^{(n)}$, where p is any variable and $n \in \mathbb{Z}$. By $p^{(n)}$ we denote $p^{\sim\sim\cdots\sim}$, where $^{\sim}$ is iterated n times, if $n \geq 0$, and $p^{--\cdots-}$, where $^{-}$ is iterated $-n$ times, if $n < 0$. $^{\sim}$ and $^{-}$ are involutive negations in NBL, but we do not consider them as connectives, because they occur only with non-constant atoms. It means that the formulas are in negation normal form. The connectives are: $\otimes$ (product), $\oplus$ (coproduct), $\wedge$ (meet) and $\vee$ (join).

We define metalanguage negations for every NBL-formula:

$$0^{\sim} = 1 \qquad 0^{-} = 1 \qquad 1^{\sim} = 0 \qquad 1^{-} = 0$$
$$(p^{(n)})^{\sim} = p^{(n+1)} \qquad (p^{(n)})^{-} = p^{(n-1)}$$
$$(A \otimes B)^{\sim} = B^{\sim} \oplus A^{\sim} \qquad (A \otimes B)^{-} = B^{-} \oplus A^{-}$$
$$(A \oplus B)^{\sim} = B^{\sim} \otimes A^{\sim} \qquad (A \oplus B)^{-} = B^{-} \otimes A^{-}$$

$$(A \wedge B)^{\sim} = A^{\sim} \vee B^{\sim} \qquad (A \wedge B)^{-} = A^{-} \vee B^{-}$$
$$(A \vee B)^{\sim} = A^{\sim} \wedge B^{\sim} \qquad (A \vee B)^{-} = A^{-} \wedge B^{-}$$

One shows: $A^{\sim -} = A^{- \sim} = A$ by induction on formulas.

Definition 1.1. We define *bunches*:

(i) The empty bunch ϵ is a bunch.

(ii) Every formula is a bunch.

(iii) If Γ and Δ are bunches, then (Γ, Δ) is also a bunch.

We assume: $(\Gamma, \epsilon) = \Gamma = (\epsilon, \Gamma)$. A *sequent* in NBL is every nonempty bunch. We often omit outer parentheses in sequents and formulas.

A context is a bunch containing a special atomic formula x. Contexts are denoted by capital Greek letters and square brackets, e.g. $\Gamma[\,], \Delta[\,]$, etc. By $\Gamma[\Delta]$ we mean the substitution of Δ for x in $\Gamma[\,]$.

Now we briefly describe the algebraic models of NBL.

Definition 1.2. An algebra $\mathbf{M} = (M, \otimes, \wedge, \vee, {}^{\sim}, {}^{-}, 1)$, where $\otimes, \wedge, \vee$ are binary operators, ${}^{\sim}, {}^{-}$ are unary operators and 1 is a constant, is called a *lattice ordered (l.o.) involutive unital groupoid*, if:

(i) $(M, \wedge, \vee)$ is a lattice;

(ii) $(M, \otimes, 1)$ is a unital groupoid;

(iii) if $a \otimes b \leq c$, then $c^{-} \otimes a \leq b^{-}$ and $b \otimes c^{\sim} \leq a^{\sim}$, for all $a, b, c \in M$;

(iv) $a^{\sim -} = a^{- \sim} = a$, for all $a \in M$.

In the above definition $\leq$ stands for the lattice order. We can prove for all $a, b \in M$ that $(b^{\sim} \otimes a^{\sim})^{-} = (b^{-} \otimes a^{-})^{\sim}$, so we define $a \oplus b = (b^{\sim} \otimes a^{\sim})^{-}$. One proves that $1^{\sim} = 1^{-}$, hence we define $0 = 1^{\sim}$. One can define residuals (implications) $a \setminus b = a^{\sim} \oplus b$, $a/b = a \oplus b^{-}$, satisfying the residuation laws: $a \otimes b \leq c$ iff $a \leq c/b$ iff $b \leq a \setminus c$. One also gets $a^{\sim} = 0 \setminus a$, $a^{-} = 0/a$.

The condition (iii) is referred to as the compatibility condition. One also proves the implications coverse to (iii).

$$\text{if } c^{-} \otimes a \leq b^{-}, \text{ then } a \otimes b \leq c$$
$$\text{if } b \otimes c^{\sim} \leq a^{\sim}, \text{ then } a \otimes b \leq c$$

It follows from (iii) that negations are antitone: if $a \leq b$, then $b^- \leq a^-$ and $b^\sim \leq a^\sim$.

There hold De Morgan laws.

$$(a \otimes b)^- = b^- \oplus a^- \qquad (a \wedge b)^- = a^- \vee b^-$$

$$(a \otimes b)^\sim = b^\sim \oplus a^\sim \qquad (a \wedge b)^\sim = a^\sim \vee b^\sim$$

$$(a \oplus b)^- = b^- \otimes a^- \qquad (a \vee b)^- = a^- \wedge b^-$$

$$(a \oplus b)^\sim = b^\sim \otimes a^\sim \qquad (a \vee b)^\sim = a^\sim \wedge b^\sim$$

The following laws will be useful.

$$a^- \otimes (a \oplus b) \leq b$$

$$(a \oplus b) \otimes b^\sim \leq a$$

$$b \leq a^\sim \oplus (a \otimes b)$$

$$a \leq (a \otimes b) \oplus b^-$$

$$\text{if } a \leq b, \text{ then } a \otimes c \leq b \otimes c \text{ and } a \oplus c \leq b \oplus c$$

$$\text{if } a \leq b, \text{ then } c \otimes a \leq c \otimes b \text{ and } c \oplus a \leq c \oplus b$$

We define a valuation μ as a homomophism of the algebra of formulas into a l.o. involutive unital groupoid. We extend it to sequents by: $\mu((\Gamma, \Delta)) = \mu(\Gamma) \otimes \mu(\Delta)$ and $\mu(\epsilon) = 1$.

We say that a sequent Γ is true in $\mathbf{M}$ for a valuation μ, if $\mu(\Gamma) \leq 0$; we write $\mathbf{M}, \mu \models \Gamma$. A sequent is said to be valid, if it is true in all algebras of this kind for all valuations.

2. Nonassociative Bilinear Logic

Now we present a one-sided sequent system for Nonassociative Bilinear Logic.

We admit axioms:

$$\text{(a-id)} \qquad p^{(n)}, p^{(n+1)} \quad \text{for any variable } p \text{ and any } n \in \mathbb{Z}$$

$$\text{(a-0)} \qquad 0$$

The rules of the cut-free NBL are:

$$(\text{r-}\otimes)\quad \frac{\Gamma[(A,B)]}{\Gamma[A\otimes B]}$$

$$(\text{r-}\oplus 1)\quad \frac{\Gamma[B]\quad \Delta, A}{\Gamma[(\Delta, A\oplus B)]}\qquad (\text{r-}\oplus 2)\quad \frac{\Gamma[A]\quad B,\Delta}{\Gamma[(A\oplus B,\Delta)]}$$

$$(\text{r-}1)\quad \frac{\Gamma[\Delta]}{\Gamma[(1,\Delta)]}\quad \frac{\Gamma[\Delta]}{\Gamma[(\Delta,1)]}$$

$$(\text{r-}\wedge)\quad \frac{\Gamma[A]}{\Gamma[A\wedge B]}\quad \frac{\Gamma[A]}{\Gamma[B\wedge A]}\qquad (\text{r-}\vee)\quad \frac{\Gamma[A]\quad \Gamma[B]}{\Gamma[A\vee B]}$$

$$(\text{r-shift})\quad \frac{(\Gamma,\Delta),\Theta}{\Gamma,(\Delta,\Theta)}$$

In this paper we show that the cut rules:

$$(\text{cut}^{\sim})\quad \frac{\Gamma[A]\quad \Delta, A^{\sim}}{\Gamma[\Delta]}\qquad (\text{cut}^{-})\quad \frac{\Gamma[A]\quad A^{-},\Delta}{\Gamma[\Delta]}$$

are admissible in the cut-free NBL.

These axioms and rules are valid. By reflexivity of the lattice order we have $0\leq 0$, which is (a-0) and $p^{(n)}\leq p^{(n)}$, which can be easily transformed into $p^{(n)}\otimes p^{(n+1)}\leq 0$ by the compatibility condition; (r-$\otimes$) is sound by definition of the valuation; (r-$\wedge$) and (r-$\vee$) express the lattice order properties; (r-1) is valid, because 1 is a neutral element of $\otimes$.

Rules (r-$\oplus$1) and (r-$\oplus$2) are sound because of the propeties $a^{-}\otimes(a\oplus b)\leq b$ and $(a\oplus b)\otimes b^{\sim}\leq a$.

We prove that (r-shift) is sound. The following are equivalent:

$$(\mu(\Gamma)\otimes\mu(\Delta))\otimes\mu(\Theta)\leq 0$$
$$0^{-}\otimes(\mu(\Gamma)\otimes\mu(\Delta))\leq(\mu(\Theta))^{-}$$
$$1\otimes(\mu(\Gamma)\otimes\mu(\Delta))\leq(\mu(\Theta))^{-}$$
$$\mu(\Gamma)\otimes\mu(\Delta)\leq(\mu(\Theta))^{-}$$
$$\mu(\Delta)\otimes(\mu(\Theta))^{-\sim}\leq(\mu(\Gamma))^{\sim}$$

$$\mu(\Delta) \otimes \mu(\Theta) \leq (\mu(\Gamma))^{\sim}$$
$$(\mu(\Delta) \otimes \mu(\Theta)) \otimes 0^{\sim} \leq (\mu(\Gamma))^{\sim}$$
$$\mu(\Gamma) \otimes (\mu(\Delta) \otimes \mu(\Theta)) \leq 0$$

The system with the cut-rules is strongly complete with respect to l.o. involutive unital groupoids. We omit a routine proof, using Lindenbaum-Tarski algebras.

DEFINITION 2.1. By the *active formula* (resp. *active bunch*) of a rule we denote the new formula (bunch) intruduced by this rule.

The rule (r-shift) would complicate our syntactic proof of cut elimination. In order to avoid that, we define an equivalent cut-free system, where (r-shift) is replaced by the following rules:

$$(\text{r-}\oplus 3)\quad \frac{A, \Gamma \quad B, \Delta}{A \oplus B, (\Delta, \Gamma)} \qquad (\text{r-}\oplus 4)\quad \frac{\Gamma, A \quad \Delta, B}{(\Delta, \Gamma), A \oplus B}$$

We assume that both Γ and Δ are nonempty. Otherwise (r-$\oplus$3) and (r-$\oplus$4) are special cases of (r-$\oplus$2) and (r-$\oplus$1). One can notice that these rules are just instances of (r-$\oplus$2) and (r-$\oplus$1) with (r-shift) applied to the conclusions. We define the cut-free NBL_0 as the cut-free NBL without (r-shift), but with (r-$\oplus$3) and (r-$\oplus$4). $\vdash_{\text{NBL}_0}$ stands for the provability in the cut-free NBL_0.

LEMMA 2.2. *The rule (r-shift) is admissible in NBL_0, i.e.* $\vdash_{NBL_0} (\Gamma, \Delta), \Theta$, *if and only if* $\vdash_{NBL_0} \Gamma, (\Delta, \Theta)$.

PROOF: We show only the left-to-right implication. The converse implication is proved analogously. We assume $\vdash (\Gamma_1, \Gamma_2), \Gamma_3$ and prove $\vdash \Gamma_1, (\Gamma_2, \Gamma_3)$ (for better readability, we skip the subscript NBL_0, unless it is necessary). We also assume that none of Γ_i, $(i = 1, 2, 3)$ is empty. Otherwise the claim is trivial. We run induction on the proof of $(\Gamma_1, \Gamma_2), \Gamma_3$.

Firstly, one can easily notice that $(\Gamma_1, \Gamma_2), \Gamma_3$ cannot be an axiom. Hence it is the conclusion of a rule.

Let us consider (r-$\otimes$), (r-$\wedge$) and (r-$\vee$). All but the last one has only one premise. The last one has two premises with the same context. The active formula must occur in one of Γ_i, $(i = 1, 2, 3)$. We apply the induction hypothesis to the premise(s) and use again the same rule.

We consider (r-1). If the active bunch occurs in one of Γ_i we proceed as above. We have only to consider the case when one of Γ_i equals 1. We consider the following instances:

$$\frac{\Gamma_2, \Gamma_3}{(1, \Gamma_2), \Gamma_3} \qquad \frac{\Gamma_1, \Gamma_3}{(\Gamma_1, 1), \Gamma_3} \qquad \frac{\Gamma_1, \Gamma_2}{(\Gamma_1, \Gamma_2), 1}$$

We replace the above instances by the following ones respectively, using (r-1) in different variant if necessary:

$$\frac{\Gamma_2, \Gamma_3}{1, (\Gamma_2, \Gamma_3)} \qquad \frac{\Gamma_1, \Gamma_3}{\Gamma_1, (1, \Gamma_3)} \qquad \frac{\Gamma_1, \Gamma_2}{\Gamma_1, (\Gamma_2, 1)}$$

We consider (r-$\oplus$1). All possible instances with conclusion $(\Gamma_1, \Gamma_2), \Gamma_3$ are the following:

$$(1) \quad \frac{(\Theta_1[B], \Theta_2), \Theta_3 \quad \Delta, A}{(\Theta_1[(\Delta, A \oplus B)], \Theta_2), \Theta_3} \qquad (2) \quad \frac{(\Theta_1, \Theta_2[B]), \Theta_3 \quad \Delta, A}{(\Theta_1, \Theta_2[(\Delta, A \oplus B)]), \Theta_3}$$

$$(3) \quad \frac{(\Theta_1, \Theta_2), \Theta_3[B] \quad \Delta, A}{(\Theta_1, \Theta_2), \Theta_3[(\Delta, A \oplus B)]} \qquad (4) \quad \frac{B, \Theta_3 \quad \Theta_1, A}{(\Theta_1, A \oplus B), \Theta_3}$$

$$(5) \quad \frac{B \quad (\Theta_1, \Theta_2), A}{(\Theta_1, \Theta_2), A \oplus B}$$

(1), (2) and (3) are similar. The active bunch occurs in one of Γ_i. We apply the induction hypothesis to the first premise and use the same rule.

For (4), we use (r-$\oplus$2) with the same premises (interchanged). For (5), we apply the induction hypothesis to the second premise and use (r-$\oplus$2).

We consider (r-$\oplus$2). We have the following instances:

$$\frac{(\Theta_1[A], \Theta_2), \Theta_3 \quad B, \Delta}{(\Theta_1[(A \oplus B, \Delta)], \Theta_2), \Theta_3} \qquad \frac{(\Theta_1, \Theta_2[A]), \Theta_3 \quad B, \Delta}{(\Theta_1, \Theta_2[(A \oplus B, \Delta)]), \Theta_3}$$

$$\frac{(\Theta_1, \Theta_2), \Theta_3[A] \quad B, \Delta}{(\Theta_1, \Theta_2), \Theta_3[(A \oplus B, \Delta)]} \qquad \frac{A, \Theta_3 \quad B, \Theta_2}{(A \oplus B, \Theta_2), \Theta_3}$$

For the first three cases we proceed like for (1)-(3) above. For the last case we use (r-$\oplus$3) with the same premises.[2]

By (r-$\oplus$3) it is not possible to obtain $(\Gamma_1, \Gamma_2), \Gamma_3$.

We consider (r-$\oplus$4). There are three cases:

[2]Here one sees an application of (r-$\oplus$3). (r-$\oplus$4) is used in the skipped part of the proof.

$$\frac{\Theta_2, A \quad \Theta_1, B}{(\Theta_1, \Theta_2), A \oplus B} \qquad \frac{A \quad (\Theta_1, \Theta_2), B}{(\Theta_1, \Theta_2), A \oplus B}$$

$$\frac{(\Theta_1, \Theta_2), A \quad B}{(\Theta_1, \Theta_2), A \oplus B}$$

For the first case we use (r-$\oplus$1) with the same premises (interchanged). For the second case we apply the induction hypothesis to the second premise and use (r-$\oplus$1). For the last case we apply the induction hypothesis to the first premise and use (r-$\oplus$2). □

COROLLARY 2.3. The cut-free NBL and NBL_0 are equivalent, i.e. they have the same theorems.

We can use NBL_0 to prove further properties of NBL.

We need the following rules (called *double negation rules*):

$$(\text{r-}^{\sim\sim}) \quad \frac{A, \Gamma}{\Gamma, A^{\sim\sim}} \qquad (\text{r-}^{--}) \quad \frac{\Gamma, A}{A^{--}, \Gamma}$$

LEMMA 2.4. *The double negation rules are admissible in the cut-free NBL_0.*

PROOF: We prove only the admissibility of (r-$^{\sim\sim}$). The proof for the second rule is similar. We assume $\vdash C, \Theta$ and show $\vdash \Theta, C^{\sim\sim}$. We use outer induction on the number of connectives in C and inner induction on the proof of C, Θ.

Let $C = p^{(n)}$. We run the inner induction. Let $p^{(n)}, \Theta$ be an axiom. Hence $\Theta = p^{(n+1)}$. Then $(\Theta, C^{\sim\sim}) = (p^{(n+1)}, (p^{(n)})^{\sim\sim}) = (p^{(n+1)}, p^{(n+2)})$, which is an axiom, too.

Now we assume that $p^{(n)}, \Theta$ is obtained by a rule. $p^{(n)}$ cannot be the active formula. Then it has to occur in one of the premises. In all but the following cases we just apply the inner induction hypothesis to the premise(s) with $p^{(n)}$ and use the same rule.

We consider the following cases:

$$\frac{B \quad p^{(n)}, A}{p^{(n)}, A \oplus B} \qquad \frac{p^{(n)}, B \quad A}{p^{(n)}, A \oplus B}$$

The first one is an instance of (r-$\oplus$1) or (r-$\oplus$2). We apply the inner induction hypothesis to the premise with $p^{(n)}$ and apply (r-$\oplus$2). The second case is an instance of (r-$\oplus$1). We apply the inner induction hypothesis to the premise with $p^{(n)}$ and use (r-$\oplus$1).

Let $C = 0$. We run the inner induction. Let $0, \Theta$ be an axiom. Then $\Theta = \epsilon$, hence $(\Theta, 0^{\sim\sim}) = 0$, which is an axiom too. Now we assume that $0, \Theta$ is obtained by a rule. Since $C = 0$ cannot the active formula, we proceed as for $p^{(n)}$.

$C = 1$. We run the inner induction. $1, \Theta$ cannot be an axiom. We assume that $1, \Theta$ is the conclusion of a rule. We have $(\Theta, C^{\sim\sim}) = (\Theta, 1)$. If 1 is not the active formula, we proceed as above. Otherwise we have only one rule to consider – (r-1) of the form:

$$\frac{\Theta}{1, \Theta}$$

We just use the other variant of (r-1).

We assume that C is not an atom. We run inner induction. Clearly, C, Θ is not an axiom. So it is obtained by a rule. If C is not the active formula, we proceed as above. Now we assume that C is the active formula.

Let $C = A \otimes B$. Hence $C^{\sim\sim} = A^{\sim\sim} \otimes B^{\sim\sim}$. The only possible rule is (r-$\otimes$) of the form:

$$\frac{(A, B), \Theta}{C, \Theta}$$

We apply Lemma 2.2 to the premise and obtain $A, (B, \Theta)$. A and B each have less connectives than C. By the outer induction hypothesis we get $(B, \Theta), A^{\sim\sim}$. By Lemma 2.2, we get $B, (\Theta, A^{\sim\sim})$, hence by the outer induction hypothesis: $(\Theta, A^{\sim\sim}), B^{\sim\sim}$. Lemma 2.2 yields $\Theta, (A^{\sim\sim}, B^{\sim\sim})$. So $\Theta, C^{\sim\sim}$ arises by (r-$\otimes$).

Let $C = A \oplus B$. Then $C^{\sim\sim} = A^{\sim\sim} \oplus B^{\sim\sim}$. The only possible rules are (r-$\oplus$1) (or (r-$\oplus$2)), (r-$\oplus$2) and (r-$\oplus$3) of the following form:

$$\frac{A \quad B, \Theta}{A \oplus B, \Theta} \qquad \frac{A, \Theta \quad B}{A \oplus B, \Theta} \qquad \frac{A, \Theta_2 \quad B, \Theta_1}{A \oplus B, (\Theta_1, \Theta_2)} \quad (\Theta = (\Theta_1, \Theta_2))$$

For the first case, we apply the outer induction hypothesis to both premises and use (r-$\oplus$1) as below:

$$\frac{A^{\sim\sim} \quad \Theta, B^{\sim\sim}}{\Theta, C^{\sim\sim}}$$

For the second case we apply the outer induction hypothesis to both premises and use (r-$\oplus$2). For the third case we apply the outer induction hypothesis for both premises and use (r-$\oplus$4).

Let $C = A \wedge B$. Then $C^{\sim\sim} = A^{\sim\sim} \wedge B^{\sim\sim}$. We have the following instances if (r-$\wedge$):

$$\frac{A, \Theta}{A \wedge B, \Theta} \qquad \frac{B, \Theta}{A \wedge B, \Theta}$$

In both cases we apply the outer induction hypothesis to the premise and use the same rule.

The last case is $C = A \vee B$. Hence $C^{\sim\sim} = A^{\sim\sim} \vee B^{\sim\sim}$. We have the following instance of (r-$\vee$):

$$\frac{A, \Theta \quad B, \Theta}{A \vee B, \Theta}$$

We apply the outer induction hypothesis to both premises and use the same rule. □

One can easily conclude the following:

Corollary 2.5. $\vdash A^{-}, \Gamma$ if and only if $\vdash \Gamma, A^{\sim}$.

3. Cut elimination

Now we are ready to prove the cut–elimination theorem. The lemmas we have already proved are very useful and with them the proof is much simpler.

Theorem 3.1. *The cut rules are admissible in the cut-free NBL_0 (NBL).*

Proof: We have to show:

(1) if $\vdash \Theta[C]$ and $\vdash \Psi, C^{\sim}$, then $\vdash \Theta[\Psi]$;

(2) if $\vdash \Theta[C]$ and $\vdash C^{-}, \Psi$, then $\vdash \Theta[\Psi]$.

By Corollary 2.5 it suffices to show (1), because (2) follows (1) immediately. As above, $\vdash$ we denote provability in the cut-free NBL_0.

The proof proceeds by the outer induction on the number of connectives in C, the intermediate induction on the proof of $\Theta[C]$ and the inner induction on the proof of $\Psi, C^{\sim}$.

We run the outer induction.

1°. $C = p^{(n)}$. Then $C^{\sim} = p^{(n+1)}$. We run the intermediate induction.

Let $\Theta[C]$ be an axiom. We have two possibilities: $p^{(n)}, p^{(n+1)}$ and $p^{(n-1)}, p^{(n)}$. We run the inner induction.

If $\Psi, C^{\sim}$ is an axiom, then $\Psi = p^{(n)} = C$. Now let $\Psi, C^{\sim}$ be the conclusion of a rule. $C^{\sim}$ cannot be the active formula of any rule. We apply the inner induction hypothesis to the premise(s) with $C^{\sim}$ and use the same rule.

We consider the following special case:

$$\frac{A \quad B, C^{\sim}}{A \oplus B, C^{\sim}},$$

with $\Psi = A \oplus B$. This may be obtained by (r-$\oplus$1) or (r-$\oplus$2). We apply the inner induction hypothesis to the premise $B, C^{\sim}$ and use (r-$\oplus$1).

We assume that $\Theta[C]$ is not an axiom, hence it is obtained by a rule. C cannot be the active formula of any rule. Hence it occurs in at least one premise, so we apply the intermediate induction hypothesis to the premise(s) with C and use the same rule.

2°. $C = 0$. Then $C^{\sim} = 1$. We run the intermediate induction.

Let $\Theta[0]$ be an axiom, then $\Theta[C] = C = 0$ and $\Theta[\Psi] = \Psi$. We run the inner induction. $\Psi, 1$ cannot be an axiom, hence it is obtained by a rule. If $C^{\sim} = 1$ is not the active formula of a rule, we proceed as for $C = p^{(n)}$. If 1 is the active formula, then the rule is (r-1) of the form:

$$\frac{\Psi}{\Psi, 1},$$

The premise is $\Psi = \Theta[\Psi]$.

Now let $\Theta[C]$ be the conclusion of a rule. $C = 0$ cannot be the active formula of any rule. We apply the intermediate induction hypothesis to the premise(s) with $C = 0$ and use the same rule.

3°. $C = 1$. Then $C^{\sim} = 0$. We run the intermediate induction.

$\Theta[1]$ cannot be an axiom, hence it is obtained by a rule. If $C = 1$ is an active formula, then $\Theta[1]$ is obtained by (r-1) admitting $\Delta = \epsilon$ in $\Theta[\Delta]$ as the premise. We run the inner induction. If $\Psi, 0$ is an axiom, then $\Psi = \epsilon$ and $\Theta[\Psi] = \Theta[\epsilon]$. Let $\Psi, 0$ be obtained by a rule. $C^{\sim} = 0$ cannot be the active formula of any rule, so we proceed as for $C = p^{(n)}$.

4°. C is not an atomic formula. We run the intermediate induction.

Since C is not atomic, $\Theta[C]$ cannot be an axiom, hence it has to be the conclusion of a rule. If C is not the active formula, we apply the intermediate induction hypothesis to the premise(s) with C and use the same rule. We assume that C is the active formula.

4.1°. $C = A \otimes B$. So $C^{\sim} = B^{\sim} \oplus A^{\sim}$ and $\Theta[C]$ arises by (r-$\otimes$):

$$\frac{\Theta[(A, B)]}{\Theta[A \otimes B]}$$

We run the inner induction. $\Psi, C^{\sim}$ is not an axiom, hence it is the conclusion of a rule.

In the cases when $C^{\sim}$ does not occur in the active bunch, we apply the inner induction hypothesis to $\Theta[C]$ and the premise(s) with $C^{\sim}$, and use the same rule.

For example:

$$\frac{\Gamma[(D, E)], C^{\sim}}{\Gamma[D \otimes E], C^{\sim}}$$

changes into:

$$\frac{\Theta[\Gamma[(D, E)]]}{\Theta[\Gamma[D \otimes E]]},$$

where $\Psi = \Gamma[D \otimes E]$.

We consider cases when $C^{\sim}$ occurs in the active bunch, but is not the active formula.

$$\frac{D \quad E, C^{\sim}}{D \oplus E, C^{\sim}} \qquad \frac{D, C^{\sim} \quad E}{D \oplus E, C^{\sim}}$$

We apply the inner induction hypothesis to the premise with $C^{\sim}$ and use (r-$\oplus$1).

Let $C^{\sim}$ be the active formula:

$$\frac{\Psi, A^{\sim} \quad B^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi, B^{\sim} \quad A^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi_2, B^{\sim} \quad \Psi_1, A^{\sim}}{(\Psi_1, \Psi_2), C^{\sim}}$$

The first case is obtained by (r-$\oplus$1). We apply the outer induction hypothesis to $\Theta[(A, B)]$ and $\Psi, A^{\sim}$ and then to $\Theta[(\Psi, B)]$ and $B^{\sim}$, obtaining $\Theta[\Psi]$. The second one is obtained by (r-$\oplus$1) or (r-$\oplus$2). We proceed as above: we apply twice the outer induction hypothesis to both premises.

The third case is obtained by (r-$\oplus$4), where $\Psi = (\Psi_1, \Psi_2)$. We apply the outer induction hypothesis twice, obtaining $\Theta[(\Psi_1, \Psi_2)] = \Theta[\Psi]$.

4.2°. $C = A \oplus B$, then $C^{\sim} = B^{\sim} \otimes A^{\sim}$. We have to consider four cases, one for each (r-$\oplus$i).

(1) $\dfrac{\Gamma[B] \quad \Delta, A}{\Gamma[(\Delta, A \oplus B)]}$

We run the inner induction. $\Psi, C^{\sim}$ is not an axiom. We skip cases when $C^{\sim}$ is not the active formula of a rule (in these cases we proceed as above). We consider (r-$\otimes$) as the only possibility:

$$\frac{\Psi, (B^{\sim}, A^{\sim})}{\Psi, C^{\sim}}$$

We apply Lemma 2.2 (admissibility of (r-shift)) to $\Psi, (B^{\sim}, A^{\sim})$, then we apply the outer induction hypothesis to Δ, A and $(\Psi, B^{\sim}), A^{\sim}$ and obtain: $\Delta, (\Psi, B^{\sim})$. By Lemma 2.2 and the outer induction hypothesis applied to $\Theta[B]$ and $(\Delta, \Psi), B^{\sim}$ we obtain $\Gamma[(\Delta, \Psi)] = \Theta[\Psi]$.

(2) $\dfrac{\Gamma[A] \quad B, \Delta}{\Gamma[(A \oplus B, \Delta)]}$

We run the inner induction and consider the same instance as above. We apply Lemma 2.2 to $\Psi, (B^{\sim}, A^{\sim})$, obtaining $(\Psi, B^{\sim}), A^{\sim}$. By Corollary 2.5 we get $A^{-}, (\Psi, B^{\sim})$. We use Lemma 2.2 and apply the outer induction hypothesis to $(A^{-}, \Psi), B^{\sim}$ and B, Δ. We obtain $(A^{-}, \Psi), \Delta$ and apply Lemma 2.2 and Corollary 2.5. We use the outer induction hypothesis with $(\Psi, \Delta), A^{\sim}$ and $\Gamma[A]$, obtaining $\Gamma[(\Psi, \Delta)] = \Theta[\Psi]$.

(3) $\dfrac{A, \Gamma \quad B, \Delta}{A \oplus B, (\Delta, \Gamma)}$

We run the inner induction and consider the same instance as above. We apply Lemma 2.2 to $\Psi, (B^{\sim}, A^{\sim})$ and obtain $(\Psi, B^{\sim}), A^{\sim}$. We apply Corollary 2.5 and get $A^{-}, (\Psi, B^{\sim})$. We use Lemma 2.2 and apply the outer induction hypothesis to $(A^{-}, \Psi), B^{\sim}$ and B, Δ. We have $(A^{-}, \Psi), \Delta$. We apply Lemma 2.2 and Corollary 2.5. We use the outer induction hypothesis to $(\Psi, \Delta), A^{\sim}$ and A, Γ, obtaining $(\Psi, \Delta), \Gamma$. We use Lemma 2.2.

(4) $\dfrac{\Gamma, A \quad \Delta, B}{(\Delta, \Gamma), A \oplus B}$

We run the inner induction and consider the same instance as above. We apply Lemma 2.2 to $\Psi, (B^{\sim}, A^{\sim})$, obtaining $(\Psi, B^{\sim}), A^{\sim}$. We apply the outer induction hypothesis to $(\Psi, B^{\sim}), A^{\sim}$ and Γ, A. We get $\Gamma, (\Psi, B^{\sim})$. We use Lemma 2.2 and apply the outer induction hypothesis to $(\Gamma, \Psi), B^{\sim}$ and Δ, B. We obtain $\Delta, (\Gamma, \Psi)$ and use Lemma 2.2.

4.3°. $C = A \wedge B$. So $C^{\sim} = A^{\sim} \vee B^{\sim}$. We have the following instances:

$$\frac{\Theta[A]}{\Theta[C]} \qquad \frac{\Theta[B]}{\Theta[C]}$$

We run the inner induction. $\Psi, C^{\sim}$ is not an axiom. We skip the cases with $C^{\sim}$ not being the active formula. We have only one possibility:

$$\frac{\Psi, A^{\sim} \quad \Psi, B^{\sim}}{\Psi, C^{\sim}}$$

We apply the outer induction hypothesis to $\Theta[A]$ and $\Psi, A^{\sim}$ or to $\Theta[B]$ and $\Psi, B^{\sim}$, depending on the proof of $\Theta[C]$. In both cases we obtain $\Theta[\Psi]$.

4.4°. $C = A \vee B$. So $C^{\sim} = A^{\sim} \wedge B^{\sim}$. We have the following case:

$$\frac{\Theta[A] \quad \Theta[B]}{\Theta[C]}$$

We run the inner induction. $\Psi, C^{\sim}$ cannot be an axiom. We consider only the cases with $C^{\sim}$ as the active formula:

$$\frac{\Psi, A^{\sim}}{\Psi, C^{\sim}} \qquad \frac{\Psi, B^{\sim}}{\Psi, C^{\sim}}$$

In the first case we apply the outer induction hypothesis to $\Theta[A]$ and $\Psi, A^{\sim}$ and in the second case to $\Theta[B]$ and $\Psi, B^{\sim}$. □

One can easily prove the strong completness of NBL (with the cut rules) with respect to l.o. involutive unital groupoids. Let X be any set of bunches. We say that formulas A, B are equivalent ($A \simeq B$) if and only if $X \vdash A, B^{\sim}$ and $X \vdash A^{-}, B$. It is easy to check that $\simeq$ is a congruence. The quotient algebra is l.o. involutive unital groupoid. We define $\mu(p^{(0)}) = [p]_{\simeq}$, $\mu(p^{(n+1)}) = \mu(p^{(n)})^{\sim}$ for $n \geq 0$ and $\mu(p^{(n-1)}) = \mu(p^{(n)})^{-}$ for $n \leq 0$. One proves $\mu(A) = [A]_{\simeq}$, hence $\mu((\Gamma, \Delta)) = [\Gamma]_{\simeq} \otimes [\Delta]_{\simeq}$. If $X \not\vdash \Gamma$, then $\mu(\Gamma) \not\leq 0$.

4. Other systems

4.1. The additive constants

The presented Nonassociative Bilinear Logic admits only multiplicative constants. We can extend this system by additive constants $\top$ and $\bot$. In the sense of algebraic models, they are the greatest and the smallest elements in the l.o. unital groupoids, respectively, i.e. for all a:

$$\bot \leq a, \qquad a \leq \top$$

In particular $\bot \leq 0$, hence it should be a theorem.

We extend NBL-language by two constants: $\top$ and $\bot$. We also add an axiom:

$$\text{(a-}\bot\text{)} \quad \Gamma[\bot]$$

which is valid because one proves $a \otimes \bot = \bot$ and $\bot \otimes a = \bot$.

This is the only axiom we add. We do not extend NBL with any new rules. It is interesting that $\top$ does not appear explicitly in any axiom nor any rule, but it is still in the language.

In the metalanguage we also add the following:

$$\bot^{\sim} = \bot^{-} = \top, \qquad \top^{\sim} = \top^{-} = \bot$$

All results presented in this paper are also true for NBL with these constants. $\top$ and $\bot$ cannot be the active formulas of any rule, so the presented reasoning remains valid. In every proof we need to add an additional case for the new axiom.

In the proof of Lemma 2.2 we consider the case when $(\Gamma_1, \Gamma_2), \Gamma_3$ is an instance of (a-$\bot$). In this case $\bot$ occurs in one of Γ_i. Then $\Gamma_1, (\Gamma_2, \Gamma_3)$ is an instance of (r-$\bot$), too.

In the proof of Lemma 2.4 we have the outer induction and the inner induction. In the outer one we consider the case when $C = \bot$ or $C = \top$. Neither can be the active formula of a rule, hence we proceed as in similar cases for other atomic formulas. In the inner induction we have an additional axiom to consider. We assume that $A, \Gamma[\bot]$ is an axiom, hence $\Gamma[\bot], A^{\sim\sim}$ is an axiom, too. Similarly if $\Gamma[\bot], A$ is an axiom, then $A^{--}, \Gamma[\bot]$ is an axiom.

Our main result is the cut–elimination theorem. In its proof we use three inductions: the outer, the intermediate and the inner. We consider three cases.

1°. $C = \bot$. Then $C^{\sim} = \top$. We run the intermediate induction. Let $\Theta[\bot]$ be an axiom. We run the inner induction. If $\Psi, \top$ is an axiom, then $\Theta[\Psi]$ is also an axiom.

If $\Theta[\bot]$ or $\Psi, \top$ is not an axiom, we proceed as for $C = p^{(n)}$.

2°. $C = \top$. Then $C^{\sim} = \bot$. We run the intermediate induction. If $\Theta[\top]$ is an axiom, then $\Theta[\Psi]$ is an axiom, too. Let $\Theta[\top]$ be obtained by a rule. $\top$ is not the active formula of any rule. We apply the intermediate induction hypothesis to the premise(s) with $C = \top$ and use the same rule.

3°. $C \neq \bot$ and $C \neq \top$. We notice that if $\Theta[C]$ is an instance of (a-$\bot$), then $\Theta[\Psi]$ is an axiom. Also, if $\Psi, C^{\sim}$ is an axiom, then $\Theta[\Psi]$ is an axiom, too.

Corollary 4.1. NBL with the additive constants is a conservative extension of NBL.

4.2. The right-sided system

The presented system is left-sided, but we can consider right-sided and two-sided systems of that logic. A two-sided system for NBL was considered in [6]. It is denoted InGL – Involutive Groupoid Logic. InGL treats negations as connectives, so the logic is much more complex than our system. It is also non-standard, because in the language there is no coproduct and the right side of the sequent serves only for technical purposes.

Our system for NBL can be easily translated into right-sided system. It is dual in the sense that product and coproduct are exchanged, similary meet and join or 1 and 0. All the results proved here can be translated into the right-sided system, remaining true.

The language and models remain the same as for the left-sided system. We modify the definition of valuation. Now $\mu((\Gamma, \Delta)) = \mu(\Gamma) \oplus \mu(\Delta)$ and $\mu(\epsilon) = 0$. We say that the sequent Γ is true in $\mathbf{M}$ for valuation μ, if $1 \leq \mu(\Gamma)$.

We admit axioms:

$$\text{(a*-id)} \qquad p^{(n+1)}, p^{(n)} \quad \text{for any variable } p \text{ and any } n \in \mathbb{Z}$$

$$\text{(a*-1)} \qquad 1$$

The rules are:

$$\text{(r*-}\oplus\text{)} \quad \frac{\Gamma[(A, B)]}{\Gamma[A \oplus B]}$$

$$\text{(r*-}\otimes 1\text{)} \quad \frac{\Gamma[B] \quad \Delta, A}{\Gamma[(\Delta, A \otimes B)]} \qquad \text{(r*-}\otimes 2\text{)} \quad \frac{\Gamma[A] \quad B, \Delta}{\Gamma[(A \otimes B, \Delta)]}$$

$$\text{(r*-0)} \quad \frac{\Gamma[\Delta]}{\Gamma[(0, \Delta)]} \quad \frac{\Gamma[\Delta]}{\Gamma[(\Delta, 0)]}$$

$$\text{(r*-}\vee\text{)} \quad \frac{\Gamma[A]}{\Gamma[A \vee B]} \quad \frac{\Gamma[A]}{\Gamma[B \vee A]} \qquad \text{(r*-}\wedge\text{)} \quad \frac{\Gamma[A] \quad \Gamma[B]}{\Gamma[A \wedge B]}$$

$$\text{(r*-shift)} \quad \frac{(\Gamma, \Delta), \Theta}{\Gamma, (\Delta, \Theta)}$$

The cut rules obtain the form:

$$\text{(cut*}^{\sim}\text{)} \quad \frac{\Gamma[A] \quad A^{\sim}, \Delta}{\Gamma[\Delta]} \qquad \text{(cut*}^{-}\text{)} \quad \frac{\Gamma[A] \quad \Delta, A^{-}}{\Gamma[\Delta]}$$

Now we show the way of translating the left-sided system for NBL to the right-sided system. We extend the metalanguage negations $^{\sim}, ^{-}$ to bunches by $(\Gamma, \Delta)^{\sim} = (\Delta^{\sim}, \Gamma^{\sim})$, $\epsilon^{\sim} = \epsilon$ and similarly for $^{-}$. Clearly $(\Gamma^{\sim})^{-} = \Gamma = (\Gamma^{-})^{\sim}$. We also extend these negations for contexts by setting: $x^{\sim} = x^{-} = x$. We obtain $\Gamma[\Delta]^{\sim} = \Gamma^{\sim}[\Delta^{\sim}]$ and similarly for $^{-}$.

LEMMA 4.2. *The sequent Θ is provable in the left-sided system if and only if $\Theta^{\sim}$ (resp. Θ^{-}) is provable in the right-sided system.*

THEOREM 4.3. *The cut rules (cut*$^{\sim}$) and (cut*$^{-}$) are admissible in the cut-free right-sided system for NBL.*

We prove the theorem for (cut$^{*\sim}$). The proof for (cut^{*-}) is similar.

PROOF: Let $\vdash_L, \vdash_R$ denote the provability in the left-sided system and in the right-sided system, respectively. Assume $\vdash_R \Gamma[A]$ and $\vdash_L A^{\sim}, \Delta$. By Lemma 4.2, $\vdash_L \Gamma^{\sim}[A^{\sim}]$ and $\vdash_L \Delta^{\sim}, A^{\sim\sim}$. By Theorem 1, $\vdash_L \Gamma^{\sim}[\Delta^{\sim}]$. So $\vdash_R (\Gamma^{\sim}[\Delta^{\sim}])^{-}$, which yields $\vdash_R \Gamma[\Delta]$. □

We can also extend the right-sided system with the additive constants $\bot$ and $\top$. We add the axiom (a*-$\top$) $\Gamma[\top]$ and we define $\bot^{\sim} = \top = \bot^{-}$ and $\top^{\sim} = \bot = \top^{-}$. One proves the lemma above for that extended system.

Since NBL is a conservative extension of NL1 (Nonassociative Lambek Calculus with 1), InNL1 (Involutive Nonassociative Lambek Calculus with 1, i.e. the multiplicative fragment of NBL) and FNL1 (Full Nonassociative Lambek Calculus with 1), all the results remain true for these weaker logics.

5. PTime complexity

In this section we prove the PTime complexity of the multiplicative fragment of NBL, i.e. MNBL. This system is denoted InNL1 in [4], which proves the PTime complexity of InNL and claims the same for InNL1. We provide a proof.

By MNBL we mean NBL without $\wedge, \vee, \bot$ and $\top$ in the language and without the corresponding axioms and rules: (a-$\bot$), (r-$\wedge$), (r-$\vee$) (resp. (a*-$\top$), (r*-$\wedge$), (r*-$\vee$) for right-sided system). All results proved before remain true for MNBL, because NBL is a conservative extension. We focus on the left-sided system. Since we consider only the multiplicative fragment of NBL, we define MNBL_0 as NBL_0 without additive connectives and constants.

DEFINITION 5.1. Let T be a set of formulas. Any sequent built from formulas of T is called *T–sequent*. A *T–proof* is a proof consisting only of T–sequents.

In NBL-language we do not treat the negations as connectives, but all formulas of the form $p^{(n)}$ are atoms. Hence p is not a subformula of $p^{\sim}$ or p^{-} etc. By theorem 3.1 one obtains the following corollary:

Corollary 5.2. For every sequent Γ, if Γ is provable in cut-free NBL, then it has T–proof, where T is the subformula closure of the set of all formulas in Γ.

The above corollary is called *subformula property*. Because NBL is a conservative extension of MNBL, MNBL possess the subformula property. Hence we can consider only T–proofs for any sequent Γ, where T is a set of all subformulas of formulas in Γ. In order to prove the PTime complexity we consider restricted sequents. A sequent is called *restricted* if it consists of at most three formulas. The restricted sequents are of the form: $A; (A, B); (A, (B, C)); ((A, B), C)$.

We define $c(T) = T \cup T^{\sim} \cup T^{-}$, where $T^{\sim} = \{A^{\sim} : A \in T\}, T^{-} = \{A^{-} : A \in T\}$. Now let T be any subformula closed set of formulas, containing 0 and 1.

By MNBL_1^T we mean a new system, defined as follows. The axioms are 0 and all sequents A^{-}, A and $A, A^{\sim}$ for $A \in T$. The inference rules are all rules of MNBL_0 limited to restricted $c(T)$–sequents with the active formula in T and the cut rules (cut$^{\sim}$), (cut^{-}) limited to $c(T)$–sequents. We assume that $\Delta \neq \epsilon$ in the cut rules. Notice that we do not limit cut rules to restricted sequents.

Since T is fixed, we write MNBL_1 for MNBL_1^T. The provability in MNBL_1 is denoted by $\vdash_1$. The system MNBL_1 posseses an interpolation property.

Lemma 5.3. *If $\vdash_1 \Theta[\Psi]$, $\Theta[\Psi] \neq \Psi$ and $\Psi \neq \epsilon$, then there exists $D \in c(T)$ such that $\vdash_1 \Theta[D]$ and either $\vdash_1 D^{-}, \Psi$ or $\vdash_1 \Psi, D^{\sim}$.*

Proof: We proceed by induction on proofs of $\Theta[\Psi]$ in MNBL_1.

Let Ψ be a formula. We put $D = \Psi$. Clearly $\vdash_1 \Theta[D]$. If $\Psi \in T$, then $\Psi, \Psi^{\sim}$ is an axiom. If $\Psi \in T^{-}$, then $\Psi^{\sim} \in T$, hence $\Psi^{\sim -}, \Psi^{\sim}$ is an axiom. The case for $\Psi \in T^{\sim}$ is analogous.

We assume Ψ is not a formula. $\Theta[\Psi]$ cannot be an axiom. We consider a case for each rule of MNBL_1.

(r-$\otimes$). The only possibilities are:

$$\frac{A, B}{A \otimes B} \qquad \frac{A, (B, C)}{A, B \otimes C} \qquad \frac{(A, B), C}{A \otimes B, C}$$

In all cases all bunches properly contained in the conclusion are formulas.

(r-$\oplus$1). We have the following cases:

(1) $\dfrac{B \quad A}{A \oplus B}$ (2) $\dfrac{B \quad C_1, A}{C_1, A \oplus B}$ (3) $\dfrac{C_1, B \quad A}{C_1, A \oplus B}$

(4) $\dfrac{B, C_1 \quad A}{A \oplus B, C_1}$ (5) $\dfrac{B \quad (C_1, C_2), A}{(C_1, C_2), A \oplus B}$ (6) $\dfrac{C_1, B \quad C_2, A}{C_1, (C_2, A \oplus B)}$

(7) $\dfrac{B, C_1 \quad C_2, A}{(C_2, A \oplus B), C_1}$ (8) $\dfrac{(C_1, C_2), B \quad A}{(C_1, C_2), A \oplus B}$ (9) $\dfrac{C_1, (C_2, B) \quad A}{C_1, (C_2, A \oplus B)}$

(10) $\dfrac{(C_1, B), C_2 \quad A}{(C_1, A \oplus B), C_2}$ (11) $\dfrac{C_1, (B, C_2) \quad A}{C_1, (A \oplus B, C_2)}$ (12) $\dfrac{(B, C_1), C_2 \quad A}{(A \oplus B, C_1), C_2}$

(13) $\dfrac{B, (C_1, C_2) \quad A}{A \oplus B, (C_1, C_2)}$

In cases (1)–(4) all bunches properly contained in the conslusion are formulas.

We consider (5). $\Psi = (C_1, C_2)$. We put $D = A^-$, hence $\Psi, D^\sim$ equals the second premise. Since $A \in T$, A^-, A is an axiom. By (r-$\oplus$1) we obtain $A^-, A \oplus B (= \Theta[D])$ from this axiom and the first premise.

In (6) and (7) $\Psi = (C_2, A \oplus B)$. We put $D = B$, so $\Theta[D]$ equals the first premise. We use (r-$\oplus$1) to obtain $\Psi, D^\sim$ from the second premise and the axiom $B, B^\sim$, since $B \in T$.

In (8) $\Psi = (C_1, C_2)$. We put $D = B^-$ and proceed in the similar way as in (5).

We consider (9). Here $\Psi = (C_2, A \oplus B)$. By the induction hypothesis there is a formula E, such that $\vdash_1 C_1, E$ and $\vdash_1 E^-, (C_2, B)$ or $\vdash_1 (C_2, B), E^\sim$. We put $D = E$. By (r-$\oplus$1) with the second premise A we obtain $E^-, (C_2, A \oplus B)$ or $(C_2, A \oplus B), E^\sim$. Note that $\Theta[D] = (C_1, E)$.

In (10)–(12) we proceed analogously as in (9). In (13) we proceed as in (8).

(r-$\oplus$2). The cases are symmetrical and the arguments are similar to those of (r-$\oplus$1).

(r-$\oplus$3). We have only one possibility:

$$\frac{A, C_1 \quad B, C_2}{A \oplus B, (C_2, C_1)}$$

Hence $\Psi = (C_2, C_1)$. Since $A \oplus B \in T$, then $A \oplus B, (A \oplus B)^\sim$ is an axiom. We put $D = (A \oplus B)^\sim$, hence $\Theta[D]$ is the axiom and D^-, Ψ is the conclusion.

(r-$\oplus$4). We have only one possibility:

$$\frac{C_1, A \quad C_2, B}{(C_2, C_1), A \oplus B}$$

Hence $\Psi = (C_2, C_1)$. Since $A \oplus B \in T$, so $(A \oplus B)^-, A \oplus B$ is an axiom. We put $D = (A \oplus B)^-$, hence $\Theta[D]$ is this axiom and $\Psi, D^\sim$ is the conclusion.

(r-1). We consider the following cases:

$$(1)\ \frac{C}{1, C} \qquad (2)\ \frac{C}{C, 1} \qquad (3)\ \frac{C_1, C_2}{1, (C_1, C_2)}$$

$$(4)\ \frac{C_1, C_2}{(1, C_1), C_2} \qquad (5)\ \frac{C_1, C_2}{C_1, (1, C_2)} \qquad (6)\ \frac{C_1, C_2}{(C_1, 1), C_2}$$

$$(7)\ \frac{C_1, C_2}{(C_1, C_2), 1} \qquad (8)\ \frac{C_1, C_2}{C_1, (C_2, 1)}$$

In the first two cases conclusions have only formulas as properly contained bunches.

In (3) and (7) we have $\Psi = (C_1, C_2)$. We put $D = 0$, hence $\Theta[D]$ can be obtained from the axiom 0 by (r-1) and D^-, Ψ is the conclusion.

In (4)–(6) we put $D = C_1$, hence $\Theta[D]$ is the premise. If $C_1 \in T$, then $C_1, C_1^\sim$ is an axiom; if $C_1 \in T^-$, then $C_1, C_1^\sim$ is an axiom; if $C_1 \in T^\sim$, then C_1^-, C_1 is an axiom. We apply (r-1) to one of those axiom (depending on C_1) and we obtain $\Psi, D^\sim$ or D^-, Ψ.

In (8) we put $D = C_2$ and proceed analogously.

(cut$^\sim$).

$$\frac{\Gamma[A] \quad \Delta, A^\sim}{\Gamma[\Delta]}$$

We have $\Theta[\Psi] = \Gamma[\Delta]$. If Ψ occurs in $\Gamma[\ ]$, then $\Gamma[A] = \Xi[\Psi][A]$ and, by the induction hypothesis, there exists E, such that $\vdash_1 \Xi[E][A]$ and $\vdash_1 E^-, \Psi$ or $\vdash_1 \Psi, E^\sim$. We put $D = E$, then $\Theta[D] = \Xi[D][\Delta]$, which we obtain by (cut$^\sim$) from $\Xi[D][A]$ and $\Delta, A^\sim$.

If Ψ occurs in Δ, the we use the induction hypothesis for $\Delta, A^{\sim}$ and proceed as above.

Now let $\Gamma[\Delta] = \Gamma_1[\Gamma_2[\Delta]]$ and $\Psi = \Gamma_2[\Delta]$ and $\Psi \neq \Delta$. Hence $\Gamma[A] = \Gamma_1[\Gamma_2[A]]$ and $\Gamma_2[A] \neq \Gamma[A]$. We use the induction hypothesis for $\Gamma_2[A]$ in $\Gamma_1[\Gamma_2[A]]$ to obtain D. We have $\vdash_1 \Gamma_1[D]$ and $\vdash_1 D^{-}, \Gamma_2[D]$ or $\vdash_1 \Gamma_2[D], D^{\sim}$. We obtain $\vdash_1 D^{-}, \Gamma_2[\Delta]$ or $\vdash_1 \Gamma_2[\Delta], D^{\sim}$ by (cut$^{\sim}$) applied with $\Delta, D^{\sim}$.

(cut^{-}). We proceed analogously as for (cut$^{\sim}$). □

For a sequent Γ we take T as the subformula closure of the set of all formulas appearing in Γ, also containing 0 and 1. We define MNBL_1^T as above.

Lemma 5.4. *Γ is provable in MNBL if and only if $\vdash_1 \Gamma$.*

Proof: All axioms of MNBL_1 are provable in MNBL, also all rules are valid in MNBL, since they are the instances of original MNBL rules or are admissible. Hence if $\vdash_1 \Gamma$, then $\vdash \Gamma$ in MNBL.

Now assume that Γ is provable in MNBL. We show that it is provable in MNBL_1. By Corollary 2.3, Γ is provable in MNBL_0. Also, because of the subformula property, Γ has a T–proof in MNBL_0. It suffices to show that all T–axioms (axioms which are T–sequents) of MNBL_0 are provable in MNBL_1 and all rules of MNBL_0 limited to T–sequents are admissible in MNBL_1.

If $p^{(n)} \in T$, then $p^{(n)}, p^{(n+1)}$ is the axiom $A, A^{\sim}$ of MNBL_1. Also 0 is an axiom of MNBL_1.

(r-$\otimes$). We assume that $\vdash_1 \Gamma[(A, B)]$ and $\Gamma[A \otimes B]$ is a T–sequent. If $\Gamma[(A, B)] = (A, B)$, then $\vdash_1 \Gamma[A \otimes B]$ ($\Gamma[A \otimes B] = A \otimes B$). We assume $\Gamma[(A, B)] \neq (A, B)$. By Lemma 5.3, there exists $D \in c(T)$, such that $\vdash_1 \Gamma[D]$ and $\vdash_1 D^{-}, (A, B)$ or $\vdash_1 (A, B), D^{\sim}$. We apply (r-$\otimes$) (in MNBL_1) and obtain $D^{-}, A \otimes B$ or $A \otimes B, D^{\sim}$. And by one of the cut rules: $\vdash_1 \Gamma[A \otimes B]$.

(r-$\oplus$1). We assume that $\vdash_1 \Gamma[B]$ and $\vdash_1 \Delta, A$ and $\Gamma[(\Delta, A \oplus B)]$ is a T–sequent. We consider two cases:

$\Delta = \epsilon$. From A and the axiom $B, B^{\sim}$ we obtain $A \oplus B, B^{\sim}$ by (r-$\oplus$1) in MNBL_1. By (cut$^{\sim}$) we get $\Gamma[A \oplus B]$.

$\Delta \neq \epsilon$. By Lemma 5.3 we have $D \in c(T)$, such that $\vdash_1 D, A$ and $\vdash_1 D^{-}, \Delta$ or $\vdash_1 \Delta, D^{\sim}$. From D, A and the axiom $B, B^{\sim}$ we obtain $(D, A \oplus$

$B), B^{\sim}$ by (r-$\oplus$1) in MNBL_1. By one of the cut rules we obtain $(\Delta, A \oplus B), B^{\sim}$ and by (cut$^{\sim}$) we get $\Gamma[(\Delta, A \oplus B)]$.

(r-$\oplus$2). The argument is similar as for (r-$\oplus$1).

(r-$\oplus$3). We assume that $\vdash_1 A, \Gamma$ and $\vdash_1 B, \Delta$ and $A \oplus B, (\Delta, \Gamma)$ is a T–sequent. We apply Lemma 5.3 twice, obtaining D_1 for Γ in A, Γ and D_2 for Δ in B, Δ. We have $\vdash_1 A, D_1$ and $\vdash_1 B, D_2$. We use (r-$\oplus$3) in MNBL_1 and get $A \oplus B, (D_2, D_1)$. We apply appropriate cut rules for both D_1 and D_2 and get $A \oplus B, (\Delta, \Gamma)$.

(r-$\oplus$4). The argument is analogous to that for (r-$\oplus$3).

(r-1). We assume that $\vdash_1 \Gamma[\Delta]$ and $\Gamma[(1, \Delta)]$ is a T–sequent. By Lemma 5.3 there is $D \in c(T)$, such that $\vdash_1 \Gamma[D]$ and $\vdash_1 D^{-}, \Delta$ or $\vdash_1 \Delta, D^{\sim}$. We assume that $\vdash D^{-}, \Delta$. The other case is analogous. By Lemma 5.3 we have $E \in c(T)$, such that $\vdash_1 D^{-}, E$ and $\vdash_1 E^{-}, \Delta$ or $\vdash_1 \Delta, E^{\sim}$. We apply (r-1) to D^{-}, E and get $D^{-}, (1, E)$. Now we use one of the cut rules and obtain $D^{-}, (1, \Delta)$. We use (cut^{-}) and obtain $\Gamma[(1, \Delta)]$. The argument for the other variant of (r-1) is the same. □

We notice that in MNBL_1 if the conclusion is restricted, then the premises are also restricted. Hence every restricted sequent Γ provable in MNBL has a proof in MNBL_1^T, where T is defined above.

For every sequent Γ we define $f(\Gamma)$ as follows: $f(A) = A$, if A is a formula and $f((\Gamma_1, \Gamma_2)) = f(\Gamma_1) \otimes f(\Gamma_2)$. It is clear, that $\vdash \Gamma$ if and only if $\vdash f(\Gamma)$. We see that $f(\Gamma)$ is a restricted sequent.

Let Γ be a restricted sequent. We define *the size of* Γ as follows: $s(p^{(n)}) = |n| + 1, s(0) = s(1) = 1, s(A \otimes B) = s(A) + s(B) + 1, s(A \oplus B) = s(A) + s(B) + 1, s((\Gamma_1, \Gamma_2)) = s(\Gamma_1) + s(\Gamma_2)$. By $|n|$ we can take either the absolute value of n or the length of its binary representation.

We provide an algorithm verifying the provability of this sequent. If we put $n = s(\Gamma)$, the complexity is polynomial with respect to n.

First, we compute T in $O(n^2)$ time and then $c(T)$ in $O(n)$ time. Notice that $c(T)$ has at most $3n$ elements, and hence there are $O(n^3)$ restricted $c(T)$–sequents.

Now we compute the list of all provable sequents of MNBL_1. We put $\Gamma_0 = 0, \Gamma_1, \Gamma_2, \ldots, \Gamma_k$, being the sequence of all axioms of MNBL_1.

We iterate over $i = 1, 2, \ldots$. For every i we extend the list with the new sequents, being the conclusions of of (r-$\oplus$1), (r-$\oplus$2), (r-$\oplus$3), (r-$\oplus$4) and the cut rules with the premises Γ_i and Γ_j, if applicable, where $j < i$. We do not add the sequents, which are already in the list. We also apply (r-$\otimes$) and (r-1) to every sequent, if applicable, and extend the list with the conclusions. Since there are $O(n^3)$ restricted $c(T)$–sequents, the procedure always stops. Assuming that one rule is executed in time $O(n)$, we have the time $O(ni)$ for every i.

The rough estimation of the complexity is $O(n^7)$.

THEOREM 5.5. *MNBL is* PTIME*.*

If we add the additive connectives $\wedge, \vee$ to MNBL, we obtain NBL, which is PSPACE. We don't know the lower bound of complexity. Pentus [11] proves that MBL (associative MNBL) is NP-complete. BL is PSPACE-complete. Since MNBL is a conservative extension of NL1 (Nonassociative Lambek Calculus with 1), theorem 5.5 remains true for NL1.

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Paweł Płaczek
Adam Mickiewicz University
Faculty of Mathematics and Computer Science
Uniwersytetu Poznańskiego 4
61-614 Poznań, Poland
e-mail: pawel.placzek@amu.edu.pl