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Majority Vote Model on Multiplex Networks

A. Krawiecki*, T. Gradowski and G. Siudem

Faculty of Physics, Warsaw University of Technology, Koszykowa 75, PL-00662 Warsaw, Poland

Majority vote model on multiplex networks with two independently generated layers in the form of scale-free networks is investigated by means of Monte Carlo simulations and heterogeneous mean-field approximation. In a version of the model under study each agent with probability $1-q$ ($0 \le q \le 1/2$) follows the opinions of the majorities of her neighbors within both layers if these opinions are identical; otherwise, she makes decision randomly. The model exhibits second-order ferromagnetic transition as q, the parameter measuring the level of internal noise, is decreased, with critical exponents depending on the details of the degree distributions in the layers. The critical value q_c of the parameter q evaluated in the heterogeneous mean-field approximation shows quantitative agreement with that obtained from numerical simulations for a broad range of parameters characterizing the degree distributions of the layers.

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1. Introduction

The majority vote (MV) model [1] (for review see [2]) is a stochastic model for the opinion formation devised as a nonequilibrium version of the Ising model. In its most popular version agents update their opinions at discrete time steps following the opinion of the majority of their neighbors with certain probability $1-q$, where the parameter q, $0 \le q \le 1/2$, controls the degree of internal noise in the system dynamics. The MV model on regular two- [1, 3] and three-dimensional [4] lattices was shown to exhibit a second-order ferromagnetic (FM) phase transition with decreasing q, with the critical exponents belonging to the universality class of the corresponding Ising model; it was also shown to follow the mean-field (MF) critical behavior above a finite upper critical dimension [5]. The MV model was also studied on various complex networks [6–14], in particular on scale-free (SF) networks [9–11] which reflect heterogeneity of human social and economic interactions [15, 16]. In these cases, in general, the MV model turned out to belong to a different universality class than the corresponding Ising model. Recently it has been recognized that even more complex and heterogeneous structures are typical of social systems which has prompted interest in the study of interacting systems on "networks of networks" [17]. In this context much attention has been devoted to multiplex networks (MNs) which consist of a fixed set of nodes connected by various sets of edges called layers [17–19]. MNs naturally emerge in various social systems (e.g., transportation or communications networks), and interacting systems on MNs exhibit rich variety of collective behaviors and critical phenomena. For example, percolation transition [20–23], cascading failures [24], threshold cascades [25, 26], diffusion processes [27, 28], epidemic spreading [29, 30], opinion formation [31, 32], FM transition in the Ising model [33], etc., were studied on MNs.

In this paper the MV model is investigated on MNs which consist of a fixed set of agents interacting via independently generated heterogeneous SF layers corresponding to different communications channels. In this way the related study of the Ising model on MNs [33] is extended to the nonequilibrium case. In the version of the MV model under study the agents are assumed to obey the update rule according to which if the majorities of the agent's neighbors within each layer share the same opinion the agent follows it with probability $1-q$; otherwise, she makes decision randomly. Conceptually, this update rule can be treated as a variant of the AND dynamics for agents on MNs [26, 31, 32] according to which the agent is more inclined to change her opinion if information received via all communications channels suggest change. It is demonstrated via Monte Carlo (MC) simulations that this model exhibits second-order FM transition with decreasing q, and critical exponents for this transition are calculated using the finite size scaling (FSS) method. Besides, critical value of the internal noise level is obtained analytically in the heterogeneous mean-field (MF) approximation and shown to exhibit quantitative agreement with results of MC simulations.

2. The model

MNs consist of a fixed set of nodes connected by several sets of edges; the set of nodes with each set of edges forms a network which is called a layer of a MN [18, 19]. In the following, for simplicity, MNs with N nodes and only two layers denoted as $G^{(A)}$, $G^{(B)}$ are considered (so-called duplex networks). It is assumed that the layers (strictly speaking, the sets of edges within each layer) are generated independently. As a result, multiple connections between nodes are not allowed within the same layer, but the same nodes can be connected by multiple edges belonging to different layers. The nodes $i = 1, 2, \ldots N$ are characterized by their degrees $k_i^{(A)}$, $k_i^{(B)}$ within each layer, i.e., the number of edges attached to them within each layer. The distributions of the degrees of nodes

*corresponding author

within each layer are denoted as $p_{k^{(A)}}$, $p_{k^{(B)}}$ and their mean values as $\langle k^{(A)}\rangle$, $\langle k^{(B)}\rangle$; since the layers are generated independently the joint distribution of the degrees of nodes in the MN is $p_{k^{(A)},k^{(B)}} = p_{k^{(A)}}p_{k^{(B)}}$. In this paper only fully overlapping MNs are considered, with all N nodes belonging to both layers; generalization of the results to the case of partly overlapping MNs, with only a fraction of nodes belonging to both layers, is straightforward but omitted for the sake of brevity. The process of generation of a MN with two layers is illustrated in Fig. 1.

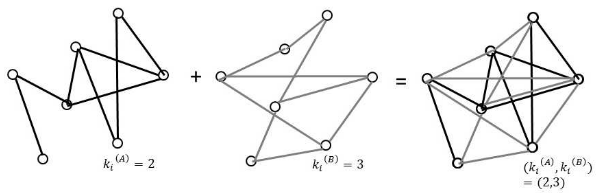

Fig. 1. Generation of a MN with two layers. The set of nodes is the same for both layers. The layers are generated independently by connecting the nodes with edges, black lines correspond to the edges of the layer $G^{(A)}$ and gray lines to the edges of the layer $G^{(B)}$. The nodes $i = 1, 2 \dots N$ within layers are characterized by the degrees $k_i^{(A)}$, $k_i^{(B)}$. Finally, both layers are superimposed by identifying the corresponding nodes to form a MN.

The MV model considered in this paper consists of agents represented by two-state spins $s_i = \pm 1$, $i = 1, 2, \dots N$ located in the nodes of a MN. The agents exchange information via edges of the layers and update their opinions (orientations) according to the opinions of the majorities of their neighbors within consecutive layers. Different layers represent different communications channels used by the agents (e.g., the Internet, e-mail, social interactions with friends, family members, co-workers) with possibly different topologies of interactions. Since the set of nodes is common for all layers the agents present the same opinion to their neighbors within each layer, i.e., they are non-schizophrenic. The probability of the agent's opinion flip per unit time (rate) is

$$w_i(s_i) = \frac{1}{2}\left[1 - (1-2q)s_i \mathrm{sgn}_i\right], \tag{1}$$

where $0 < q < 1/2$ is the internal noise parameter,

$$\mathrm{sgn}_i = \mathrm{sgn}\left(\sum_{j:(i,j)\in G^{(A)}} s_j, \sum_{j':(i,j')\in G^{(B)}} s_{j'}\right), \tag{2}$$

where the summations run over the nearest neighbors of the node i within the layers $G^{(A)}$, $G^{(B)}$ and

$$\mathrm{sgn}(x,y) = \begin{cases} -1 & \text{for} \quad x<0, y<0 \\ +1 & \text{for} \quad x>0, y>0, \\ \ \ 0 & \text{else,} \end{cases} \tag{3}$$

denotes the signum function with two arguments. The assumed form of the opinion flip rate, Eq. (1), is such that the agent with probability $1-q$ follows the opinions of the majorities of her neighbors within all layers only if these opinions are identical; otherwise, she makes decision randomly. In other words, if the opinion on a given subject gathered from different communications channels is identical the agent follows it with higher probability; otherwise, the agent becomes frustrated and acts randomly (independently). This update rule, conceptually related to the AND update rule for MNs [26, 31, 32], is reasonable and simplifies evaluation of the critical intensity of the internal noise for the FM transition based on the heterogeneous MF approximation (Sect. 5) but, of course, it is not the only possible choice and other rules can be used if necessary. It should be also emphasized that in Eq. (2) the majority opinions within each layer are calculated separately, thus the MV model on a MN cannot be reduced to the MV model on a complex network with the same set of nodes and with a set of edges being a superposition of the sets of edges from all layers of the MN.

Theoretical analysis of the model described above in the MF approximation in Sect. 5 is quite general and not restricted to any particular sort of MNs. However, MC simulations as well as detailed analytic calculations are performed for the model on MNs in which the layers are independently generated SF networks with the distributions of the degrees of nodes

$$p_{k^{(A)}} \propto \left(k^{(A)}\right)^{-\gamma^{(A)}} \quad \text{for } k^{(A)} > \tilde{m}^{(A)},$$

$$p_{k^{(B)}} \propto \left(k^{(B)}\right)^{-\gamma^{(B)}} \quad \text{for } k^{(B)} > \tilde{m}^{(B)},$$

where $\tilde{m}^{(A)}$ ($\tilde{m}^{(B)}$) is the minimum degree of nodes within the layer $G^{(A)}$ ($G^{(B)}$). Such SF networks partly reflect topology of empirical networks of human interactions and communications characterized by the presence of heterogeneous agents, from typical ones with a limited number of contacts to "hubs" with large number of contacts and high social influence [16], thus they are often used to reproduce networks of interactions in the models for social behavior.

3. Methods of simulation and analysis

The MNs under study are constructed by generating separately and independently the SF layers from the configuration model [34] for a fixed set of N nodes. To generate the first layer $G^{(A)}$, the algorithm starts with assigning to each node i, in a set of N nodes, a degree, i.e., a random number $k_i^{(A)}$ of ends of edges drawn from a given probability distribution $p_{k^{(A)}}$, with $\tilde{m}^{(A)} < k_i^{(A)} < N$ (the minimum degree of node is $\tilde{m}^{(A)}$, and the maximum one $N-1$), with the condition that the sum $\sum_i k_i^{(A)}$ is even. The layer is completed by connecting pairs of ends of edges chosen uniformly at random to make complete edges, respecting the preassigned sequence $k_i^{(A)}$ and under the condition that multiple and self-connections are forbidden. The next layer $G^{(B)}$ is generated in a similar way, with the degrees $k_i^{(B)}$ assigned randomly (and thus independently of the degrees in the first layer) to the nodes from the probability distribution $p_{k^{(B)}}$.

MC simulations of the MV model are performed using simulated annealing algorithm with random sequential updating of the agents' opinions: each simulation is started with high value of q corresponding to the disordered phase and with random initial conditions, the internal noise is decreased in small steps toward zero, and for each intermediate value of q time series of the average opinion $m = N^{-1}\sum_{i=1}^{N} s_i$ are collected after initial transient. Next, the order parameter, i.e., the magnetization M, susceptibility χ and the fourth-order Binder cumulant U_L are calculated as functions of q,

$$M(q) = [\langle|m|\rangle_t]_{av}\,, \tag{4}$$

$$\chi(q) = N\left[\left(\langle m^2\rangle_t - \langle|m|\rangle_t^2\right)\right]_{av}\,, \tag{5}$$

$$U_L(q) = \frac{1}{2}\left[3 - \frac{\langle m^4\rangle_t}{\langle m^2\rangle_t^2}\right]_{av}\,, \tag{6}$$

where $\langle\cdot\rangle_t$ denotes time average for a given realization of the MN (usually over 2.5×10^4 MC simulation steps, each corresponding to updating N nodes) and $[\cdot]_{av}$ denotes averaging over different realizations of the MN with given parameters such as the number of nodes and the degree distributions within each layer; usually results are averaged over 60–100 such realizations, depending on N ranging from 2×10^4 to 10^3, respectively. The above quantities are expected to obey FSS relations analogous to those valid for systems on complex heterogeneous networks [35],

$$M = N^{-\beta/\nu} f_m\left(N^{1/\nu}(q-q_c)\right) \tag{7}$$

$$\chi = N^{\gamma/\nu} f_\chi\left(N^{1/\nu}(q-q_c)\right) \tag{8}$$

$$q_c - q^\star(N) \propto N^{-1/\nu}. \tag{9}$$

The critical value q_c of the internal noise can be obtained from the intersection point of the Binder cumulants for different sizes N of the MN [36]. Next, from Eqs. (7, 8) the exponents β/ν and γ/ν, respectively, can be determined. Furthermore, Eq. (9) can be used to calculate the exponent $1/\nu$ using the value $q^\star(N)$ for which the susceptibility χ of the model on the MN with N nodes has a maximum value. Finally, it is checked if the obtained exponents fulfil the hyperscaling relation,

$$2\frac{\beta}{\nu} + \frac{\gamma}{\nu} = D_{\mathrm{eff}}, \tag{10}$$

where the effective dimension $D_{eff} = 1$ is expected in the case of systems on complex networks (and, consequently, on MNs) which do not have any particular spatial dimension [35].

4. Critical exponents

MC simulations of the MV model on MNs were performed for SF layers characterized by various exponents $\gamma^{(A)}$, $\gamma^{(B)}$ and minimum degrees of nodes $\tilde{m}^{(A)}$, $\tilde{m}^{(B)}$; exemplary results are shown in Fig. 2. MC simulations show that the model exhibits FM transition, as expected, characterized by the increase of the magnetization (Fig. 2a) and maximum of the susceptibility (Fig. 2b) as q is decreased. The Binder cumulants cross at one point corresponding to q_c and are monotonically decreasing functions of q (Fig. 2c), which is typical of the second-order transition [36]. Critical exponents for this transition can be obtained using the FSS relations, Eq. (7)–(9); these relations are fulfilled well (Fig. 2d) and the properly rescaled magnetizations and susceptibilities coincide for different N as functions of rescaled q (Fig. 2a,b, insets).

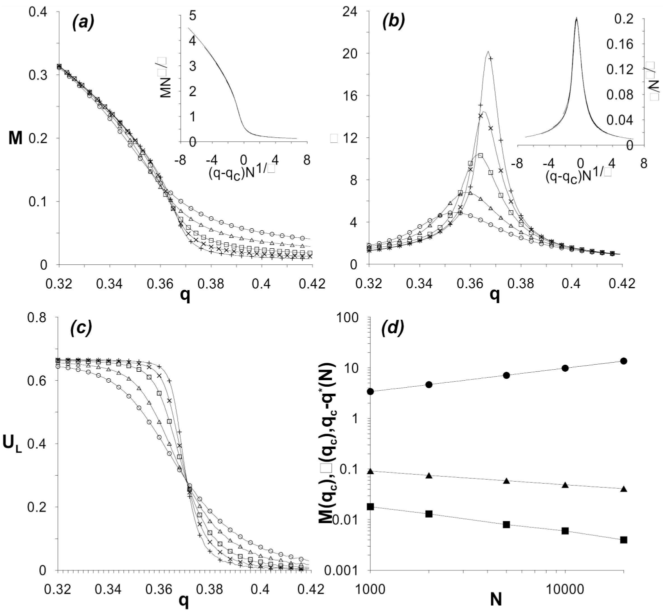

Fig. 2. Results of MC simulations of the MV model on a MN with SF layers with $\gamma^{(A)} = 4.0$, $\tilde{m}^{(A)} = 20$, $\gamma^{(B)} = 5.5$, $\tilde{m}^{(B)} = 20$: (a) magnetization M vs. q for $N = 1000$ (circles), $N = 2000$ (triangles), $N = 5000$ (squares), $N = 10000$ (crosses), $N = 20000$ (pluses), inset shows results rescaled according to Eq. (7) with critical exponents from Table I; (b) susceptibility χ vs. q, symbols as in (a), inset shows results rescaled according to Eq. (8) with critical exponents from Table I; (c) Binder cumulants U_L vs. q, symbols as in (a); (d) log–log plots of $M(q_c)$ vs. N (filled circles), $\chi(q_c)$ vs. N (filled triangles) and $q_c - q^\star(N)$ vs. N (filled squares), straight lines are least-squares fits to the FSS relations, Eq. (7), Eq. (8), Eq. (9), respectively..

The exponents β/ν, γ/ν, $1/\nu$ obtained for the model with fixed $\gamma^{(B)} = 5.5$, $\tilde{m}^{(A)} = \tilde{m}^{(B)} = 20$ and different $\gamma^{(A)}$ are summarized in Table I. It can be seen that the universality class of the model depends substantially on $\gamma^{(A)}$. This dependence is similar to that of the MV model on SF networks, e.g., the exponent $1/\nu$ increases as a function of $\gamma^{(A)}$ approximately up to 0.5 [11]. Besides, for a strongly heterogeneous layer $G^{(A)}$ with $\gamma^{(A)} < 5$, the MV model on MNs belongs to a different universality class than the corresponding Ising model. First, even for $\gamma^{(A)} = 3$ the FM transition occurs at $q_c < 0.5$ whereas for the corresponding Ising model the critical temperature diverges [33]. Second, the critical exponents for the MV

TABLE I

Critical intensity of internal noise q_c, critical exponents β/ν, γ/ν, $1/\nu$ and effective dimension D_{eff} for the MV model on MNs with SF layers with different $\gamma^{(A)}$ and fixed $\gamma^{(B)}=5.5$, $\tilde{m}^{(A)}=\tilde{m}^{(B)}=20$.

$\gamma^{(A)}$	q_c	β/ν	γ/ν	$1/\nu$	$D_{\rm eff}$
3.0	0.390(2)	0.3071(4)	0.3870(8)	0.3538(60)	1.0012(16)
3.5	0.370(2)	0.2821(1)	0.4385(1)	0.4547(36)	1.0027(3)
4.0	0.371(2)	0.2688(1)	0.4649(1)	0.4975(25)	1.0025(3)
4.5	0.368(2)	0.2790(1)	0.4563(1)	0.4958(101)	1.0143(3)
5.0	0.365(2)	0.2553(1)	0.4859(2)	0.5258(144)	0.9965(4)
5.5	0.363(2)	0.2442(3)	0.5003(5)	0.5424(61)	0.9887(11)

model and the Ising model are different. For example, assuming that $\gamma^{(B)}>\gamma^{(A)}$, for the Ising model on MNs $\beta=1/\left(\gamma^{(A)}-3\right)$ for $3<\gamma^{(A)}<5$ [33], and these values are significantly higher than those resulting from Table I.

In general, for high values of $\tilde{m}^{(A)}$, $\tilde{m}^{(B)}$, as $\gamma^{(A)}$, $\gamma^{(B)}$ are increased the critical exponents approach the values $\beta/\nu\approx 0.25$, $\gamma/\nu\approx 0.5$, $1/\nu\approx 0.5$ which were obtained for the MV model on random graphs [8]; these values are also MF values of the critical exponents for the Ising model [35]. On the one hand, this result could be expected since for large $\gamma^{(A)}$, $\gamma^{(B)}$ the SF layers approach random graphs [16]. On the other hand, it is not obvious since the update rule, Eq. (1), is qualitatively different from, e.g., the Glauber or Metropolis spin flip rates for the kinetic Ising model on MNs [33]: it depends on the majority opinions evaluated separately within each layer rather than on the majority opinion obtained by summing the opinions of all neighbors of a given agent in all layers. Finally, from Table I it follows that the critical exponents for the MV model on MNs with SF layers obey with good accuracy the hyperscaling relation, Eq. (10), with $D_{\rm eff}=1$, as expected.

5. Heterogeneous mean-field theory for the ferromagnetic transition

In this section the critical value of the internal noise level q_c for the FM transition in the model under study is obtained analytically in the heterogeneous MF approximation and compared with that obtained from the MC simulations. The analytic approach is analogous to that used in the studies of epidemic spreading [29, 30] and FM transition in the Ising model [33] on MNs.

5.1. Equations for the order parameters

The starting point for the heterogeneous MF approximation is the master equation for the probability $\Pr(s_i=+1)=1-\Pr(s_i=-1)$ that at time t the agent in node i has opinion $s_i=+1$:

$$\frac{\partial}{\partial t}\Pr(s_i=+1) = w_i(-1)\Pr(s_i=-1)-w_i(+1)\Pr(s_i=+1)\,. \tag{11}$$

Taking into account that

$$\Pr(s_i=+1)=\frac{1+\langle s_i\rangle}{2},\qquad \Pr(s_i=-1)=\frac{1-\langle s_i\rangle}{2},$$

where $\langle s_i\rangle$ is the average opinion, and applying the MF approximation, the master equation becomes [5]:

$$\frac{\partial\langle s_i\rangle}{\partial t}=-\langle s_i\rangle+(1-2q){\rm sgn}_i \to \frac{\partial\langle s_i\rangle}{\partial t}=-\langle s_i\rangle+(1-2q)\langle {\rm sgn}_i\rangle, \tag{12}$$

with

$$\langle {\rm sgn}_i\rangle=(+1)\Pr({\rm sgn}_i=+1)+0\Pr({\rm sgn}_i=0) +(-1)\Pr({\rm sgn}_i=-1)\,. \tag{13}$$

In particular, in the model under study from Eq. (2) there is

$$\Pr({\rm sgn}_i=\mp 1)= \tag{14}$$

$$\left\{\sum_{m=0}^{\left[\frac{k_i^{(B)}}{2}\right]}\binom{k_i^{(B)}}{m}\prod_{j''}^{m}\Pr(s_{j''}=\pm 1)\prod_{j'''}^{k_i^{(B)}-m}\Pr(s_{j'''}=\mp 1)\right\} \times\left\{\sum_{l=0}^{\left[\frac{k_i^{(A)}}{2}\right]}\binom{k_i^{(A)}}{l}\prod_{j}^{l}\Pr(s_j=\pm 1)\prod_{j'}^{k_i^{(A)}-l}\Pr(s_{j'}=\mp 1)\right\},$$

where $\left[\frac{k}{2}\right]=\frac{k}{2}-1$ for k even and $\left[\frac{k}{2}\right]=\frac{k-1}{2}$ for k odd.

The basic assumption of the heterogeneous MF theory for interacting systems on MNs is that the nodes are divided into classes according to their degrees within each layer and that the average values of the states of interacting units (spins, agents, etc.) in nodes belonging to the same class are equal [33]. In particular, in the case of the MV model on a MN consisting of two layers $G^{(A)}$, $G^{(B)}$ the nodes are divided into classes according to their degrees $\left(k^{(A)},k^{(B)}\right)$ and it is assumed that the average opinion of agents in nodes belonging to each such class is equal to $\langle s_{k^{(A)},k^{(B)}}\rangle$. Further analytic results can be obtained if correlations between the degrees of nodes within each layer are vanishingly small (this is the case of SF layers generated from the configuration model with $\gamma^{(A)}>3$, $\gamma^{(B)}>3$) or are neglected. Then for independent layers the probability that the edge of the layer $G^{(A)}$ attached at one end to the node i is linked at the other end to the node with degrees $\left(k^{(A)},k^{(B)}\right)$ is

$$\frac{p_{k^{(A)}}p_{k^{(B)}}k^{(A)}}{\sum_{k^{(A)},k^{(B)}}p_{k^{(A)}}p_{k^{(B)}}k^{(A)}}=\frac{p_{k^{(A)}}p_{k^{(B)}}k^{(A)}}{\langle k^{(A)}\rangle},$$

and similarly for the layer $G^{(B)}$ (note that these probabilities must be evaluated separately for each layer). Thus, the number of nodes with degrees $\left(k^{(A)},k^{(B)}\right)$ connected to the node i by edges of the layer $G^{(A)}$ is

$$k_i^{(A)} \frac{p_{k^{(A)}} p_{k^{(B)}} k^{(A)}}{\langle k^{(A)} \rangle},$$

and similarly for the layer $G^{(B)}$. As a result, in the case of MNs with independent SF layers in Eq. (14) the products of probabilities over the indices of nodes can be replaced with products over the classes of nodes,

$$\prod_j^l \Pr(s_j = \pm 1) = \prod_{k^{(A)},k^{(B)}} \Pr\left(s_{k^{(A)},k^{(B)}} = \pm 1\right)^{l \frac{k^{(A)} p_{k^{(A)}} p_{k^{(B)}}}{\langle k^{(A)} \rangle}}, \quad (15)$$

etc.

Another assumption of the heterogeneous MF theory is that the system on a MN undergoing phase transition should be characterized by several (mutually dependent) order parameters related to the consecutive layers [33]. In the case of the MV model these parameters are weighted average opinions, with weights associated with each node and equal to the degrees of nodes within consecutive layers. For the two layers $G^{(A)}$, $G^{(B)}$ the order parameters are $\langle S^{(A)} \rangle$, $\langle S^{(B)} \rangle$, where

$$\langle S^{(A)} \rangle = \frac{1}{N \langle k^{(A)} \rangle} \sum_{i=1}^N k_i^{(A)} \langle s_i \rangle = \sum_{k^{(A)},k^{(B)}} \frac{k^{(A)} p_{k^{(A)}} p_{k^{(B)}}}{\langle k^{(A)} \rangle} \langle s_{k^{(A)},k^{(B)}} \rangle, \quad (16)$$

and similarly for $\langle S^{(B)} \rangle$.

From the master Eq. (12), using Eqs. (13)–(15) it is possible to obtain linearized equations for $\langle S^{(A)} \rangle$, $\langle S^{(B)} \rangle$ valid for small values of the order parameters, i.e., in the disordered phase and just below the transition point to the FM phase. For this purpose let us take into account that $\Pr\left(s_{k^{(A)},k^{(B)}} = \pm 1\right) = \frac{1 \pm \langle s_{k^{(A)},k^{(B)}} \rangle}{2}$, substitute this into Eq. (15), expand for small $\langle s_{k^{(A)},k^{(B)}} \rangle$,

$$\left(1 \pm \langle s_{k^{(A)},k^{(B)}} \rangle\right)^{l \frac{k^{(A)} p_{k^{(A)}} p_{k^{(B)}}}{\langle k^{(A)} \rangle}} \approx 1 \pm l \frac{k^{(A)} p_{k^{(A)}} p_{k^{(B)}}}{\langle k^{(A)} \rangle} \langle s_{k^{(A)},k^{(B)}} \rangle,$$

etc., up to linear terms, substitute resulting Eq. (15) into Eq. (14) and retain only terms linear with respect to $\langle s_{k^{(A)},k^{(B)}} \rangle$. The resulting sums over the classes of nodes in Eq. (14) can be written in a concise form by introducing the quantity

$$B_{k_i} = 2^{k_i} \sum_{l=0}^{\left[\frac{k_i}{2}\right]} \binom{k_i}{l} (k_i - 2l), \quad (17)$$

which is evaluated in Appendix, Eq. (25). Then it can be written that

$$\sum_{l=0}^{\left[\frac{k_i}{2}\right]} \binom{k_i}{l} = \begin{cases} 2^{k_i - 1} - 2^{k_i} k_i^{-1} B_{k_i} & \text{for } k_i \text{ even}, \\ 2^{k_i - 1} & \text{for } k_i \text{ odd}, \end{cases} \quad (18)$$

where the first equality follows from Eq. (25) in Appendix and the second one is trivial. Finally, the right-hand side of Eq. (12) can be expressed by the order parameters (16):

$$\frac{\partial \langle s_i \rangle}{\partial t} = -\langle s_i \rangle + (1 - 2q) \times \left[\left(1 - \frac{2B_{k_i^{(B)}}}{k_i^{(B)}} \delta_{k_i^{(B)},\text{even}}\right) B_{k_i^{(A)}} \langle S^{(A)} \rangle + \left(1 - \frac{2B_{k_i^{(A)}}}{k_i^{(A)}} \delta_{k_i^{(A)},\text{even}}\right) B_{k_i^{(B)}} \langle S^{(B)} \rangle \right]. \quad (19)$$

It can be seen that the form of the equation for the mean opinion of the agent in node i depends on the parity of the degrees of this node within the layers of the MN. Such dependence is not observed in the Ising model [33].

The next step is to multiply both sides of Eq. (19) by $k_i^{(A)}/N\langle k^{(A)} \rangle$ ($k_i^{(B)}/N\langle k^{(B)} \rangle$) and perform summation over the indices of nodes $i = 1, 2 \ldots N$. When doing this it should be remembered that approximately half of nodes has even (odd) degrees within each layer and the degree distributions within different layers are independent, thus, e.g.,

$$\sum_{i=1}^N k_i^{(A)} \left(1 - \frac{2B_{k_i^{(B)}}}{k_i^{(B)}} \delta_{k_i^{(B)},\text{even}}\right) B_{k_i^{(A)}} \approx \frac{1}{2} \sum_{i=1}^N k_i^{(A)} \left(1 - \frac{2B_{k_i^{(B)}}}{k_i^{(B)}}\right) B_{k_i^{(A)}} + \frac{1}{2} \sum_{i=1}^N k_i^{(A)} B_{k_i^{(A)}},$$

etc. The above sums on the right-hand sides of the two resulting equations can be evaluated using the approximate equality in Eq. (25),

$$\frac{1}{N} \sum_{i=1}^N k_i^{(A)} B_{k_i^{(A)}} = \langle k^{(A)} B_{k^{(A)}} \rangle \approx \frac{1}{\sqrt{2\pi}} \langle \left(k^{(A)}\right)^{3/2} \rangle$$

$$\frac{1}{N} \sum_{i=1}^N \left(k_i^{(B)}\right)^{-1} B_{k_i^{(B)}} k_i^{(A)} B_{k_i^{(A)}} = \langle \left(k^{(B)}\right)^{-1} B_{k^{(B)}} k^{(A)} B_{k^{(A)}} \rangle = \langle \left(k^{(B)}\right)^{-1} B_{k^{(B)}} \rangle \langle k^{(A)} B_{k^{(A)}} \rangle \approx \frac{1}{2\pi} \langle \left(k^{(B)}\right)^{-1/2} \rangle \langle \left(k^{(A)}\right)^{3/2} \rangle,$$

etc., where the brackets $\langle \cdot \rangle$ denote the appropriate moments of the degree distributions $p_{k^{(A)}}$, $p_{k^{(B)}}$.

Eventually, the following system of two linear equations for the time dependence of the order parameters is obtained,

$$\frac{\partial \langle S^{(A)} \rangle}{\partial t} = \left[-1 + \frac{1-2q}{\sqrt{2\pi}} \frac{\langle \left(k^{(A)}\right)^{3/2} \rangle}{\langle k^{(A)} \rangle} \left(1 - \frac{\langle \left(k^{(B)}\right)^{-1/2} \rangle}{\sqrt{2\pi}}\right)\right] \langle S^{(A)} \rangle + \frac{1-2q}{\sqrt{2\pi}} \langle \left(k^{(B)}\right)^{1/2} \rangle \left(1 - \frac{1}{\sqrt{2\pi}} \frac{\langle \left(k^{(A)}\right)^{1/2} \rangle}{\langle k^{(A)} \rangle}\right) \langle S^{(B)} \rangle, \quad (20)$$

and a complementary equation for $\langle S^{(B)}\rangle$ which can be obtained from Eq. (20) by exchanging the indices A, B.

5.2. *Critical value of the internal noise intensity for the ferromagnetic transition*

In the stationary state with $\partial\langle S^{(A)}\rangle/\partial t = 0$, $\partial\langle S^{(B)}\rangle/\partial t = 0$ the system of Eq. (20) and the complementary equation has a trivial solution $\langle S^{(A)}\rangle = \langle S^{(B)}\rangle = 0$ corresponding to a disordered phase. For the occurrence of non-zero solutions, corresponding to the FM phase, the determinant of the right-hand side of this system of equations must be zero. This condition leads to a quadratic equation for q, and the solution of this equation with a higher value corresponds to the critical noise intensity q_c:

$$q_c = \frac{1}{2}\left(1 + \frac{b+\sqrt{\Delta}}{2a}\right), \tag{21}$$

where

$$a = \frac{1}{2\pi}\left[\frac{\langle (k^{(A)})^{3/2}\rangle}{\langle k^{(A)}\rangle}\frac{\langle (k^{(B)})^{3/2}\rangle}{\langle k^{(B)}\rangle}\left(1-\frac{\langle (k^{(A)})^{-1/2}\rangle}{\sqrt{2\pi}}\right)\right. \times\left(1-\frac{\langle (k^{(B)})^{-1/2}\rangle}{\sqrt{2\pi}}\right) - \langle (k^{(A)})^{1/2}\rangle\langle (k^{(B)})^{1/2}\rangle \times\left.\left(1-\frac{1}{\sqrt{2\pi}}\frac{\langle (k^{(A)})^{1/2}\rangle}{\langle k^{(A)}\rangle}\right)\left(1-\frac{1}{\sqrt{2\pi}}\frac{\langle (k^{(B)})^{1/2}\rangle}{\langle k^{(B)}\rangle}\right)\right],$$

$$b = \frac{1}{\sqrt{2\pi}}\left[\frac{\langle (k^{(A)})^{3/2}\rangle}{\langle k^{(A)}\rangle}\left(1-\frac{\langle (k^{(B)})^{-1/2}\rangle}{\sqrt{2\pi}}\right) + \frac{\langle (k^{(B)})^{3/2}\rangle}{\langle k^{(B)}\rangle}\left(1-\frac{\langle (k^{(A)})^{-1/2}\rangle}{\sqrt{2\pi}}\right)\right],$$

$$\Delta = \frac{1}{2\pi}\left[\frac{\langle (k^{(A)})^{3/2}\rangle}{\langle k^{(A)}\rangle}\left(1-\frac{\langle (k^{(B)})^{-1/2}\rangle}{\sqrt{2\pi}}\right) - \frac{\langle (k^{(B)})^{3/2}\rangle}{\langle k^{(B)}\rangle}\left(1-\frac{\langle (k^{(A)})^{-1/2}\rangle}{\sqrt{2\pi}}\right)\right]^2 + \frac{2}{\pi}\langle (k^{(A)})^{1/2}\rangle\langle (k^{(B)})^{1/2}\rangle\left(1-\frac{1}{\sqrt{2\pi}}\frac{\langle (k^{(A)})^{1/2}\rangle}{\langle k^{(A)}\rangle}\right) \times\left(1-\frac{1}{\sqrt{2\pi}}\frac{\langle (k^{(B)})^{1/2}\rangle}{\langle k^{(B)}\rangle}\right).$$

In particular, for SF layers with $p_k \propto k^{-\gamma}$ for $k > \tilde{m}$ there is

$$\langle k^{-1/2}\rangle = \frac{\gamma-1}{\gamma-\frac{1}{2}}\tilde{m}^{-1/2}, \qquad \langle k^{1/2}\rangle = \frac{\gamma-1}{\gamma-\frac{3}{2}}\tilde{m}^{1/2},$$
$$\langle k\rangle = \frac{\gamma-1}{\gamma-2}\tilde{m}, \qquad \langle k^{3/2}\rangle = \frac{\gamma-1}{\gamma-\frac{5}{2}}\tilde{m}^{3/2}.$$

Substituting these expressions with $\gamma = \gamma^{(A)}$, $\gamma = \gamma^{(B)}$, respectively, into Eq. (21) the critical intensity of the internal noise q_c for the MV model on MNs with independent SF layers can be obtained. In particular, if both layers have identical degree distributions with $\gamma^{(A)} = \gamma^{(B)} = \gamma$ and $\tilde{m}^{(A)} = \tilde{m}^{(B)} = \tilde{m} \gg 1$ there is

$$\frac{\langle k^{-1/2}\rangle\langle k^{3/2}\rangle}{\langle k\rangle} \propto O(1) \ll \frac{\langle k^{3/2}\rangle}{\langle k\rangle}, \quad \langle k^{1/2}\rangle \propto \tilde{m}^{1/2}.$$

This yields the following expression for the critical intensity of the internal noise:

$$q_c = \frac{1}{2}\left[1 - \sqrt{2\pi}\left(\frac{\langle k^{3/2}\rangle}{\langle k\rangle} + \langle k^{1/2}\rangle\right)^{-1}\right], \tag{22}$$

which can be compared with that for the MV model on heterogeneous SF networks [11].

In Fig. 3 predictions of the MF approximation, Eq. (21), are compared with the results of MC simulations. It can be seen that these predictions are quantitatively correct in the case of MNs with high mean degrees of nodes within SF layers, and small deviations from the values of q_c obtained numerically occur in the case of SF layers with smaller mean degrees of nodes, as expected. Besides, from Eq. (21) follows that in the model under study the FM transition occurs at $q_c < 1/2$ if $\gamma^{(A)} > 2.5$, $\gamma^{(B)} > 2.5$, which is confirmed by numerical simulations.

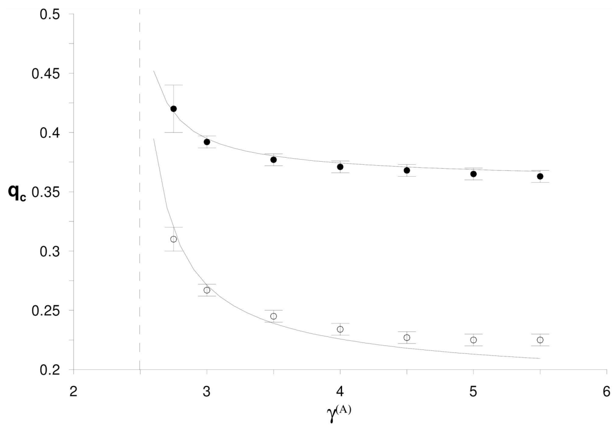

Fig. 3. Theoretical (solid lines) and obtained from Monte Carlo simulations (symbols) critical values of q_c vs. $\gamma^{(A)}$ for the MV model on MN with SF layers with fixed $\gamma^{(B)} = 5.5$ and $\tilde{m}^{(A)} = \tilde{m}^{(B)} = 5$ (circles), $\tilde{m}^{(A)} = \tilde{m}^{(B)} = 20$ (dots).

6. Summary and conclusions

A nonequilibrium model for the opinion formation, the MV model, was investigated on MNs with two independently generated layers by means of MC simulations and theoretically in the heterogeneous MF approximation. Detailed calculations were performed and their results were compared with numerical ones for the layers in the form of SF networks. In the version of the MV model under study it was assumed that the agents obey the update rule according to which if the majorities of the agent's neighbors within each layer share the same opinion the agent follows it with probability $1-q$; otherwise, she makes decision randomly. It was shown that the model exhibits continuous FM transition as the level of the internal noise is decreased, and that its universality class depends on the details of the degree distributions in the layers and for strongly heterogeneous layers is different from the universality class of the corresponding equilibrium Ising model. Heterogeneous MF theory for the model under study was presented, and the evaluated critical value q_c of the internal noise intensity for the FM transition showed quantitative agreement with that obtained from MC simulations for a broad range of parameters characterizing the degree distributions of the layers.

In contemporary efforts to create agent-based models for social and economic phenomena it is more and more understood that the underlying networks of human interactions (for direct contacts, information spreading, transport, etc.) are not only heterogeneous, but often have a multilayer structure, with different layers influencing each other, which can be taken into account by using the concept of MNs [18, 19]. For example, in this paper the multiplex structure of the network of interactions affects the model under study via the update rule for the agents' opinions, Eq. (1), in which the effect of information received by agents from their neighbors via different channels need not be equivalent: the bias from a (possibly larger) majority of agent's contacts within one layer can be balanced by the opposite bias from a (possibly smaller) majority of contacts within the other layer. By changing this update rule it is possible, e.g., to investigate the effect of the relative importance of different layers of the network of interactions on the process of opinion formation, both numerically and in the MF approximation, by changing appropriately Eq. (14). Another straightforward extension of the present study is to investigate the effects on q_c of partial multiplexity and of correlation between the degrees of nodes within different layers, in analogy with similar studies for the Ising model on MNs [33]. Finally, it is known that the multiplex structure of the network of interactions can lead to discontinuous FM transition in certain models for the opinion formation [31, 32]. The present work opens a way to look for similar phenomena in the (possibly modified) MV model on MNs.

Appendix

In this Appendix the quantity B_{k_i} given by Eq. (17) is evaluated. For k_i even, taking into account that

$$\sum_{l=0}^{\frac{k_i}{2}-1}\binom{k_i}{l} = \frac{1}{2}\left[\sum_{l=0}^{k_i}\binom{k_i}{l} - \binom{k_i}{\frac{k_i}{2}}\right] = 2^{k_i-1} - \frac{1}{2}\binom{k_i}{\frac{k_i}{2}}, \tag{23}$$

$$\begin{aligned}\sum_{l=0}^{\frac{k_i}{2}-1}\binom{k_i}{l}l = \sum_{l=0}^{\frac{k_i}{2}}\binom{k_i}{l}l - \frac{k_i}{2}\binom{k_i}{\frac{k_i}{2}} = \\ k_i\sum_{l=0}^{\frac{k_i}{2}-1}\binom{k_i-1}{l} - \frac{k_i}{2}\binom{k_i}{\frac{k_i}{2}} = \\ \frac{k_i}{2}\sum_{l=0}^{k_i-1}\binom{k_i-1}{l} - \frac{k_i}{2}\binom{k_i}{\frac{k_i}{2}} = \frac{k_i}{2}2^{k_i-1} - \frac{k_i}{2}\binom{k_i}{\frac{k_i}{2}},\end{aligned} \tag{24}$$

it is obtained that

$$B_{k_i} = 2^{-k_i}\frac{k_i}{2}\binom{k_i}{\frac{k_i}{2}} \approx \frac{1}{\sqrt{2\pi}}k_i^{\frac{1}{2}}, \tag{25}$$

where the approximate equality, valid for large k_i, results from the Stirling formula. From this equation the first equality in Eq. (18) follows. In a similar way, the approximate equality, Eq. (25), can be proved for k_i odd.

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