Proc. XII Int. School on Theoretical Physics — Symmetry and Structural Properties of Condensed Matter

Aspects of Applicability King–St. Clair Approximation in a Non-Newtonian Fluid Mechanics

A. Włoch*, H. Czyż and T. Jasiński

Rzeszów University of Technology, Faculty of Mathematics and Applied Physics, Rzeszów, Poland

This studyis a contribution to research on the biomedical and clinical applications of ultrasound. Our research concerns the procedure for the separation of human blood fractions — erythrocytes. Ultrasonic waves can be used for the separation of cells in human blood. From a physical point of view, the human blood is a suspension of liquids and solids (cell elements), and behaves like a non-Newtonian fluid. Our work is devoted to the problem of the motion of red cells in human blood under the influence of ultrasonic wave. It defines a range of the applicability of approximation consisting in neglecting the nonlinear term in the friction force. It also analyzes the general properties of the equation of motion of the cell in the case of large attenuation constants, corresponding to the values of the drift forces for the cells with radii of a few μm. Finally, it defines the applicability criterion of the so-called King–St Clair approximation consisting in the assumption of equilibrium between the drift and the Stokes viscosity forces, neglecting the term representing inertia. This approximation permits analytical estimation of the time constants for the cell transport to points of stable equilibrium in an ultrasonic standing wave field.

DOI: 10.12693/APhysPolA.132.164

PACS/topics: acoustic radiation force, acoustic standing wave, medical applications of ultrasound, separation of red blood cells from human blood

1. Introduction

We present some results from our analysis of ultrasound standing waves as a potential separation mechanism for blood cells [1,2]. We analyze the motion of the particle corresponding to a molecule of human erythrocytes in an ultrasonic standing wave field. Human blood is a tissue composed of about 4,000 different kinds of components. The red cells, white cells, platelets, and plasma are the most important ones. Red cells, or erythrocytes, are relatively large microscopic cells. They make up half of the total blood volume. White cells, or leukocytes make up a small part of blood volume, about one percent in adult people. The platelets are about 1/3 the size of red cells. They have a lifespan of 9–10 days. Plasma is the clear yellow fluid (more than ninety percent is water), which carries the red cells [3]. From a physical point of view, the human blood behaves like a non-Newtonian fluid. The viscosity of non-Newtonian fluids is dependent on shear rate. In the ultrasonic standing wave cells contained in the liquid are set in a characteristic motion, referred to as drift [4]. This motion consists of monotonically approaching to the stable equilibrium point or quasi periodical vibration with amplitude suppression. The drift forces result from the interaction between cells and fluid vibration. This phenomenon can be used for separating the cells from the liquid [5, 6]. This paper considers the motion of a cell in human blood caused by the drift forces in a standing ultrasonic wave. These forces, a consequence of interaction between the cell and the vibrating medium, result from such phenomena as the radiation pressures F_{DR}, the asymmetric vibration motion of the cell F_{DA} or periodic changes in the viscosity of the fluid F_{DL} [3]:

$$F_{DR} = \frac{8}{3}\pi k r^3 \mu_g^2 \bar{E} \sin(2kx), \tag{1}$$

$$F_{DA} = -\frac{1}{2} m_p \rho_g^{-1} k \bar{E} \sin(2kx), \tag{2}$$

$$F_{DL} = 3\pi(\kappa - 3)\mu_g^2 \eta r (\rho_g c)^{-1} \bar{E} \sin(2kx), \tag{3}$$

where symbols denote k — wave number, r — red cell radius, $\bar{E}$ — the mean energy of the wave, m_p — mass of a particle (red cell), x — position measured along wave propagation direction, κ — the exponent of the adiabatic, η — kinematic viscosity of the liquid, ρ_g — density of the fluid, ω — angular frequency, τ — the relaxation time μ_p is the coefficient of particle entrainment known from orthokinetic theory, μ_g is the so-called coefficient of the particle flow

$$\mu_p = \frac{1}{\sqrt{1+\omega^2\tau^2}},$$
$$\mu_g = \frac{\omega\tau}{\sqrt{1+\omega^2\tau^2}},$$
$$\tau = \frac{2\rho_p r_p^2}{9\eta}. \tag{4}$$

The different kinds of drift have a common property; the forces F_D applied on the cell depend in the same way on the position with respect to the nodes and loops of the standing wave, $F_D(x) = F_0 \sin(2kx)$, where F_0 is the amplitude of the drift force. The drift forces are proportional to the density $\bar{E}$ of the wave energy.

2. The equation of the motion of red cells affected by the drift forces

Consideration is given to the problem of the motion of a single cell (erythrocyte), taking into account the resis-

*corresponding author; e-mail: awloch@prz.edu.pl

tance forces of the part of the fluid and the drift force. This equation has the form

$$m_p\ddot{x} = -6\pi\eta r_p\dot{x} - \frac{4}{9}\pi r^2\rho_g\dot{x}|\dot{x}| + F_D\sin(2kx). \qquad (5)$$

On the right side of the equation the first term presents the Stokes force related to the viscosity of the fluid, the second one presents the nonlinear Oseen correction. It is significant for large Reynolds numbers. This differential equation has no elementary solution. By dividing both sides of the equation by the mass the resulting equation of motion is

$$\ddot{x} + \frac{1}{\tau}\dot{x} + \frac{27\rho_g}{16\rho_p}\dot{x}|\dot{x}| - \frac{F_0}{m_p}\sin(2kx) = 0. \qquad (6)$$

The relaxation time τ characterizes the effect of the Stokes force. The coefficient $A_D = \frac{F_0}{m_p}$ equals the acceleration of the cell in the absence of the friction forces. Applying the expressions (1), (2), (3) of the drift forces of different kinds it is possible to calculate the corresponding accelerations, see Fig. 1.

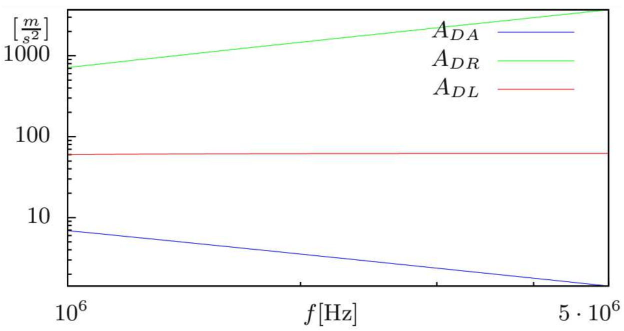

Fig. 1. Plots of the acceleration for the drift forces of type R, L and A as the function of the frequency of the wave at $r = 4 \times 10^{-6}$ m radius of the cell.

In calculating the numerical values for all graphs we have adopted the following values for the parameters describing the cells, the fluid (plasma) and the ultrasonic field: density red blood cells $\rho_p = 1080$ kg/m^3, radius of the cell $r = 4 \times 10^{-6}$ m, density of plasma $\rho_g = 1070$ kg/m^3, $c = 1550$ m/s, $\kappa = 1.33$, $\eta = 4.62 \times 10^{-3}$ (N s)/m^2, energy density $\bar{E} = 100$ J/m^3, frequencies: 10^6 to 5×10^6 Hz. On the basis of plots it should be noted that for the cells with radius at 10^{-6} m the radiation drift is the strongest. Now we evaluate other components of the equation of motion. We reduce to a minimum the number of constants in Eq. (5). We replace the position and the time by the variable

$$y = \pi - 2kx, \quad \theta = t\sqrt{2kA_D}. \qquad (7)$$

For these variables the equation of motion is obtained

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} + \alpha\frac{\mathrm{d}y}{\mathrm{d}\theta} + \beta\frac{\mathrm{d}y}{\mathrm{d}\theta}\left|\frac{\mathrm{d}y}{\mathrm{d}\theta}\right| + \sin y = 0, \qquad (8)$$

where the following was introduced: $\alpha = \frac{1}{\tau\sqrt{2kA_D}}$ and $\beta = \frac{27\rho_g}{32kr\rho_p}$.

Equation (8) contains two constants α and β. The constant α describes the contribution of the Stokes resistance force to the cell motion, and the constant β presents the Oseen correction to this force. The constants α and β depend on the parameters describing the ultrasonic wave and cells. The constant α depends on the acceleration of the drift force. This means that it depends on the type of drift. Now, we estimate the constants α and β and thus evaluate the two terms presenting friction in Eq. (8). Assuming that the same numerical values which were used in calculating the quantity A_D it can estimate the constants α and β.

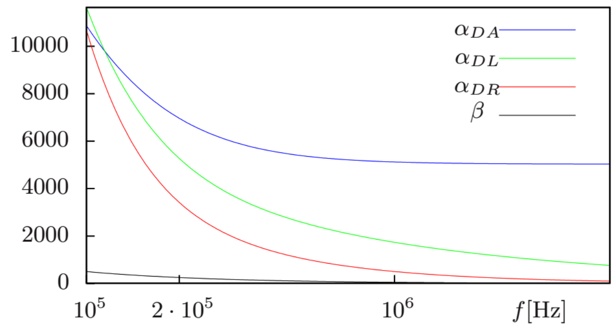

Fig. 2. Plots of the constants α and β of Eq. (8) as a functions of the wave frequency at $r = 4 \times 10^{-6}$ m radius of the cell. The symbols R, L, A distinguish the types of drift.

Analysis of plots in Fig. 2 shows that the nonlinear friction term, represented by the constant β, can play the role assuming that $|\frac{\mathrm{d}y}{\mathrm{d}\theta}| < 1$. It is expected that a component representing the inertia of Eq. (8) can be omitted. The equation of motion (8) after the rejection of the Oseen correction has the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}\theta^2} + \alpha\frac{\mathrm{d}y}{\mathrm{d}\theta} + \sin y = 0. \qquad (9)$$

This equation has no elementary solution. The solution of this equation can be obtained numerically.

3. Analysis of the King–St. Clair approximation

For sufficiently large values of the constant α in Eq. (9) solutions do not show an oscillatory character, see Fig. 3.

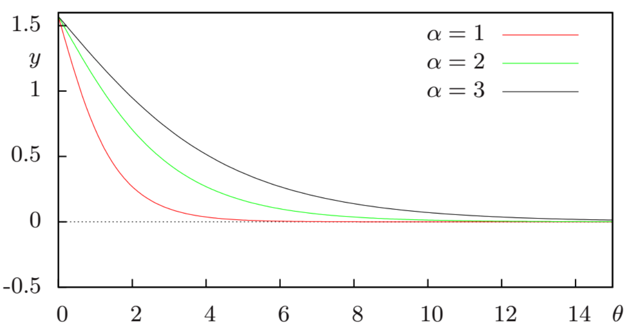

Fig. 3. Numerical solutions of equation of motion (9) for coefficients $\alpha = 1, 2, 3$.

Cells tend to the position of equilibrium asymptotically for $\alpha \geq 2$. This means that the force of inertia is negligible compared with other forces: of friction and that of drift. For large α in Eq. (9) it is possible to neglect the second derivative presenting the inertia force

$$\alpha \frac{\mathrm{d}y}{\mathrm{d}\theta} + \sin y = 0. \tag{10}$$

It is a nonlinear differential equation of the first order with separating variables. With the initial condition $y(0) = y_0$ the solution is in the form, Fig. 4:

$$y = 2 \arctan\left(\tan\left(\frac{y_0}{2}\right) \mathrm{e}^{\frac{\theta}{\alpha}}\right). \tag{11}$$

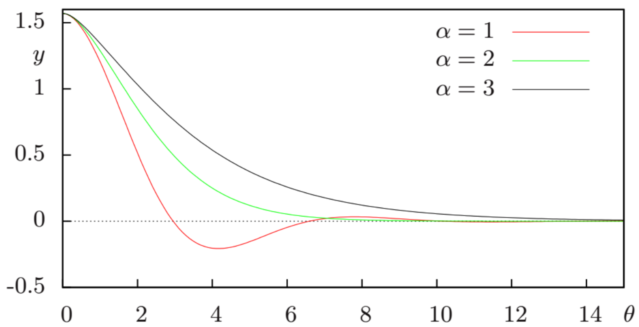

Fig. 4. The analytical solutions of Eq. (10) for coefficients $\alpha = 1, 2, 3$.

For the variables x and t we obtain the following:

$$x = \frac{1}{k} \tan^{-1}\left(\tan kx_0 \mathrm{e}^{2\tau A_D kt}\right). \tag{12}$$

The analytic form of the King–St. Clair approximation (11), (12) allows accurate estimate the time constants of force drift. It was discovered that nonoscillatory solutions could be obtained for $\alpha > 2$. This condition may be adopted as the criterion of the applicability of the King–St. Clair approximation. Taking into consideration formula (12) we receive the inequality $\frac{1}{\tau\sqrt{2kA_D}} > 2$. After transformations we obtained the condition $A_D < \frac{81\eta^2 c}{64\pi r^4 \rho_p^2 f}$. The last inequality defines for what acceleration of the drift forces the cell motion do not have an oscillatory character and the King–St. Clair approximation can be applied.

4. Conclusions

This paper presents the theoretical analysis of the mechanisms of the action of drift forces on cells in fluid in a standing ultrasonic wave. The results of our research are the next step to a controlled ultrasonic separation of red cells in human blood [7–9]. Summing up the considerations it can be stated that:

a) The radiation drift force directs cells to nodes or loops of the ultrasonic wave.

b) It is possible to neglect the nonlinear correction for friction in the equation of motion.

c) It is possible to apply the King–St. Clair for $\alpha \geq 2$. It allows to studythe motion analytically.

The approximations presented here are useful in studies on this motion in the ultrasonic wave.

References

[1] H. Czyż, *Dispersed Phase Acoustics in Liquid*, Rzeszów 2003, p. 82 (in Polish).

[2] G.T. Clement, *Ultrasonics* **42**, 1087 (2004).

[3] S.B. Barnett, *Ultrasound Med. Biol.* **26** (Suppl. 1), 68 (2000).

[4] H. Li, T. Kenny, *High Speed Particles Separation Using Ultrasound for Micro TAS and lab-4*, 2004.

[5] X. Ding, Z. Peng, S.-C. S. Lin, M. Geri, S. Li, P. Li, Y. Chen, M. Dao, S. Suresh, T.J. Huang, *Proc. PNAS USA* **111**, 12992 (2014).

[6] T.N. Pashovkin, D.G. Sadikova, *Acoust. Phys.* **55**, 584 (2009).

[7] D.G. Sadikova, T.N. Pashovkin, *Open J. Biophys.* **3**, 70 (2013).

[8] A. Włoch, H. Czyż, T. Jasiński, *Acta Phys. Pol. A* **128**, 234 (2015).

[9] M.W. Miller, L.F. Battaglia, *Ultrasound Med. Biol.* **29**, 1479 (2003).