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Ergodic averages and free $\mathbb{Z}^2$ actions

by

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Abstract. If the ergodic transformations S, T generate a free $\mathbb{Z}^2$ action on a finite non-atomic measure space $(X, \mathcal{S}, \mu)$ then for any $c_1, c_2 \in \mathbb{R}$ there exists a measurable function f on X for which $(N+1)^{-1}\sum_{j=0}^{N} f(S^j x) \to c_1$ and $(N+1)^{-1}\sum_{j=0}^{N} f(T^j x) \to c_2$ μ-almost everywhere as $N \to \infty$. In the special case when S, T are rationally independent rotations of the circle this result answers a question of M. Laczkovich.

Introduction. The problem discussed in this paper was originally motivated by non-absolute integration, that is, by generalizations of the Lebesgue integral which integrate functions f for which $|f|$ is not necessarily Lebesgue integrable (for details of such methods we refer to [P]). We were interested in how Birkhoff's Ergodic Theorem is related to generalized integration procedures. It follows from the main result of this paper that one encounters serious problems even in the classical situation of rotations of the unit circle equipped with the Lebesgue measure. In fact, it follows from our result that given any two irrationals α and β for which α/β is also irrational there exists a Lebesgue measurable function f defined on the circle for which

$$\frac{1}{N+1}\sum_{j=0}^{N} f(x+j\alpha) \to 1 \quad \text{and} \quad \frac{1}{N+1}\sum_{j=0}^{N} f(x+j\beta) \to 0 \quad \text{for a.e. } x.$$

Of course, by the ergodic theorem f is not Lebesgue integrable. This also shows that if a generalized integral of f is defined, then either the α ergodic average or the β average does not converge to the value of this integral.

Answering a less specific question of this author, P. Major [M] has constructed a function $f : X \to \mathbb{R}$ and ergodic transformations $S, T : X \to X$ on a Lebesgue space $(X, \mathcal{S}, \mu)$ such that $\lim_{N\to\infty}(N+1)^{-1}\sum_{j=0}^{N} f(S^j x) = 0$ a.e. and $\lim_{N\to\infty}(N+1)^{-1}\sum_{j=0}^{N} f(T^j x) = 1$ a.e. In Major's example T is

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a shift on a suitable Lebesgue space and S is conjugate to T. The definition of S is quite involved.

M. Laczkovich raised the question whether X in the above example can be the unit circle with S and T being irrational rotations. In this paper an affirmative answer to this question is given in a somewhat more general setting. Since the transformations in Major's example were conjugate, and two conjugate, orientation preserving homeomorphisms of the circle have the same rotation number, Major's example differs substantially from the rotation case.

Working on M. Laczkovich's problem, in [Bu] we obtained the following result: Suppose that f is a measurable function defined on the circle and

$$\Gamma_f = \left\{ \alpha : \frac{1}{N+1} \sum_{j=0}^{N} f(x + j\alpha) \text{ converges a.e.} \right\}.$$

We verified that Γ_f is of positive Lebesgue measure if and only if f is Lebesgue integrable, and in that case, by the ergodic theorem, all the limits equal almost everywhere the integral of f. Furthermore, given a sequence $\{\alpha_j\}_{j=1}^{\infty}$ of rationally independent irrationals, there exists a non-Lebesgue integrable f such that each $\alpha_j \in \Gamma_f$. This result implies that Γ_f can be dense for non-integrable functions. In [S] R. Svetic improves this result by showing that there exists a non-integrable f for which $\Gamma_f \cap I$ is of cardinality continuum for any non-empty open subinterval I of the circle. It is still an open question whether there exists a non-Lebesgue integrable measurable function f such that the Hausdorff dimension of Γ_f is positive.

If α and β are independent over the rationals then $Tx = x + \alpha$ and $Sx = x + \beta$ generate a free $\mathbb{Z}^2$ action on the circle. The main result of this paper shows that if S, T are ergodic transformations of a non-atomic Lebesgue measure space $(X, \mathcal{S}, \mu)$ and they generate a free $\mathbb{Z}^2$ action then for any $c_1, c_2 \in \mathbb{R}$ there exists a measurable function $f : X \to \mathbb{R}$ such that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} f(S^j x) = c_1,$$
$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} f(T^j x) = c_2 \quad \text{for } \mu\text{-a.e. } x.$$

Preliminaries. In this paper, whenever we use the symbol $\sum_{\gamma\in\Gamma} a_\gamma$ and Γ is empty then by definition $\sum_{\gamma\in\Gamma} a_\gamma = 0$.

Free $\mathbb{Z}^2$ actions on Lebesgue spaces are natural generalizations of independent rotations of the circle. Assume that a $\mathbb{Z}^2$ action is generated by S and T on a finite non-atomic Lebesgue measure space $(X, \mathcal{S}, \mu)$, and

T^jS^k for all $(j,k)\in\mathbb{Z}^2$ is a measure preserving transformation on X. We say that the group action generated by T and S is *free* if $T^jS^kx\neq x$ for $(j,k)\neq(0,0)$ and μ-a.e. x. Given a number N denote by R_N the rectangle $\{(j,k): 1\le j\le N,\ 1\le k\le 2N\}$. Observe that translated copies of R_N form a partition of $\mathbb{Z}^2$, that is, R_N is a *tiling* set in the sense of [OW]. By Theorem 2 of [OW] Rokhlin's lemma is valid for the above free $\mathbb{Z}^2$ actions and R_N. This means the following:

For any $\varepsilon>0$ there is a set $B\in\mathcal{S}$ such that

(i) $\{T^jS^kB:(j,k)\in R_N\}$ are disjoint sets, and
(ii) $\mu(\bigcup_{(j,k)\in R_N}T^jS^kB)>1-\varepsilon$.

Main result

THEOREM. *Assume that $(X,\mathcal{S},\mu)$ is a finite non-atomic Lebesgue measure space and $S,T:X\to X$ are two μ-ergodic transformations which generate a free $\mathbb{Z}^2$ action on X. Then for any $c_1,c_2\in\mathbb{R}$ there exists a μ-measurable function $f:X\to\mathbb{R}$ such that*

$$M_N^Sf(x)=\frac{1}{N+1}\sum_{j=0}^{N}f(S^jx)\to c_1,$$

$$M_N^Tf(x)=\frac{1}{N+1}\sum_{j=0}^{N}f(T^jx)\to c_2\quad \textit{for } \mu\textit{-almost every } x \textit{ as } N\to\infty.$$

Proof. If $c_1=c_2$ then any function with $\int_X f\,d\mu=c_1$ is suitable. Without limiting generality we can assume that $\mu(X)=1$, $c_1=0$, and $c_2=1$. Given a μ-measurable function $g:X\to\mathbb{R}$ and an $\varepsilon>0$ we say that it is (S,ε)-*good* if there exists a measurable set $X_{\varepsilon,S}$ such that $\mu(X\setminus X_{\varepsilon,S})<2\varepsilon$ and $|M_N^Sg(x)|<\varepsilon$ for all $x\in X_{\varepsilon,S}$ and $N=0,1,\ldots$ Denote by E the support of g.

CLAIM 1. *Given an integer N_0 assume that $\mu(\bigcup_{k=0}^{N_0}S^{-k}E)<2\varepsilon$ and*

$$\Big|\sum_{k=0}^{N}g(S^kx)\Big|<N_0\varepsilon\quad \textit{for all } x\in X \textit{ and } N=0,1,\ldots \tag{1}$$

Then g is (S,ε)-good.

Proof. Let $X_{\varepsilon,S}=X\setminus\bigcup_{k=0}^{N_0}S^{-k}E$. If $x\in X_{\varepsilon,S}$ then $g(S^kx)=0$ for $k=0,\ldots,N_0$; hence $M_N^Sg(x)=0$ for $N=0,\ldots,N_0$. Furthermore by using (1) for $N>N_0$ we have

$$|M_N^Sg(x)|=\Big|\frac{1}{N+1}\sum_{k=0}^{N}g(S^kx)\Big|<\Big|\frac{1}{N_0}\sum_{k=0}^{N}g(S^kx)\Big|<\varepsilon.$$

This shows that Claim 1 is true.

Assume that $\varepsilon_0 > \varepsilon_1 > \ldots > 0$, $\sum_{j=0}^{\infty} \varepsilon_j < \infty$, and $1/\varepsilon_j$ is an integer for all j. We also suppose that the bounded measurable functions $f_j : X \to \mathbb{R}$ have the following properties:

(i) if E^j denotes the support of f_j then $\mu(E^j) < 2\varepsilon_j$,
(ii) $\int_X f_j \, d\mu = 1$,
(iii) $f_{2j+1} - f_{2j}$ is (S, ε_{2j})-good and
(iv) $f_{2j+2} - f_{2j+1}$ is (T, ε_{2j+1})-good for $j = 0, 1, \ldots$

Later we show that such functions exist. Now we verify that the existence of such functions implies the theorem. Set $f = \sum_{j=0}^{\infty} (-1)^j f_j$. From (i) and $\sum_j \varepsilon_j < \infty$ it follows that the sum defining f converges μ-almost everywhere.

We first show that $M_N^T f(x) \to 1$ μ-almost everywhere. Given $\varepsilon > 0$ choose N_0 such that $\sum_{j=2N_0+1}^{\infty} \varepsilon_j < \varepsilon/4$. Since $f_{2j+2} - f_{2j+1}$ is (T, ε_{2j+1})-good for each j there exists $X_{\varepsilon_{2j+1},T}$ such that $\mu(X \setminus X_{\varepsilon_{2j+1},T}) < 2\varepsilon_{2j+1}$ and $|M_N^T(f_{2j+2} - f_{2j+1})(x)| < \varepsilon_{2j+1}$ for all $N = 0, 1, \ldots$ and $x \in X_{\varepsilon_{2j+1},T}$. Observe that letting

$$g_{N_0} = \sum_{j=0}^{2N_0} (-1)^j f_j = f_0 + \sum_{j=0}^{N_0-1} f_{2j+2} - f_{2j+1}$$

we have $\int_X g_{N_0} \, d\mu = 1$ and by the ergodic theorem we can choose a measurable set X_{N_0} and a number $N_1 > N_0$ such that $\mu(X \setminus X_{N_0}) < \varepsilon/2$ and $|M_N^T g_{N_0}(x) - 1| < \varepsilon/2$ for $x \in X_{N_0}$ and $N \ge N_1$.

Set $\widehat{X} = X_{N_0} \cap \bigcap_{j=N_0}^{\infty} X_{\varepsilon_{2j+1},T}$. Then $\mu(X \setminus \widehat{X}) < \varepsilon$ and for $x \in \widehat{X}$ and $N \ge N_1$ we have

$$\begin{aligned} |M_N^T f(x) - 1| &\le |M_N^T g_{N_0}(x) - 1| + \sum_{j=N_0}^{\infty} |M_N^T(f_{2j+2} - f_{2j+1})(x)| \\ &< \varepsilon/2 + \sum_{j=N_0}^{\infty} \varepsilon_{2j+1} < \varepsilon. \end{aligned}$$

Since this estimate is valid for all $\varepsilon > 0$ this implies $M_N^T f(x) \to 1$ μ-almost everywhere. The argument showing $M_N^S f(x) \to 0$ is similar and is based on the fact that if we set $g_{N_0} = \sum_{j=0}^{N_0-1} f_{2j+1} - f_{2j}$ then $\int_X g_{N_0} \, d\mu = 0$.

To complete the proof of the Theorem we need to show that functions f_j with properties (i)–(iv) exist. This is based on the following lemma.

Lemma. *Suppose that the transformations S, T satisfy the assumptions of the Theorem. Assume that K and N are arbitrary positive integers and g_0 is a bounded measurable function with support E^0. Set $\varepsilon = 1/K$. Then there exists another bounded measurable function g_1 such that*

(a) $\int_X g_1 \, d\mu = \int_X g_0 \, d\mu$,

(b) *if* E^1 *denotes the support of* g_1 *then* $\mu(\bigcup_{k=-N}^{N} S^{-k}E^1) < 2\varepsilon$,

(c) $\sup_{x\in X}|g_1(x)| \le \varepsilon^{-1}\sup_{x\in X}|g_0(x)|$,

(d) $|\sum_{k=0}^{M}(g_1-g_0)(T^k x)| \le 2\varepsilon^{-1}\sup_{x\in X}|g_0(x)|$ *for* $M = 0,1,\ldots$ *and all* $x\in X$, *and*

(e) *if* $E_{1,0}$ *denotes the support of* g_1-g_0 *we have* $E_{1,0}\subset\bigcup_{k=0}^{K} T^kE^0$.

We prove the Lemma later. Next we use it repeatedly to find the functions f_j. Let $K_j = 1/\varepsilon_j$. Since the even and odd steps are slightly different, we now state what properties we want to satisfy at these steps.

The even case:

(a_{2j}) $\int_X f_{2j}\,d\mu = \int_X f_{2j-1}\,d\mu = 1$,

(b_{2j}) $\mu(\bigcup_{k=-N_{2j}}^{N_{2j}} S^{-k}E^{2j}) < 2\varepsilon_{2j}$,

(c_{2j}) $\sup_{x\in X}|f_{2j}(x)| \le \varepsilon_{2j}^{-1}\sup_{x\in X}|f_{2j-1}(x)|$,

(d_{2j}) $|\sum_{k=0}^{M}(f_{2j}-f_{2j-1})(T^kx)| \le 2\varepsilon_{2j}^{-1}\sup_{x\in X}|f_{2j-1}(x)|$ for $M = 0,1,\ldots,$ and all $x\in X$,

(e_{2j}) if $E_{2j,2j-1}$ denotes the support of $f_{2j}-f_{2j-1}$ we have

$$E_{2j,2j-1}\subset\bigcup_{k=0}^{K_{2j}} T^kE^{2j-1}.$$

The odd case:

(a_{2j+1}) $\int_X f_{2j+1}\,d\mu = \int_X f_{2j}\,d\mu = 1$,

(b_{2j+1}) $\mu(\bigcup_{k=-N_{2j+1}}^{N_{2j+1}} T^{-k}E^{2j+1}) < 2\varepsilon_{2j+1}$,

(c_{2j+1}) $\sup_{x\in X}|f_{2j+1}(x)| \le \varepsilon_{2j+1}^{-1}\sup_{x\in X}|f_{2j}(x)|$,

(d_{2j+1}) $|\sum_{k=0}^{M}(f_{2j+1}-f_{2j})(S^kx)| \le 2\varepsilon_{2j+1}^{-1}\sup_{x\in X}|f_{2j}(x)|$ for $M = 0,1,\ldots$ and all $x\in X$,

(e_{2j+1}) if $E_{2j+1,2j}$ denotes the support of $f_{2j+1}-f_{2j}$ we have

$$E_{2j+1,2j}\subset\bigcup_{k=0}^{K_{2j+1}} S^kE^{2j}.$$

Set $f_{-1}(x) = 1$ for all $x\in X$. Let

$$N_0 = \frac{2}{\varepsilon_1^2}\cdot\frac{1}{\varepsilon_0} = \frac{2}{\varepsilon_1^2}\cdot\frac{1}{\varepsilon_0}\sup_{x\in X}|f_{-1}|.$$

Apply the Lemma with $K = K_0 = 1/\varepsilon_0$, $N = N_0$, and $g_0 = f_{-1}$ to obtain a bounded measurable function f_0 such that properties (a_0)–(d_0) are satisfied.

The general odd step: Assume that f_{2j} is defined for a $j = 0,1,\ldots$ Set

$$N_{2j+1} = \frac{2}{\varepsilon_{2j+2}^2}\cdot\frac{1}{\varepsilon_{2j+1}}\sup_{x\in X}|f_{2j}|.$$

Apply the Lemma by reversing the role of S and T with $K = K_{2j+1} = 1/\varepsilon_{2j+1}$, $N = N_{2j+1}$ and $g_0 = f_{2j}$. This yields a function f_{2j+1} with properties (a_{2j+1})–(e_{2j+1}).

The general even step: Assume that f_{2j+1} is defined for a $j = 0, 1, \ldots$ Set

$$N_{2j+2} = \frac{2}{\varepsilon_{2j+3}^2} \cdot \frac{1}{\varepsilon_{2j+2}} \sup_{x\in X} |f_{2j+1}|.$$

Apply the Lemma for S and T with $K = K_{2j+2} = 1/\varepsilon_{2j+2}$, $N = N_{2j+2}$ and $g_0 = f_{2j+1}$. This yields a function f_{2j+2} satisfying (a_{2j+2})–(e_{2j+2}).

It is clear that the functions f_j defined above have properties (i)–(ii). Next we verify (iii). From $\int_X f_{2j-1}\,d\mu = 1$ it follows that $\sup_{x\in X} |f_{2j-1}(x)| \ge 1$; hence $1/\varepsilon_{2j+1} = K_{2j+1} < N_{2j}$. Thus using (e_{2j+1}) we infer $E_{2j+1,2j} \subset \bigcup_{k=0}^{K_{2j+1}} S^k E^{2j} \subset \bigcup_{k=0}^{N_{2j}} S^k E^{2j}$. Therefore

$$\bigcup_{k=0}^{N_{2j}} S^{-k} E_{2j+1,2j} \subset \bigcup_{k=-N_{2j}}^{N_{2j}} S^{-k} E^{2j}.$$

Now, (b_{2j}) implies

$$\mu\Big(\bigcup_{k=0}^{N_{2j}} S^{-k} E_{2j+1,2j}\Big) \le \mu\Big(\bigcup_{k=-N_{2j}}^{N_{2j}} S^{-k} E^{2j}\Big) < 2\varepsilon_{2j}.$$

From (d_{2j+1}) and (c_{2j}) we obtain

$$\begin{aligned}\Big|\sum_{k=0}^{M} (f_{2j+1} - f_{2j})(S^k x)\Big| &\le \frac{2}{\varepsilon_{2j+1}} \sup_{x\in X} |f_{2j}(x)| \le \frac{2}{\varepsilon_{2j+1}} \cdot \frac{1}{\varepsilon_{2j}} \sup_{x\in X} |f_{2j-1}(x)| \\ &= N_{2j}\varepsilon_{2j+1} < N_{2j}\varepsilon_{2j}\end{aligned}$$

for all $M = 0, 1, \ldots$ and $x \in X$. Claim 1 implies that $g = f_{2j+1} - f_{2j}$ is (S, ε_{2j})-good. A similar argument shows (iv). This completes the proof of the Theorem.

Proof of the Lemma. Let $N_0 = (2/\varepsilon)N$ and $\varepsilon_0 = \varepsilon/(2(2N+1))$. Using Rokhlin's Lemma with ε_0 and R_{N_0} choose a measurable set B such that

(i) the sets $\{T^j S^k B : (j,k) \in R_{N_0}\}$ are disjoint, and
(ii) letting $E_0^1 = \bigcup_{(j,k)\in R_{N_0}} T^j S^k B$ we have $\mu(E_0^1) > 1 - \varepsilon_0$.

Observe that from (i)–(ii) it follows that $1 - \varepsilon_0 < 2N_0^2 \mu(B) \le 1$.

We will call the system $\{T^j S^k B : (j,k) \in R_{N_0}\}$ a *Rokhlin tower* corresponding to ε_0 and R_{N_0}. The set $C_j = \bigcup_{k=1}^{2N_0} T^j S^k B$ is called the jth *column* of the tower.

If $j \in \{1, \ldots, N_0\varepsilon\}$ and $x \in C_{jK}$ then set $g_1(x) = \sum_{k=0}^{K-1} g_0(T^{-k}x)$; at other points of E_0^1 set $g_1(x) = 0$. If $x \notin E_0^1$ set $g_1(x) = g_0(x)$. From this definition it follows that $|g_1(x)| \le K \sup_{y\in X} |g_0(y)|$ for all $x \in X$. This proves property (c). Since T^{-k} is measure preserving it is not difficult to see that $\int_{E_0^1} g_1\,d\mu = \int_{E_0^1} g_0\,d\mu$. On the other hand, for $x \notin E_0^1$, $g_1(x) = g_0(x)$. This implies (a).

Set $E_{00}^1 = \bigcup_{j=1}^{N_0\varepsilon} C_{jK}$ and $E_1^1 = X \setminus E_0^1$. The definition of g_1 implies that its support, E^1, is covered by $E_{00}^1 \cup E_1^1$. We also have $\mu(E_1^1) < \varepsilon_0$, and $\mu(E_{00}^1) = 2\varepsilon N_0^2 \mu(B)$.

If $x \in E_{00}^1$ and $g_1(x) \neq 0$ then there exists $0 \le k < K$ such that $g_0(T^{-k}x) \neq 0$; hence $x \in T^k E^0$ for this k. Since $g_1 - g_0$ is 0 on E_1^1, its support, $E_{1,0}$, is a subset of $\bigcup_{k=0}^{K} T^k E^0$. This shows (e).

On the other hand

$$\bigcup_{k=-N}^{N} S^k E_{00}^1 = \bigcup_{k=-N}^{N} \bigcup_{j=1}^{N_0\varepsilon} \bigcup_{l=1}^{2N_0} S^k T^{jK} S^l B = \bigcup_{j=1}^{N_0\varepsilon} \bigcup_{l=-N+1}^{2N_0+N} T^{jK} S^l B;$$

hence

$$\mu\Big(\bigcup_{k=-N}^{N} S^k E_{00}^1\Big) \le N_0\varepsilon(2N_0 + 2N)\mu(B) = \varepsilon 2N_0^2\Big(1 + \frac{\varepsilon}{2}\Big)\mu(B) < \frac{3}{2}\varepsilon.$$

Clearly $\mu(\bigcup_{k=-N}^{N} S^k E_1^1) = (2N+1)\varepsilon_0 < \varepsilon/2$. Since $E^1 \subset E_{00}^1 \cup E_1^1$ the above inequalities imply that (b) also holds.

Assume that $T^{k'}x \in C_{jK-(K-1)}$ for a $j \in \{1,\ldots,N_0\varepsilon\}$. Then $T^{k'+K-1}x \in C_{jK}$ and hence

$$(2)\qquad \sum_{k=k'}^{k'+K-1} (g_1 - g_0)(T^k x) = g_1(T^{k'+K-1}x) - \sum_{k=0}^{K-1} g_0(T^{-k}(T^{k'+K-1}x)) = 0.$$

Given $x \in X$ choose $k_0 \ge 0$ such that $x, \ldots, T^{k_0-1}x \notin E_0^1$ but $T^{k_0}x \in E_0^1$. If there is no such k_0 then $(g_1 - g_0)(T^k x) = 0$ for all k and this implies property (d). If there is such a k_0 then choose $k_0 \le k_1 < k_0 + K$ such that $T^{k_1}x \in C_{j_1K-(K-1)}$ for a $j_1 \in \{1,\ldots,N_0\varepsilon\}$, or $T^{k_1}x \notin E_0^1$ and $T^{k'}x \in E_0^1$ for $k_0 \le k' < k_1$.

Next we choose a sequence $k_1 < k_2 < \ldots$ such that for each n either $T^{k_n}x \notin E_0^1$, or if $T^{k_n}x \in E_0^1$ then there exists $j_n \in \{1,\ldots,N_0\varepsilon\}$ such that $T^{k_n}x \in C_{j_nK-(K-1)}$. If $j_n < N_0\varepsilon$ then set $k_{n+1} = k_n + K$ and $j_{n+1} = j_n + 1$. In this case $T^{k_{n+1}}x \in C_{j_{n+1}K-(K-1)}$.

If $j_n = N_0\varepsilon$ then again set $k_{n+1} = k_n + K$. Observe that $T^{k_{n+1}-1}x \in C_{KN_0\varepsilon} = C_{N_0}$, which is the "last column" of the tower. Since $C_j = T^{-1}C_{j+1}$ when $j < N_0$, if $T^{k_{n+1}}x \in E_0^1$ then $T^{k_{n+1}}x \in C_1 = C_{K-(K-1)}$ and we can set $j_{n+1} = 1$.

Now assume that for some n, $T^{k_n}x \notin E_0^1$. Then $(g_1 - g_0)(T^{k_n}x) = 0$. Set $k_{n+1} = k_n + 1$. If $T^{k_{n+1}}x \notin E_0^1$ then repeat the above process. If $T^{k_{n+1}}x \in E_0^1$ then it is again easy to see that $T^{-1}(T^{k_{n+1}}x) = T^{k_n}x \notin E_0^1$ implies that $T^{k_{n+1}}x \in C_1 = C_{K-(K-1)}$. Set again $j_{n+1} = 1$.

If $n \geq 1$ and $T^{k_n}x \in E_0^1$ then (2) used with $k' = k_n$ implies

$$\sum_{k=k_n}^{k_{n+1}-1} (g_1 - g_0)(T^k x) = 0. \tag{3}$$

If $T^{k_n}x \notin E_0^1$ then $k_{n+1} - 1 = k_n$ and from $(g_1 - g_0)(T^{k_n}x) = 0$ it follows that (3) holds in this case as well. Therefore we have (3) for $n = 1, 2, \dots$

It is also clear that $k_{n+1} - k_n \leq K$ and if $k_n < M < k_{n+1}$ then

$$\Big| \sum_{k=k_n}^{M} (g_1 - g_0)(T^k x) \Big| = \Big| \sum_{k=k_n}^{M} g_0(T^k x) \Big| \leq K \sup_{x \in X} |g_0(x)|.$$

One can easily see that

$$\Big| \sum_{k=k_0}^{k_1-1} (g_1 - g_0)(T^k x) \Big| = \Big| \sum_{k=k_1-K}^{k_1-1} g_0(T^k x) - \sum_{k=k_0}^{k_1-1} g_0(T^k x) \Big| \leq K \sup_{x \in X} |g_0(x)|.$$

Finally for $0 \leq k < k_0$ we have $(g_1 - g_0)(T^k x) = 0$. As $K = 1/\varepsilon$ we obtain, for any M,

$$\Big| \sum_{k=0}^{M} (g_1 - g_0)(T^k x) \Big| \leq \frac{2}{\varepsilon} \sup_{x \in X} |g_0(x)|.$$

This proves (d) and concludes the proof of the Lemma.

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