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Analytic determinacy and $0^{\#}$
A forcing-free proof of Harrington's theorem

by

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Abstract. We prove the following theorem: *Given* $a \subseteq \omega$ *and* $1 \le \alpha < \omega_1^{\mathrm{CK}}$, *if for some* $\eta < \aleph_1$ *and all* $u \in \mathbf{WO}$ *of length* η, *a is* $\Sigma^0_\alpha(u)$, *then a is* Σ^0_α. We use this result to give a new, forcing-free, proof of Leo Harrington's theorem: Σ^1_1-*Turing-determinacy implies the existence of* $0^{\#}$.

A major step in delineating the precise connections between large cardinals and game-determinacy hypotheses is the well-known theorem: *For any real a,* $\Sigma^1_1(a)$ *games are determined if and only if* $a^{\#}$ *exists.* The "if" part is due to D. A. Martin [Mr2], and the "only if" part is Leo Harrington's [Hg] (1). Harrington's proof of this result is quite complex, relying on a fine analysis due to John Steel [Sl] of the ordinal-collapse forcing relation (a variant of this proof is given in Mansfield and Weitkamp's [MW].)

We propose here a new, forcing-free and quite elementary proof, Theorem 3.9. Our proof is built upon a new ordinal-definability theorem, for reals, which is interesting in its own right, namely Theorem 2.4:

For $\alpha < \omega_1^{\mathrm{CK}}$, *if a real is* Σ^0_α *in* (*all codes of*) *some countable ordinal*, *it is* Σ^0_α.

A further simplification is brought about by the use of an easily defined game (Definition 3.2) avoiding metamathematical notions. In §4, using the same techniques, we sketch a proof of a related result of Harrington.

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(1) For an excellent mathematical and chronological account of the context of this last result, describing *inter alia* the important contributions of H. Friedman and D. A. Martin, see Kanamori's [Kn], §31.

1. Preliminaries and background

1.1. We refer to Moschovakis' [Ms] for (effective) descriptive set theory and for the theory of infinite games. $\mathcal{C} = \mathcal{P}(\omega)$ is the Cantor space or the set of *reals*, $\mathcal{R} = \mathcal{P}(\omega \times \omega)$ is the space of relations on ω and S_∞ is the space of permutations of ω, each equipped with its usual recursively presented Polish topology. Basic hyperarithmetic theory and the connection with admissible sets and ordinals are assumed (see Sacks' [Sc2] or [MW]). We will make use of the effective Borel hierarchy Σ^0_α, $\alpha < \omega_1^{\mathrm{CK}}$, and its relativizations. The reader who is averse to the effective hierarchy can easily recast all statements and proofs below in terms of Δ^1_1 sets. This leads to slightly shorter proofs of somewhat less transparent statements. (We have stated, in Remarks 2.5(c) and 3.7, "Δ^1_1 versions" of the key steps towards the main result.)

1.2. Let $R \subseteq \mathcal{X} \times \mathcal{Y}$ where $\mathcal{Y}$ is a topological space. Recall that the category quantifier "$\exists^* y(R(x,y))$" stands for: *the set* $\{y \in \mathcal{Y} \mid R(x,y)\}$ *is non-meager in* $\mathcal{Y}$. We will make use of the category computations from Kechris' [Kc]: *For* $R \subseteq \mathcal{X} \times \mathcal{Y}$, *where* $\mathcal{X}$ *and* $\mathcal{Y}$ *are recursively presented Polish spaces and* R *is* Σ^0_α *with* $\alpha < \omega_1^{\mathrm{CK}}$ [*resp.* R *is* Δ^1_1], *the relation* $\exists^* y(R(-,y))$ *is* Σ^0_α [*resp.* Δ^1_1].

1.3. Linear orderings will be taken to be reflexive, that is, non-strict. $\mathbf{LO} = \{r \subseteq \omega \times \omega \mid r \text{ is a linear ordering of its field}\}$. For $r \in \mathbf{LO}$, $\leqslant_r$ is just r and $<_r$ has the usual meaning. Next, $\mathbf{WO} = \{r \in \mathbf{LO} \mid <_r \text{ is well founded}\}$. For $r \in \mathbf{WO}$, $|r|$ will denote its length and for $\alpha < \aleph_1$, $\mathbf{WO}_\alpha = \{r \in \mathbf{WO} \mid |r| = \alpha\}$. Given $r, u \in \mathbf{LO}$, with the same order-type, it is not necessarily the case that $(\omega, r) \cong (\omega, u)$; we will implicitly use the easy fact that there is $u' \leqslant_{\mathrm{T}} u$ such that $(\omega, r) \cong (\omega, u')$. For $k \in \omega$, define the *restriction* $r{\restriction}k = \{(m,n) \mid m <_r k \;\&\; n <_r k \;\&\; m \leqslant_r k\}$. Note that if $k \notin \mathrm{Field}(r)$, $r{\restriction}k = \emptyset$. The function $(r,k) \mapsto r|k$ is recursive.

1.4. The following result, due to J. Silver, is instrumental to the proof. Martin was the first to use it to derive the existence of $0^\#$ from determinacy hypotheses. A proof can be found in [MW, 7.22] or in [Hg, §1].

THEOREM. *If there is a real* c *such that every* c*-admissible ordinal is an* L*-cardinal then* $0^\#$ *exists.*

2. Reals simply defined from ordinals

2.1. Recall that $r \in \mathbf{LO}$ is called a *pseudo-well-ordering* if any non-empty $\Delta^1_1(r)$ subset of $\mathrm{Field}(r)$ has an r-least element. $\mathbf{pWO}$ will denote the set of such orderings. Obviously, $\mathbf{pWO} \supseteq \mathbf{WO}$ and, by a standard computation, $\mathbf{pWO}$ is Σ^1_1. Harrison in [Hn] has shown that, for any $u \in \mathbf{pWO} - \mathbf{WO}$, $\mathrm{OrderType}(u) = \omega_1^u \cdot (1 + \boldsymbol{\eta}) + \varrho_u$, where $\boldsymbol{\eta}$ is the order-type of the rationals, and $\varrho_u < \omega_1^u$.

2.2. LEMMA. *Any* $r \in \mathbf{pWO}$ *for which* $\omega_1^r = \omega_1^{\mathrm{CK}}$ *has an isomorphic recursive copy.*

Proof. If r is a well-ordering, then $|r| < \omega_1^{\mathrm{CK}}$. Thus the conclusion, by definition of ω_1^{CK}.

If, instead, $r \in \mathbf{pWO} - \mathbf{WO}$, then $\mathrm{OrderType}(r) = \omega_1^{\mathrm{CK}} \cdot (1 + \boldsymbol{\eta}) + \varrho_r$, where $\varrho_r < \omega_1^{\mathrm{CK}}$. An easy boundedness argument shows that $\{u \in \mathbf{WO} \mid u \text{ is recursive}\}$ is not Σ_1^1, whereas $\{u \in \mathbf{pWO} \mid u \text{ is recursive}\}$ is Σ_1^1. Pick a recursive $u \in \mathbf{pWO} - \mathbf{WO}$. By trimming some excess, if needed, we may assume $\mathrm{OrderType}(u) = \omega_1^{\mathrm{CK}} \cdot (1 + \boldsymbol{\eta})$. Informally, then, by stringing together u and a recursive well-ordering of length ϱ_r, one constructs a recursive copy of r. ■

2.3. Given $f \in S_\infty$ and $r \subseteq \omega \times \omega$ we denote by $f \cdot r$ the isomorphic copy of r by f. Note that $(f, r) \mapsto f \cdot r$ is a recursive map $S_\infty \times \mathcal{R} \to \mathcal{R}$. Suppose $r, u \subseteq \omega \times \omega$ are isomorphic, say via $g : (\omega, r) \to (\omega, u)$. For any $Z \subseteq \mathcal{R}$, $\{f \mid f \cdot r \in Z\} = \{f \mid f \cdot u \in Z\} \circ g$. Right multiplication by g being a homeomorphism of S_∞, the topological properties of $\{f \mid f \cdot r \in Z\}$ and $\{f \mid f \cdot u \in Z\}$ are identical.

2.4. THEOREM. *Given* $a \in \mathcal{C}$ *and* $1 \leqslant \alpha < \omega_1^{\mathrm{CK}}$, *if for some* $\eta < \aleph_1$ *and all* $u \in \mathbf{WO}_\eta$, a *is* $\Sigma_\alpha^0(u)$, *then* a *is*, *in fact*, Σ_α^0.

Proof. Let $U \subseteq \omega \times \mathcal{R} \times \omega$ be ω-universal for the Σ_α^0 subsets of $\mathcal{R} \times \omega$. Fix $r \in \mathbf{WO}_\eta$. From the hypothesis, for all $f \in S_\infty$ there is $e \in \omega$ such that $a = U(e, f \cdot r, -)$. The Baire Category Theorem yields an $e_0 \in \omega$ such that $\{f \mid a = U(e_0, f \cdot r, -)\}$ is non-meager in S_∞. Set $U_0 = U(e_0, -, -)$. Assume now—towards a contradiction—that a is not Σ_α^0. Consider the set

$$A = \{(x, v) \mid x \in \mathcal{C} \text{ is not } \Sigma_\alpha^0 \ \&\ v \in \mathbf{pWO} \ \&\ \exists^* f \in S_\infty (x = U_0(f \cdot v, -))\}.$$

We first check that A is Σ_1^1. Indeed, "x is Σ_α^0" is a Δ_1^1 property of x, $\mathbf{pWO}$ is Σ_1^1. Finally, "$x = U_0(f \cdot v, -)$" is a Δ_1^1 property of (x, f, v), thus, by the category computations of 1.2, the third conjunct in the definition of A is Δ_1^1. Further since $(a, r) \in A$, A is not empty. By the Gandy Basis Theorem [Gn], let $(x_0, v_0) \in A$ be such that $\omega_1^{(x_0, v_0)} = \omega_1^{\mathrm{CK}}$. It follows, *a fortiori*, that $\omega_1^{v_0} = \omega_1^{\mathrm{CK}}$. Let now, by 2.2 above, w_0 be a recursive copy of v_0. By 2.3, we have $\exists^* f \in S_\infty(x_0 = U_0(f \cdot w_0, -))$, since $\{f \mid x_0 = U_0(f \cdot w_0, -)\}$ is a translate in S_∞ of $\{f \mid x_0 = U_0(f \cdot v_0, -)\}$. Let $V \subseteq S_\infty$ be a non-empty basic open set such that $\{f \mid x_0 = U_0(f \cdot w_0, -)\}$ is comeager in V. A straightforward category argument now yields

$$k \in x_0 \Leftrightarrow \exists^* f \in V(U_0(f \cdot w_0, k)).$$

Note that, since w_0 is recursive, "$U_0(f \cdot w_0, k)$", as a relation in (f, k), is Σ_α^0. The category computations of 1.2 now yield that the R.H.S. is Σ_α^0; yet, by the definition of A, x_0 is not Σ_α^0. This contradiction finishes the proof. ■

2.5. Remarks. (a) The case $\alpha = 1$ of this result was proved, by a different method, in the author's [Sm, 2.5]. It was used there to establish a weak precursor of Harrington's theorem.

(b) Our proof shows: *If for some* $u \in \mathbf{pWO}$, $\{f \mid a \text{ is } \Sigma^0_\alpha(f \cdot u)\}$ *is non-meager in* S_∞, *then* a *is*, *in fact*, Σ^0_α. This less quotable version of the theorem could be easier to apply.

(c) The "Δ^1_1 version" of 2.4 should read: *Given* $a \in C$, *if there is* $u \in \mathbf{WO}$ *and a* Δ^1_1 *relation* $D \subseteq \mathcal{R} \times \omega$ *such that* $\forall f \in S_\infty(a = D(f \cdot u, -))$, *then* a *is* Δ^1_1.

3. Harrington's theorem

3.1. As usual, $\leqslant_{\mathrm{T}}$ and $\leqslant_{\mathrm{h}}$ denote respectively Turing and hyperarithmetic reducibility. A set of reals is said to be *Turing-closed* if it is closed under Turing equivalence $=_{\mathrm{T}}$. Harrington's theorem proceeds from the, *a priori* weaker, hypothesis of determinacy of Σ^1_1 games with Turing-closed payoff sets (henceforth: Σ^1_1-*Turing-determinacy*). For $c \in \mathcal{C}$, define the *Turing cone* $\mathrm{Cone}(c) = \{x \in \mathcal{C} \mid c \leqslant_{\mathrm{T}} x\}$. Recall Martin's Lemma [Mr1]: *For a Turing-closed set* A, *the infinite game over* A *is determined if and only if* A *or its complement includes a cone*.

3.2. Definition. For $a, b \in \mathcal{C}$, set

$$a \sqsubset b \Leftrightarrow \forall x \leqslant_{\mathrm{h}} a(x \leqslant_{\mathrm{T}} b) \ \&\ \omega_1^a = \omega_1^b$$

and let $\mathcal{S} = \{z \in \mathcal{C} \mid \exists y(y \sqsubset z)\}$.

It is clear, by a direct computation, that the relation $\sqsubset$ is Σ^1_1. The set $\mathcal{S}$ is the payoff set of the game we are going to use to derive the existence of $0^{\#}$.

3.3. Proposition. $\mathcal{S}$ *is* Σ^1_1, *Turing-closed and cofinal in the Turing degrees.*

Proof. That $\mathcal{S}$ is Turing-closed and Σ^1_1 is immediate from its definition and the complexity of the relation $\sqsubset$. To prove that $\mathcal{S}$ is cofinal, let $a \in \mathcal{C}$ and set $A = \{z \in \mathcal{C} \mid \forall x \leqslant_{\mathrm{h}} a(x \leqslant_{\mathrm{T}} z)\}$. Then A is $\Sigma^1_1(a)$ and non-empty. By Gandy's Basis Theorem, let $b \in A$ be such that $\omega_1^b \leqslant \omega_1^a$. Note that $a \leqslant_{\mathrm{T}} b$; thus one gets $\omega_1^a = \omega_1^b$ and hence $a \sqsubset b$. Consequently, $b \in \mathcal{S}$. ∎

We shall need the following well-known complexity computations; a proof is sketched for the reader's convenience. (The bound here is quite loose, for optimal results see Stern's [Sr].)

3.4. Lemma. *For* $\alpha < \aleph_1$,

(a) $\mathbf{WO}_\alpha$ *is* $\mathbf{\Sigma}^0_{\alpha+2}$.

(b) *Given* $r \in \mathbf{WO}_\alpha$, *the relation "*$u \in \mathbf{WO}_{|r\restriction k|}$*" in* (u, k) *is* $\Sigma^0_{\alpha+2}(r)$.

P r o o f. (a) is proved by induction on α. First, $\mathbf{WO}_0$ is Π^0_1. Now, if α is a limit ordinal, then

$$u \in \mathbf{WO}_\alpha \Leftrightarrow \bigwedge_{\xi<\alpha} \bigvee_{k<\omega} (u{\restriction}k \in \mathbf{WO}_\xi) \ \& \ \bigwedge_{k<\omega} \bigvee_{\xi<\alpha} (u{\restriction}k \in \mathbf{WO}_\xi)$$

(this holds even if $\mathrm{Field}(u) \neq \omega$). Using the inductive hypothesis, $\mathbf{WO}_\alpha$ is computed to be in $\mathbf{\Pi}^0_{\alpha+1} \subseteq \mathbf{\Sigma}^0_{\alpha+2}$. Finally, if $\alpha = \beta + 1$, then

$$u \in \mathbf{WO}_\alpha \Leftrightarrow u \in \mathbf{LO} \ \& \ \exists k (k \text{ is } \leqslant_u\text{-maximum} \ \& \ u{\restriction}k \in \mathbf{WO}_\beta)$$

and the R.H.S. is readily checked to be in $\mathbf{\Sigma}^0_{\beta+2} \subseteq \mathbf{\Sigma}^0_{\alpha+2}$.

(b) is just the effective version of (a). ■

3.5. Given $\alpha < \aleph_1$, $r \in \mathbf{WO}_\alpha$ and $X \subseteq \alpha$, let $\iota_r : (\mathrm{Field}(r), r) \to (\alpha, \leqslant)$ be the canonical isomorphism, and set $\mathrm{Code}(X, r) = \iota_r^{-1}[X]$. Observe that if M is an admissible set and $r \in M$, then $\iota_r \in M$ and thus $X \in M \Leftrightarrow \mathrm{Code}(X, r) \in M$.

We can now state the key technical property of the elements of $\mathcal{S}$.

3.6. LEMMA. *Let* $a \in \mathcal{S}$, $\alpha < \omega_1^a$ *and* $r \in \mathbf{WO}_\alpha$. *For all* $X \in \mathcal{P}(\alpha) \cap \mathsf{L}_{\omega_1^a}$, $\mathrm{Code}(X, r)$ *is* $\Sigma^0_{\alpha+2}(a, r)$.

P r o o f. Let $a' \sqsubset a$. Since $\omega_1^{a'} = \omega_1^a$, we have $\alpha < \omega_1^{a'}$; let then $r' \in \mathbf{WO}_\alpha$ be recursive in a' and such that $(\omega, r') \cong (\omega, r)$. Set $x = \mathrm{Code}(X, r)$ and $x' = \mathrm{Code}(X, r')$. Since $X, r' \in \mathsf{L}_{\omega_1^{a'}}[a']$, it follows by 3.5 that $x' \in \mathsf{L}_{\omega_1^{a'}}[a']$. Consequently, $x' \leqslant_{\mathrm{h}} a'$ and, since $a' \sqsubset a$, $x' \leqslant_{\mathrm{T}} a$. Now, for $k \in \omega$, one can easily verify that

$$k \in x \Leftrightarrow \exists k' (k' \in x' \ \& \ (\omega, r', k') \cong (\omega, r, k)).$$

We claim that the R.H.S. is $\Sigma^0_{\alpha+2}(a, r)$. Indeed, since $x' \leqslant_{\mathrm{T}} a$, "$k' \in x'$" is a $\Sigma^0_1(a)$ property of k'. Set now $I_r(r', k', k) \Leftrightarrow (\omega, r', k') \cong (\omega, r, k)$. Since $(\omega, r') \cong (\omega, r)$, $I_r(r', k', k)$ is equivalent to $[k \in \mathrm{Field}(r) \Leftrightarrow k' \in \mathrm{Field}(r')]$ & $r'{\restriction}k' \in \mathbf{WO}_{|r{\restriction}k|}$. By 3.4(b), I_r is $\Sigma^0_{\alpha+2}(r)$, and since $r' \leqslant_{\mathrm{T}} a$, $I_r(r', -, -)$ is $\Sigma^0_{\alpha+2}(a, r)$. Thus the claim follows. ■

3.7. REMARK. The, somewhat less intuitive, "Δ^1_1 version" of this last result should read: *Given* $r \in \mathbf{WO}$, *and setting* $\alpha = |r|$, *there is a* $\Delta^1_1(r)$ *set* $D_r \subseteq \mathcal{C} \times \omega \times \omega$ *such that*, *for any* $a \in \mathcal{S}$, *if* $\omega_1^a > \alpha$ *then for all* $X \in \mathcal{P}(\alpha) \cap \mathsf{L}_{\omega_1^a}$, *there is* $e \in \omega$ *such that* $\mathrm{Code}(X, r) = D_r(a, e, -)$.

The next proposition is the heart of the proof we are aiming at. Its proof makes essential use of Theorem 2.4.

3.8. PROPOSITION. *If a Turing cone* $\mathrm{Cone}(c)$ *is included in* $\mathcal{S}$ *then every* c*-admissible ordinal is an* L*-cardinal.*

P r o o f. By a standard downward Löwenheim–Skolem argument, it suffices to verify that every *countable* c-admissible ordinal is an L-cardinal. Further, we know that by Sacks' Theorem [Sc1] every countable c-admissible ordinal $> \omega$ has the form ω_1^d, for some $d \in \mathrm{Cone}(c)$. For such a d, $\mathrm{Cone}(d) \subseteq \mathcal{S}$. It suffices, thus, to show that $\mathrm{Cone}(c) \subseteq \mathcal{S}$ implies that ω_1^c is an L-cardinal.

Assume the contrary. Thus there is $\varrho < \omega_1^c$ and $W \subseteq \varrho \times \varrho$, a *constructible* well-ordering of ϱ, of length ω_1^c. Fix $r \in \mathbf{WO}_\varrho$ recursive in c. *Via* some simple constructible bijection $\varrho \to \varrho \times \varrho$, code W as a subset $A \subseteq \varrho$. Say $A \in \mathsf{L}_\sigma$, where $\sigma < \aleph_1$. Pick any $s \in \mathbf{WO}_\sigma$; since $\sigma < \omega_1^s \leqslant \omega_1^{c \oplus s}$, $A \in \mathsf{L}_{\omega_1^{c\oplus s}}$. Now $c \oplus s \in \mathrm{Cone}(c) \subseteq \mathcal{S}$, thus, applying Lemma 3.6 to $c \oplus s$, $\mathrm{Code}(A, r)$ is $\Sigma^0_{\varrho+2}(c \oplus s, r)$. Consequently, since $r \leqslant_{\mathrm{T}} c$, $\mathrm{Code}(A, r)$ is $\Sigma^0_{\varrho+2}(c \oplus s)$. This being true for every $s \in \mathbf{WO}_\sigma$, Theorem 2.4 relativized to c yields that $\mathrm{Code}(A, r)$ is $\Sigma^0_{\varrho+2}(c)$. Thus $\mathrm{Code}(A, r) \in \mathsf{L}_{\omega_1^c}[c]$ and, since $r \leqslant_{\mathrm{T}} c$, this entails that $A \in \mathsf{L}_{\omega_1^c}[c]$ and thus $W \in \mathsf{L}_{\omega_1^c}[c]$. This in turn contradicts the admissibility of $\mathsf{L}_{\omega_1^c}[c]$. ∎

Our concluding statement is now but a direct consequence of what precedes.

3.9. Theorem (Harrington [Hg]). *Σ^1_1-Turing-determinacy implies the existence of $0^\#$.*

P r o o f. Since $\mathcal{S}$ is Σ^1_1 and cofinal in the degrees, Σ^1_1-Turing-determinacy implies, *via* Martin's Lemma, that there is a cone $\mathrm{Cone}(c) \subseteq \mathcal{S}$. By 3.8, every c-admissible ordinal is an L-cardinal. Thus, by Silver's Theorem 1.4, $0^\#$ exists. ∎

4. Borel reducibility of analytic sets

4.1. For $A, B \subseteq \mathcal{C}$ let $A \leqslant_{\mathcal{B}} B$ stand for: A is many-one reducible to B *via* a Borel function. In [Hg] Harrington proves the following:

4.2. Theorem. *If for all Σ^1_1 sets A, B, $A \leqslant_{\mathcal{B}} B$ whenever B is not Borel, then $0^\#$ exists.*

The technique used in the previous section can be easily adapted to prove this result as well. We just sketch the main steps.

Let U be the closure under isomorphism of $\mathbf{pWO} - \mathbf{WO}$. Then U is Σ^1_1 and it is easily checked that neither U nor $\mathcal{S}$ is Borel. From the hypothesis, let $F : \mathcal{R} \to \mathcal{R}$ be a Borel reduction of U to $\mathcal{S}$.

Observe first that for all $\xi < \aleph_1$ there is $u \in U$ such that $\exists^* f(\xi < \omega_1^{F(f \cdot u)})$. Indeed, otherwise, one argues that for some $\xi < \aleph_1$,

$$u \in U \Leftrightarrow \forall^* f(F(f \cdot u) \in \{x \in \mathcal{S} \mid \omega_1^x \le \xi\})$$

and the R.H.S. is Borel.

F being in $\mathbf{\Delta}^1_1$, say F is $\Delta^1_1(c)$. We claim that every c-admissible ordinal is an L-cardinal. For that it suffices to show that ω_1^c is such.

Argue as in 3.8. Let $\varrho < \omega_1^c$, and $A \subseteq \varrho$ be constructible. Say $A \in \mathsf{L}_\sigma$, with σ countable. To show $A \in \mathsf{L}_{\omega_1^c}[c]$ let $u \in U$ be such that $\exists^* f(\sigma < \omega_1^{F(f\cdot u)})$. For any $r \in \mathbf{WO}_\varrho$, using 3.6 one gets

$$\exists^* f(\mathrm{Code}(A, r) \text{ is } \Sigma^0_{\varrho+2}(F(f\cdot u), r)).$$

Now we can assume ϱ to be large enough relative to the Borel rank of F and $r \leqslant_{\mathrm{T}} c$. It follows that $\exists^* f(\mathrm{Code}(A, r)$ is $\Sigma^0_{\varrho+2}(f\cdot u, c))$. Using Theorem 2.4 (as generalized in 2.5(b)) one concludes that $\mathrm{Code}(A, r)$ is $\Sigma^0_{\varrho+2}(c)$. Thus $A \in \mathsf{L}_{\omega_1^c}[c]$, as desired.

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