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When a partial Borel order is linearizable

by

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Abstract. We prove a classification theorem of the "Glimm–Effros" type for Borel order relations: a Borel partial order on the reals either is Borel linearizable or includes a copy of a certain Borel partial order $\leq_0$ which is not Borel linearizable.

NOTATION. A binary relation $\preccurlyeq$ on a set X is a *partial quasi-order*, or *p.q.-o.* in brief, on X, iff $x \preccurlyeq y \wedge y \preccurlyeq z \Rightarrow x \preccurlyeq z$, and $x \preccurlyeq x$ for any $x \in X$. In this case, $\approx$ is the associated equivalence relation, i.e. $x \approx y$ iff $x \preccurlyeq y \wedge y \preccurlyeq x$.

If in addition $x \approx x \Rightarrow x = x$ for any x then $\preccurlyeq$ is a *partial order*, or *p.o.*, so that, say, forcing relations are p.q.-o.'s, but, generally speaking, not p.o.'s in this terminology.

A p.o. is *linear* (l.o.) iff we have $x \preccurlyeq y \vee y \preccurlyeq x$ for all x, $y \in X$.

Let $\preccurlyeq$ and $\preccurlyeq'$ be p.q.-o.'s on resp. X and X'. A map $h : X \to X'$ will be called *half order preserving*, or *h.o.p.*, iff $x \preccurlyeq y \Rightarrow h(x) \preccurlyeq' h(y)$.

DEFINITION 1. A Borel p.q.-o. $\langle X; \preccurlyeq\rangle$ is *Borel linearizable* iff there is a Borel l.o. $\langle X'; \preccurlyeq'\rangle$ and a Borel h.o.p. map $h : X \to X'$ (called a *linearization map*) satisfying $x \approx y \Leftrightarrow h(x) = h(y)$ ([1]).

Introduction. Harrington, Marker, and Shelah [2] proved several theorems on Borel partial order relarions, mainly concerning *thin* p.q.-o.'s, i.e. those which do not admit uncountable pairwise incomparable subsets. In particular, they demonstrated that any such Borel p.q.-o. is Borel linearizable, and moreover the corresponding l.o. $\langle X'; \preccurlyeq'\rangle$ can be chosen as a suborder of $\langle 2^\alpha; \leq_{\text{lex}}\rangle$ for some $\alpha < \omega_1$, where $\leq_{\text{lex}}$ is the lexicographical order.

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([1]) The equivalence cannot be dropped in this definition as otherwise a one-element set X' works in any case.

As elementary examples show that thinness is not a necessary condition for Borel linearizability, this result leaves open the problem of linearization of non-thin Borel p.q.-o.'s. Harrington *et al.* wrote in [2] that "there is little to say about nonthin orderings", although there are many interesting among them like the *dominance* order on ω^ω.

Our main result will say that not all Borel p.q.-o.'s are Borel linearizable, and there exists a *minimal* one, in a certain sense, among them.

DEFINITION 2. Let $a, b \in 2^\omega$. We define $a \le_0 b$ iff either $a = b$ or $a \mathsf{E}_0 b$ ([2]) and $a(k_0) < b(k_0)$ where k_0 is the largest k such that $a(k) \ne b(k)$ ([3]).

The relation $\le_0$ is a Borel p.q.-o. on 2^ω which orders every E_0-class similarly to the integers $\mathbb{Z}$ (except for the class $[\omega \times \{0\}]_{\mathsf{E}_0}$ ordered as ω and the class $[\omega \times \{1\}]_{\mathsf{E}_0}$ ordered as ω^*, the inverted order) but leaves any two E_0-inequivalent reals incomparable.

The following is the main result of the paper.

THEOREM 3. *Suppose that $\preccurlyeq$ is a Borel p.q.-o. on $\mathcal{N} = \omega^\omega$. Then exactly one of the following two conditions is satisfied*:

(I) $\preccurlyeq$ *is Borel linearizable*; *moreover* ([4]), *there exist an ordinal $\alpha < \omega_1$ and a Borel linearization map $h : \langle \mathcal{N}; \preccurlyeq \rangle \to \langle 2^\alpha; \le_{\text{lex}} \rangle$.*

(II) *there exists a continuous* 1-1 *map $F : 2^\omega \to \mathcal{N}$ such that we have $a \le_0 b \Rightarrow F(a) \preccurlyeq F(b)$ while $a \not\mathsf{E}_0 b$ implies that $F(a)$ and $F(b)$ are $\preccurlyeq$-incomparable* ([5]).

The theorem resembles the case of Borel equivalence relations where a necessary and sufficient condition for a Borel equivalence relation E to be *smooth* is that E_0 (which is not smooth) does not continuously embed in E (Harrington, Kechris and Louveau [1]). ($\le_0$ itself is *not* Borel linearizable.)

The proof is essentially a combination of ideas and techniques in [1, 2].

1. Incompatibility. Let us first prove that (I) and (II) are incompatible.

([2]) That is, $a(k) = b(k)$ for all but finite k, the *Vitali* equivalence relation on 2^ω.

([3]) If one enlarges $<_0$ so that, in addition, $a <_0 b$ whenever $a, b \in 2^\omega$ are such that $a(k) = 1$ and $b(k) = 0$ for all but finite k then the enlarged relation can be induced by a Borel action of $\mathbb{Z}$ on 2^ω, such that $a <_0 b$ iff $a = zb$ for some $z \in \mathbb{Z}$, $z > 0$.

([4]) The "moreover" assertion is an immediate corollary of the linearizability by the above-mentioned result of [2].

([5]) Then F associates a chain $\{F(b) : b \mathsf{E}_0 a\}$ in $\langle \mathcal{N}; \preccurlyeq \rangle$ to each E_0-class $[a]_{\mathsf{E}_0}$ so that any two different chains do not contain $\preccurlyeq$ comparable elements: let us call them *fully incomparable* chains. Thus (II) essentially says that $\preccurlyeq$ admits an effectively "large" Borel family of fully incomparable chains, which is therefore necessary and sufficient for $\preccurlyeq$ to be *not* Borel linearizable.

Suppose otherwise. The superposition of the maps F and h is then a Borel h.o.p. map $\phi : \langle 2^\omega ; \le_0 \rangle \to \langle 2^\alpha ; \le_{\rm lex} \rangle$ satisfying the following: $\phi(a) = \phi(b)$ implies that $a \, \mathsf{E}_0 \, b$, i.e. a and b are $\le_0$ comparable.

Therefore, as any E_0-class is $\le_0$-ordered similarly to $\mathbb{Z}$, ω, or ω^*, the ϕ-image $X_a = \phi\text{"}[a]_{\mathsf{E}_0}$ of the E_0-class of any $a \in 2^\omega$ is $\le_{\rm lex}$-ordered similarly to a subset of $\mathbb{Z}$. If $X_a = \{x_a\}$ is a singleton then put $\psi(a) = x_a$.

Assume now that X_a contains at least two points. In this case we can effectively pick an element in X_a! Indeed, there is a maximal sequence $u \in 2^{<\alpha}$ such that $u \subseteq x$ for each $x \in X_a$. Then the set $X_a^{\rm left} = \{x \in X : u^\wedge 0 \subseteq x\}$ contains a $\le_{\rm lex}$-largest element, which we denote by $\psi(a)$.

To conclude, ψ is a Borel reduction of E_0 to the equality on 2^α, i.e. $a \, \mathsf{E}_0 \, b$ iff $\psi(a) = \psi(b)$, which is impossible because E_0 is not a smooth Borel equivalence relation (see [1]).

2. The dichotomy. As usual, it will be assumed that the p.q.-o. $\preccurlyeq$ of Theorem 3 is a $\varDelta^1_1$ relation. Let $\approx$ denote the associated equivalence.

Following [2] let, for $\alpha < \omega_1^{\rm CK}$, $\mathcal{F}_\alpha$ be the family of all h.o.p. $\varDelta^1_1$ functions $f : \langle \mathcal{N} ; \preccurlyeq \rangle \to \langle 2^\alpha ; \le_{\rm lex} \rangle$. Then $\mathcal{F} = \bigcup_{\alpha < \omega_1^{\rm CK}} \mathcal{F}_\alpha$ is a (countable) $\varPi^1_1$ set, in a suitable coding system for functions of this type. (See [2] for details.)

Define, for $x, y \in \mathcal{N}$, $x \equiv y$ iff $f(x) = f(y)$ for any $f \in \mathcal{F}$.

LEMMA 4 (see [2]). *$\equiv$ is a $\varSigma^1_1$ equivalence relation including $\approx$.*

Proof. As $\preccurlyeq$ is $\varDelta^1_1$, one gets by a rather standard argument a $\varPi^1_1$ set $N \subseteq \omega$ and a function $f_n \in \mathcal{F}$ for any $n \in N$ so that $\mathcal{F} = \{f_n : n \in N\}$ and the relations $n \in N \wedge f_n(x) \le_{\rm lex} f_n(y)$ and $n \in N \wedge f_n(x) <_{\rm lex} f_n(y)$ are presentable in the form $n \in N \wedge \mathcal{O}(x,y)$ and $n \in N \wedge \mathcal{O}'(x,y)$ where $\mathcal{O}$, $\mathcal{O}'$ are $\varSigma^1_1$ relations. Now $x \equiv y$ iff $\forall n \, (n \in N \Rightarrow f_n(x) = f_n(y))$, as required. ∎

CASE 1: *$\equiv$ coincides with $\approx$.* Let us show how this implies (I) of Theorem 3. The set

$$P = \{\langle x, y, n \rangle : x \not\approx y \wedge f_n(x) \ne f_n(y)\}$$

is $\varPi^1_1$ and, by the assumption of Case 1, its projection on x, y coincides with the complement of $\approx$. Let $Q \subseteq P$ be a $\varPi^1_1$ set uniformizing P in the sense of $\mathcal{N}^2 \times \omega$. Then Q is $\varDelta^1_1$ because

$$Q(x,y,n) \Leftrightarrow x \not\approx y \wedge \forall n' \ne n \; (\neg Q(x,y,n')).$$

It follows that $N' = \{n : \exists x, y \; Q(x,y,n)\} \subseteq N$ is $\varSigma^1_1$. Therefore by the $\varSigma^1_1$ separation theorem there is a $\varDelta^1_1$ set M such that $N' \subseteq M \subseteq N$ ([6]).

Consider a $\varDelta^1_1$ enumeration $M = \{n_l : l \in \omega\}$. For any l, $f_{n_l} \in \mathcal{F}_\alpha$ for some ordinal $\alpha = \alpha_l < \omega_1^{\rm CK}$. Another standard argument (see

([6]) Harrington *et al.* [2] use a general reflection theorem to get such a set, but a more elementary reasoning sometimes has advantage.

[2]) shows that in this case (e.g. when $M \subseteq N$ is a Δ^1_1 set) the ordinals α_l are bounded by some $\alpha < \omega_1^{\mathrm{CK}}$. It follows that the function $h(x) = f_{n_0}(x)^\wedge f_{n_1}(x)^\wedge f_{n_2}(x)^\wedge \ldots^\wedge f_{n_l}(x)^\wedge \ldots$ belongs to some $\mathcal{F}_\beta$, $\beta \le \alpha \cdot \omega$. On the other hand, by the construction we have $x \approx y \Leftrightarrow h(x) = h(y)$, hence h satisfies (I) of Theorem 3.

CASE 2: $\approx \subsetneqq \equiv$. Assuming this we work towards (II) of Theorem 3.

3. The domain of singularity. By the assumption the Σ^1_1 set $A = \{x : \exists y\ (x \approx y \wedge x \not\equiv y)\}$ is non-empty.

Define $X \equiv Y$ iff we have $\forall x \in X\ \exists y \in Y\ (x \equiv y)$ and *vice versa*.

PROPOSITION 5. *Let $X, Y \subseteq A$ be non-empty Σ^1_1 sets satisfying $X \equiv Y$. Then the sets*

$$P_+ = \{\langle x, y\rangle \in X \times Y : x \equiv y \wedge x \preccurlyeq y\}, \quad \textit{and}$$
$$P_- = \{\langle x, y\rangle \in X \times Y : x \equiv y \wedge x \not\preccurlyeq y\}$$

are non-empty Σ^1_1 sets, their projections (7) $\mathtt{pr}_1 P^+$ *and* $\mathtt{pr}_1 P^-$ *are Σ^1_1-dense in X* (8), *while the projections* $\mathtt{pr}_2 P^+$ *and* $\mathtt{pr}_2 P^-$ *are Σ^1_1-dense in Y.*

Proof. The density easily follows from the non-emptiness, so let us concentrate on the latter. *We prove that $P_+ \ne \emptyset$.*

Suppose on the contrary that $P_+ = \emptyset$. Then there is a single function $f \in \mathcal{F}$ such that the set $\{\langle x, y\rangle \in X \times Y : f(x) = f(y) \wedge x \preccurlyeq y\}$ is empty. (See the reasoning in Case 1 of Section 2.) Define

$$X_\infty = \{x : \forall y \in Y\ (f(x) = f(y) \Rightarrow x \not\preccurlyeq y)\},$$

so that X_∞ is a Π^1_1 set and $X \subseteq X_\infty$ but $Y \cap X_\infty = \emptyset$. Using separation, we can easily define an increasing sequence of sets

$$X = X_0 \subseteq U_0 \subseteq X_1 \subseteq U_1 \subseteq \ldots \subseteq X_n \subseteq U_n \subseteq \ldots \subseteq X_\infty$$

so that $U_n = \{x' : \exists x \in X_n\ (f(x) = f(x') \wedge x \preccurlyeq x')\}$ while $X_{n+1} \in \Delta^1_1$ for all n. (Note that if $X_n \subseteq X_\infty$ and U_n is defined as indicated then $U_n \subseteq X_\infty$ too.) Moreover, a proper execution of the construction (9) allows getting the final set $U = \bigcup_n U_n = \bigcup_n X_n$ in Δ^1_1. Note that $X \subseteq U$, but $Y \cap U = \emptyset$ since $U \subseteq X_\infty$.

Put $f'(x) = f(x)^\wedge 1$ whenever $x \in U$, and $f'(x) = f(x)^\wedge 0$ otherwise. *We assert that $f' \in \mathcal{F}$.* Indeed, suppose that $x' \preccurlyeq y'$; we prove $f'(x') \le_{\mathrm{lex}} f'(y')$.

(7) For a set $P \subseteq \mathcal{N}^2$, $\mathtt{pr}_1 P$ and $\mathtt{pr}_2 P$ have the obvious meaning of the projections on the resp. 1st and 2nd copy of $\mathcal{N}$.

(8) That is, each of them intersects any non-empty Σ^1_1 set $X' \subseteq X$.

(9) We refer to the proof of an "invariant" effective separation theorem in [1], which includes a similar construction.

It can be assumed that $f(x') = f(y')$. It remains to check that $x' \in U \Rightarrow y' \in U$, which easily follows from the definition of the sets U_n. Thus $f' \in \mathcal{F}$.

However, clearly $f'(x) \neq f'(y)$, hence $x \not\equiv y$, whenever $x \in X$ and $y \in Y$, which contradicts the assumption that $X \equiv Y$.

Now *we prove that* $P_- \neq \emptyset$. Consider first the case $X = Y$. Suppose on the contrary that $P_- = \emptyset$. Then, as above, there is a single function $f \in \mathcal{F}$ such that the set $\{\langle x, y\rangle \in X^2 : f(x) = f(y) \wedge x \not\preccurlyeq y\}$ is empty, so that $\equiv$ and $\approx$ coincide on X. Our plan is to find functions $f', f'' \in \mathcal{F}$ such that

$$Q' = \{\langle x, y\rangle \in X \times \mathcal{N} : f'(x) = f'(y) \wedge y \not\preccurlyeq x\},$$
$$Q'' = \{\langle x, y\rangle \in X \times \mathcal{N} : f''(x) = f''(y) \wedge x \not\preccurlyeq y\}$$

are empty sets; then $Q = \{\langle x, y\rangle \in X \times \mathcal{N} : x \equiv y \wedge y \not\approx x\} = \emptyset$, which contradicts $\emptyset \neq X \subseteq A$.

Let us find f'; the case of the other function is similar. Define

$$X_\infty = \{x : \forall x' \in X\ (f(x) = f(x') \Rightarrow x \preccurlyeq x')\},$$

so that X_∞ is Π^1_1 and $X \subseteq X_\infty$. As above there is a sequence of sets

$$X = X_0 \subseteq U_0 \subseteq X_1 \subseteq U_1 \subseteq \ldots \subseteq X_n \subseteq U_n \subseteq \ldots \subseteq X_\infty$$

such that $U_n = \{u : \exists x \in X_n\ (f(x) = f(u) \wedge u \preccurlyeq x)\}$ while $X_{n+1} \in \Delta^1_1$ for all n and the final set $U = \bigcup_n U_n = \bigcup_n X_n$ belongs to Δ^1_1.

Set $f'(x) = f(x)^\wedge 0$ whenever $x \in U$, and $f'(x) = f(x)^\wedge 1$ otherwise. Then $f' \in \mathcal{F}$. We prove that f' witnesses that $Q' = \emptyset$. Consider any $x \in X$ and $y \in \mathcal{N}$ such that $f'(x) = f'(y)$. Then in particular $f(x) = f(y)$ and $x \in U \Leftrightarrow y \in U$, so that $y \in U$ because we know that $x \in X \subseteq U$. Thus $y \in X_\infty$, so by definition $y \preccurlyeq x$, as required.

Finally, we prove $P_- \neq \emptyset$ in the general case. By the result for the case $X = Y$, the Σ^1_1 set $P' = \{\langle x, x'\rangle \in X^2 : x \equiv x' \wedge x \not\preccurlyeq x'\}$ is non-empty. Let $X' = \{x' \in X : \exists x\ P'(x, x')\}$ and $Y' = \{y \in Y : \exists x' \in X'\ (x' \equiv y)\}$, so that X', Y' are Σ^1_1 sets satisfying $X' \equiv Y'$. By the result for P_+ there exist $x' \in X'$ and $y \in Y'$ satisfying $x' \equiv y$ and $y \preccurlyeq x'$. Now there is $x \in X$ such that $x \equiv x'$ and $x \not\preccurlyeq x'$. Then $x \equiv y$ and $x \not\preccurlyeq y$, as required. ∎

4. The forcing notions involved. Our further strategy will be the following. We shall define a generic extension of the universe $\mathbf{V}$ (where Theorem 3 is being proved) in which there exists a function F which witnesses (II) of Theorem 3. However, as the existence of such a function is a Σ^1_2 statement, we obtain the result for $\mathbf{V}$ by the Shoenfield absoluteness theorem [10].

Definition 6. $\mathbb{P}$ is the collection of all non-empty Σ^1_1 sets $X \subseteq A$.

[10] In fact, the proof can be conducted without any use of metamathematics, as in [1], but at the cost of longer reasoning.

It is a standard fact that $\mathbb{P}$ (the *Gandy forcing*) forces a real which is the only real which belongs to every set in the generic set $G \subseteq \mathbb{P}$. (We identify Σ^1_1 sets in the ground universe $\mathbf{V}$ with their copies in the extension.)

Definition 7. $\mathbb{P}^+_2$ is the collection of all non-empty Σ^1_1 sets $P \subseteq A^2$ such that $P(x,y) \Rightarrow x \equiv y \wedge x \preccurlyeq y$. The collection $\mathbb{P}^-_2$ is defined similarly but with the requirement $P(x,y) \Rightarrow x \equiv y \wedge x \not\preccurlyeq y$ instead.

Both $\mathbb{P}^+_2$ and $\mathbb{P}^-_2$ are non-empty forcing notions by Proposition 5. Each of them forces a pair of reals $\langle x,y\rangle \in A^2$ satisfying resp. $x \preccurlyeq y$ and $x \not\preccurlyeq y$.

Definition 8. $\mathbb{P}^2_\equiv$ is the collection of all sets of the form $\Upsilon = X \times Y$ where X, Y are sets in $\mathbb{P}$ satisfying $X \equiv Y$.

Lemma 9. $\mathbb{P}^2_\equiv$ *forces a pair of reals* $\langle x,y\rangle$ *such that* $x \not\preccurlyeq y$.

Proof. Suppose that, on the contrary, a condition $\Upsilon_0 = X_0 \times Y_0$ in $\mathbb{P}^2_\equiv$ forces $x \preccurlyeq y$. Consider a more complicated forcing $\mathfrak{P}$ which consists of forcing conditions of the form $\mathfrak{p} = \langle \Upsilon, P, \Upsilon', Q\rangle$, where $\Upsilon = X \times Y$ and $\Upsilon' = X' \times Y'$ belong to $\mathbb{P}^2_\equiv$, $P \in \mathbb{P}^+_2$, $P \subseteq Y \times X'$, $Q \in \mathbb{P}^-_2$, $Q \subseteq X \times Y'$, and the sets $\mathtt{pr}_1\, P \subseteq Y$, $\mathtt{pr}_2\, P \subseteq X'$, $\mathtt{pr}_1\, Q \subseteq X$ and $\mathtt{pr}_2\, Q \subseteq Y'$ are Σ^1_1-dense in resp. Y, X', X, Y'.

For instance, setting $P_0 = \{\langle y,x'\rangle \in Y_0 \times X_0 : y \equiv x' \wedge y \preccurlyeq x'\}$ and $Q_0 = \{\langle x,y'\rangle \in X_0 \times Y_0 : x \equiv y' \wedge x \not\preccurlyeq y'\}$, we get a condition $\mathfrak{p}_0 = \langle \Upsilon_0, P_0, \Upsilon_0, Q_0\rangle \in \mathfrak{P}$ by Proposition 5.

It is the principal fact that if $\mathfrak{p} = \langle \Upsilon, P, \Upsilon', Q\rangle \in \mathfrak{P}$ and we strengthen one of the components within the corresponding forcing notion then this can be appropriately reflected in the other components. To be concrete assume that, for instance, $P^* \in \mathbb{P}^+_2$, $P^* \subseteq P$, and find a condition $\mathfrak{p}_1 = \langle \Upsilon_1, P_1, \Upsilon'_1, Q_1\rangle \in \mathfrak{P}$ satisfying $\Upsilon_1 \subseteq \Upsilon$, $\Upsilon'_1 \subseteq \Upsilon'$, $P_1 \subseteq P^*$, and $Q_1 \subseteq Q$.

Assume that $\Upsilon = X \times Y$ and $\Upsilon' = X' \times Y'$. Consider the non-empty Σ^1_1 sets $Y_2 = \mathtt{pr}_1\, P^* \subseteq Y$ and $X_2 = \{x \in X : \exists y \in Y_2\ (x \equiv y)\}$. It follows from Proposition 5 that $Q_1 = \{\langle x,y\rangle \in Q : x \in X_2\} \neq \emptyset$, hence Q_1 is a condition in $\mathbb{P}^-_2$ and $X_1 = \mathtt{pr}_1\, Q_1$ is a non-empty Σ^1_1 subset of $X_2 \subseteq X$.

The set $Y_1 = \{y \in Y_2 : \exists x \in X_1\ (x \equiv y)\}$ satisfies $X_1 \equiv Y_1$, therefore $\Upsilon_1 = X_1 \times Y_1 \in \mathbb{P}^2_\equiv$. Furthermore, $P_1 = \{\langle y,x\rangle \in P^* : y \in Y_1\} \in \mathbb{P}^+_2$.

Put $X'_1 = \mathtt{pr}_2\, P_1 \subseteq X'$ and $Y'_1 = \mathtt{pr}_2\, Q_1 \subseteq Y'$. Notice that $Y_1 \equiv X'_1$ because any condition in $\mathbb{P}^+_2$ is a subset of $\equiv$, similarly $X_1 \equiv Y'_1$, and $X_1 \equiv Y_1$ (see above). It follows that $X'_1 \equiv Y'_1$, hence $\Upsilon'_1 = X'_1 \times Y'_1$ is a condition in $\mathbb{P}^2_\equiv$.

Now $\mathfrak{p}_1 = \langle \Upsilon_1, P_1, \Upsilon'_1, Q_1\rangle \in \mathfrak{P}$ as required.

We conclude that $\mathfrak{P}$ forces "quadruples" of reals $\langle x,y,x',y'\rangle$ such that the pairs $\langle x,y\rangle$ and $\langle x',y'\rangle$ are $\mathbb{P}^2_\equiv$-generic, hence satisfy $x \preccurlyeq y$ and $x' \preccurlyeq y'$ provided the generic set contains Υ_0—by the assumption above. Furthermore, the pair $\langle y,x'\rangle$ is $\mathbb{P}^+_2$-generic, hence $y \preccurlyeq x'$, while the pair $\langle x,y'\rangle$ is $\mathbb{P}^-_2$-generic, hence $x \not\preccurlyeq y'$, which is a contradiction. ∎

5. The splitting construction. Let, in the universe $\mathbf{V}$, $\kappa = 2^{\aleph_0}$. Let $\mathbf{V}^+$ be a κ-collapse extension of $\mathbf{V}$.

Our aim is to define, in $\mathbf{V}^+$, a splitting system of sets which leads to a function F satisfying (II) of Theorem 3. Let us fix two points before the construction starts.

First, as the forcing notions involved are countable in $\mathbf{V}$, there exist, in $\mathbf{V}^+$, enumerations $\{D(n) : n \in \omega\}$, $\{D_2(n) : n \in \omega\}$, and $\{D^2(n) : n \in \omega\}$ of all open dense sets in resp. $\mathbb{P}$, $\mathbb{P}_2^+$, $\mathbb{P}_{\equiv}^2$, which (the dense sets) belong to $\mathbf{V}$, such that $D(n+1) \subseteq D(n)$ etc. for each n.

Second, we introduce the notion of a crucial pair. A pair $\langle u, v\rangle$ of binary sequences $u, v \in 2^n$ is called *crucial* iff $u = 1^k{}^\wedge 0^\wedge w$ and $v = 0^k{}^\wedge 1^\wedge w$ for some $k < n$ and $w \in 2^{n-k-1}$. One easily sees that the graph of all crucial pairs in 2^n is actually a chain connecting all members of 2^n.

We define, in $\mathbf{V}^+$, a system of sets $X_u \in \mathbb{P}$, where $u \in 2^{<\omega}$, and sets $P_{uv} \in \mathbb{P}_2^+$, $\langle u, v\rangle$ being a crucial pair in some 2^n, satisfying the following conditions:

(1) $X_u \in D(n)$ whenever $u \in 2^n$; $X_{u^\wedge i} \subseteq X_u$;

(2) if $\langle u, v\rangle$ is a crucial pair in 2^n then $P_{uv} \in D_2(n)$ and $P_{u^\wedge i,\, v^\wedge i} \subseteq P_{uv}$;

(3) if $u, v \in 2^n$ and $u(n-1) \neq v(n-1)$ then $X_u \times X_v \in \mathbb{P}_{\equiv}^2$, $X_u \times X_v \in D^2(n)$, and $X_u \cap X_v = \emptyset$;

(4) if $\langle u, v\rangle$ is a crucial pair in 2^n then $\mathtt{pr}_1\, P_{uv} = X_u$ and $\mathtt{pr}_2\, P_{uv} = X_v$.

Why does this imply the existence of a required function? First of all for any $a \in 2^\omega$ (in $\mathbf{V}^+$) the sequence of sets $X_{a\restriction n}$ is $\mathbb{P}$-generic over $\mathbf{V}$ by (1), therefore the intersection $\bigcap_{n\in\omega} X_{a\restriction n}$ is a singleton. Let $F(a) \in \mathcal{N}$ be its only element.

It does not take much effort to prove that F is continuous and 1-1.

Consider $a, b \in 2^\omega$ satisfying $a \not\mathsf{E}_0\, b$. Then $a(n) \neq b(n)$ for infinitely many n, hence the pair $\langle F(a), F(b)\rangle$ is $\mathbb{P}_{\equiv}^2$-generic by (3), thus $F(a)$ and $F(b)$ are $\preccurlyeq$-incomparable by Lemma 9.

Consider $a, b \in 2^\omega$ satisfying $a \leq_0 b$. We may assume that a and b are $\leq_0$-neighbours, i.e. $a = 1^k{}^\wedge 0^\wedge c$ while $b = 0^k{}^\wedge 1^\wedge c$ for some $k \in \omega$ and $c \in 2^\omega$. Then by (2) the sequence of sets $P_{a\restriction n,\, b\restriction n}$, $n > k$, is $\mathbb{P}_2^+$-generic, hence it results in a pair of reals satisfying $x \preccurlyeq y$. However, $x = F(a)$ and $y = F(b)$ by (4).

The construction of a splitting system. We argue in $\mathbf{V}^+$.

Suppose that the construction has been completed up to a level n; we will expand it to the next level. From now on s, t will denote sequences in 2^n while u, v will denote sequences in 2^{n+1}.

To start with, we set $X_{s^\wedge i} = X_s$ for all $s \in 2^n$ and $i = 0, 1$, and $P_{s^\wedge i,\, t^\wedge i} = P_{st}$ whenever $i = 0, 1$ and $\langle s, t\rangle$ is a crucial pair in 2^n.

For the "initial" crucial pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$ at this level let $P_{1^n{}^\wedge 0,\, 0^n{}^\wedge 1} = X_{1^n{}^\wedge 0} \times X_{0^n{}^\wedge 1} = X_{1^n} \times X_{0^n}$. Then $P_{1^n{}^\wedge 0,\, 0^n{}^\wedge 1} \in \mathbb{P}^2_{\equiv}$ ([11]).

This ends the definition of "initial values" at the $(n+1)$th level. The plan is to gradually "shrink" the sets in order to fulfill the requirements.

Step 1. We take care of item (1). Consider an arbitrary $u_0 = s_0{}^\wedge i \in 2^{n+1}$. As $D(n)$ is dense there is a set $X' \in D(n)$ with $X' \subseteq X_{u_0}$. The intention is to take X' as the "new" X_{u_0}. But this change has to be expanded through the chain of crucial pairs, in order to preserve (4).

Thus put $X'_{u_0} = X'$. Suppose that X'_u has been defined and is included in X_u, the "old" version, for some $u \in 2^{n+1}$, and $\langle u, v\rangle$ is a crucial pair, $v \in 2^{n+1}$ being not yet encountered. Define $P'_{uv} = (X'_u \times \mathcal{N}) \cap P_{uv}$ and $X'_v = \mathtt{pr}_2\, P'_{uv}$. Clearly (4) holds for the "new" sets X'_u, X'_v, and P'_{uv}.

The construction describes how the original change from X_{u_0} to X'_{u_0} spreads through the chain of crucial pairs in 2^{n+1}, resulting in a system of new sets, X'_u and P'_{uv}, which satisfy (1) for the particular $u_0 \in 2^{n+1}$. We iterate this construction consecutively for all $u_0 \in 2^{n+1}$, getting finally a system of sets satisfying (1) (fully) and (4), which we shall denote by X_u and P_{uv} from now on.

Step 2. We take care of item (3). Fix a pair of u_0 and v_0 in 2^{n+1} such that $u_0(n) = 0$ and $v_0(n) = 1$. By the density of $D^2(n)$, there is a set $X'_{u_0} \times X'_{v_0} \in D^2(n)$ included in $X_{u_0} \times X_{v_0}$. We may assume that $X'_{u_0} \cap X'_{v_0} = \emptyset$. (Indeed, it easily follows from Proposition 5, for P_-, that there exist reals $x_0 \in X_{u_0}$ and $y_0 \in X_{v_0}$ satisfying $x_0 \equiv y_0$ but $x_0 \neq y_0$, say $x_0(k) = 0$ while $y_0(k) = 1$. Define

$$X = \{x \in X_0 : x(k) = 0 \wedge \exists y \in Y_0\ (y(k) = 1 \wedge x \equiv y)\},$$

and Y correspondingly; then $X \equiv Y$ and $X \cap Y = \emptyset$.)

Spread the change from X_{u_0} to X'_{u_0} and from X_{v_0} to X'_{v_0} through the chain of crucial pairs in 2^{n+1}, by the method of Step 1, until the wave of spreading from u_0 meets the wave of spreading from u_0 at the "meeting" crucial pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$. This leads to a system of sets X'_u and P'_{uv} which satisfy (3) for the particular pair $\langle u_0, v_0\rangle$ and still satisfy (4) possibly except for the "meeting" crucial pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$ (for which basically the set $P'_{1^n{}^\wedge 0,\, 0^n{}^\wedge 1}$ is not yet defined at this step).

Note that Step 1 leaves $P_{1^n{}^\wedge 0,\, 0^n{}^\wedge 1}$ in the form $X_{1^n{}^\wedge 0} \times X_{0^n{}^\wedge 1}$ (where $X_{1^n{}^\wedge 0}$ and $X_{0^n{}^\wedge 1}$ are the "versions" at the end of Step 1). We now have the "new" sets, $X'_{1^n{}^\wedge 0}$ and $X'_{0^n{}^\wedge 1}$, included in resp. $X_{1^n{}^\wedge 0}$ and $X_{0^n{}^\wedge 1}$ and satisfying $X'_{0^n{}^\wedge 0} \equiv X'_{0^n{}^\wedge 1}$ (because we had $X'_{u_0} \equiv X'_{v_0}$ at the beginning of

([11]) It easily follows from (2) and (4) that $X_s \equiv X_t$ for all $s, t \in 2^n$, because s and t are connected in 2^n by a unique chain of crucial pairs.

the change). It remains to define $P'_{1^n{}^\wedge 0,\, 0^n{}^\wedge 1} = X'_{1^n{}^\wedge 0} \times X'_{0^n{}^\wedge 1}$. This ends the consideration of the pair $\langle u_0, v_0 \rangle$.

Applying this construction consecutively for all pairs of $u_0 \in P_0$ and $v_0 \in P_1$ (including the pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1 \rangle$) we finally get a system of sets satisfying (1), (3), and (4), which will be denoted still by X_u and P_{uv}.

STEP 3. We finally take care of (2). Consider a crucial pair $\langle u_0, v_0 \rangle$ in 2^{n+1}. By density, there exists a set $P'_{u_0,v_0} \in D_2(n)$ with $P'_{u_0,v_0} \subseteq P_{u_0,v_0}$. (In the case when $\langle u_0, v_0 \rangle$ is the pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1 \rangle$ we rather apply Proposition 5 to obtain the set P'_{u_0,v_0}.)

Define $X'_{u_0} = \mathtt{pr}_1 \, P'_{u_0,v_0}$ and $X'_{v_0} = \mathtt{pr}_2 \, P'_{u_0,v_0}$ and spread this change through the chain of crucial pairs in 2^{n+1}. (Note that $X'_{u_0} \equiv X'_{v_0}$ as sets in $\mathbb{P}^2_{\equiv}$ are included in $\equiv$. This keeps $X'_u \equiv X'_v$ for all $u, v \in 2^{n+1}$ through the spreading.)

Executing this step for all crucial pairs in 2^{n+1}, we finally end the construction, in $\mathbf{V}^+$, of a system of sets satisfying (1) through (4). $\blacksquare_{\text{Theorem 3}}$

References

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