

**A function space $C_p(X)$
not linearly homeomorphic to $C_p(X) \times \mathbb{R}$**

by

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Abstract. We construct two examples of infinite spaces X such that there is no continuous linear surjection from the space of continuous functions $C_p(X)$ onto $C_p(X) \times \mathbb{R}$. In particular, $C_p(X)$ is not linearly homeomorphic to $C_p(X) \times \mathbb{R}$. One of these examples is compact. This answers some questions of Arkhangel'skiĭ.

1. Introduction. All spaces under consideration are completely regular. For a space X , $C_p(X)$ denotes the space of all continuous real-valued functions on X equipped with the pointwise convergence topology. For linear topological spaces E and F , we write $E \overset{t}{\approx} F$ ($E \overset{l}{\approx} F$) if these spaces are (linearly) homeomorphic.

In functional analysis and infinite-dimensional topology, quite often factorization properties of linear spaces E are considered, i.e. properties like $E \overset{l}{\approx} E \times E$, $E \overset{t}{\approx} E \times E$, $E \overset{l}{\approx} E \times \mathbb{R}$, etc. (here, we discuss only infinite-dimensional linear spaces). In most cases, linear spaces possess some of these properties, e.g. for all Banach spaces E we have $E \overset{t}{\approx} E \times E$ and $E \overset{t}{\approx} E \times \mathbb{R}$. On the other hand, we also have numerous examples of “pathological” spaces, for which many of these factorization properties do not hold. Many examples are known of normed (or linear metric) spaces E with $E \overset{l}{\not\approx} E \times \mathbb{R}$; see [BPR], [Be], [Du], [Ro]. Quite recently, Gowers and Maurey [Go], [GM] have constructed examples of Banach spaces E such that $E \overset{l}{\not\approx} E \times \mathbb{R}$ —this is a solution to an old problem in Banach space theory. In [vM2] van Mill gave an example of a normed space E with $E \overset{t}{\not\approx} E \times \mathbb{R}$ (see also [Ma3]).

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The first example of a normed space E such that $E \stackrel{t}{\not\cong} E \times E$ was given by Pol in [Po1]. The paper [Ma1] contains a construction of a normed space E without a continuous map onto $E \times E$. We also have examples of spaces X with $C_p(X) \stackrel{t}{\not\cong} C_p(X) \times C_p(X)$ (see [Ma2], [Gu]) and examples of metric spaces X such that $C_p(X) \stackrel{1}{\not\cong} C_p(X) \times C_p(X)$ (see [Po2]).

Arkhangel'skiĭ asked whether the space $C_p(X)$ is linearly homeomorphic to $C_p(X) \times \mathbb{R}$, for every infinite (compact) space X (cf. [Ar2, Problem 56], [Ar3, Problem 1] and [Ar4, Problems 24, 27]). For a wide class of spaces the answer is affirmative, e.g. if the space X contains a nontrivial convergent sequence, or X is not pseudocompact (see [Ar4, Section 4]). However, in general, the answer is negative:

1.1. **EXAMPLE.** There exists an infinite compact space X such that there is no continuous linear surjection from the function space $C_p(X)$ onto $C_p(X) \times \mathbb{R}$. In particular, the space $C_p(X)$ is not linearly homeomorphic to the product $C_p(X) \times E$, for any nontrivial linear topological space E .

In fact, we construct two examples of spaces X with the above property of the function space $C_p(X)$. The first one (noncompact) is a subspace X of $\beta\omega$, fairly simple to describe. The construction of the second example, a compact space K , is much more involved; it uses the idea of “killing maps” devised by Kuratowski and Sierpiński [KS], [Ku].

The paper is organized as follows:

Sections 2 and 3 contain some auxiliary results. In the next two sections we describe the constructions of our examples. We give some additional comments in the last section.

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2. Linear surjections of function spaces. Following Arkhangel'skiĭ, for a set (space) X , we denote by X^+ the set (space) obtained by adding one new (isolated) point $*$ to X (we hope that ω^+ will not be confused with the cardinal successor of ω). Using this notation we may identify the products $C_p(X) \times \mathbb{R}$ and $\mathbb{R}^X \times \mathbb{R}$ with $C_p(X^+)$ and \mathbb{R}^{X^+} , respectively.

For a pseudocompact space X , the Banach space of continuous functions on X equipped with the standard supremum norm is denoted by $C(X)$. If A is a dense subset of the space X , then by $C_A(X)$ we denote the space of continuous real-valued functions on X with the topology of pointwise convergence on A . Hence, we may identify $C_A(X)$ with $\{f|A : f \text{ is continuous on } X\} \subseteq \mathbb{R}^A$ and $C_X(X) = C_p(X)$.

Let E and F be dense linear subspaces of \mathbb{R}^A and \mathbb{R}^B , respectively. Let $T : E \rightarrow F$ be a continuous linear surjection. It is well-known that every continuous linear functional on \mathbb{R}^A (or on E) is a linear combination of evaluation functionals. Therefore, for every $b \in B$, there is a finite subset of A , called the *support* of b and denoted by $\text{supp}(b)$, such that $Tf(b) = \sum_{a \in \text{supp}(b)} \lambda_{ba} f(a)$ for some nonzero $\lambda_{ba} \in \mathbb{R}$ and all $f \in E$ (we refer the reader to [Ar1] and [BdG] for more information about the supports). To simplify the notation, for every $b \in B$ and $S \subseteq A$, we define $\lambda(b, S) = \sum\{|\lambda_{ba}| : a \in \text{supp}(b) \cap S\}$. For fixed $b \in B$, one may consider $\lambda(b, \cdot)$ as a measure (with finite support) on the power set of A . Let $b \in B$, $S \subseteq A$ and $f \in E$. Obviously, if $f(a) = \lambda_{ba}/|\lambda_{ba}|$ for every $a \in \text{supp}(b) \cap S$, and $f(a) = 0$ for every $a \in \text{supp}(b) \setminus S$, then $Tf(b) = \lambda(b, S)$.

2.1. LEMMA. *Let X be a separable space and $\varphi : C_p(X) \rightarrow C_p(X) \times \mathbb{R}$ be a continuous linear surjection. Then there exists a countable dense set $D \subseteq X$ such that the map $(\pi_D \times \text{id}_{\mathbb{R}})\varphi\pi_D^{-1} : C_D(X) \rightarrow C_D(X) \times \mathbb{R}$ is a continuous linear surjection. ($\pi_D : C_p(X) \rightarrow C_D(X)$ is the standard projection.)*

PROOF. We may identify $C_p(X) \times \mathbb{R}$ with $C_p(X^+)$. Take any countable dense subset $D_0 \subseteq X^+$ (obviously, D_0 contains the isolated point $*$). By induction we construct countable sets D_n , $n \in \omega$, defined by

$$D_{n+1} = \bigcup\{\text{supp}(d) : d \in D_i, i \leq n\}.$$

It is routine to verify that the set $D = \bigcup\{D_n : n \in \omega\} \setminus \{*\}$ has the required property. ■

2.2. LEMMA. *Let $T : C_D(X) \rightarrow C_E(Y)$ be a continuous linear surjection, where D and E are dense subsets of the spaces X and Y , respectively. Let $e \in E$ and U be a neighborhood of the nonempty set $S \subseteq \text{supp}(e)$ in X . Then there exists a neighborhood V of e in Y such that, for every $b \in V \cap E$, we have $\lambda(b, U \cap D) > \lambda(e, S)/2$.*

PROOF. We may assume that $\text{supp}(e) \cap U = S$. Let $f : X \rightarrow [-1, 1]$ be a continuous function such that $f(a) = \lambda_{ea}/|\lambda_{ea}|$ for every $a \in S$, and f takes the value 0 outside U . We have $Tf(e) = \lambda(e, S)$. Let V be a neighborhood of e such that, for every $b \in V \cap E$, $Tf(b) > \lambda(e, S)/2$. This easily implies that $\lambda(b, U \cap D) > \lambda(e, S)/2$. ■

Let A be a countable infinite set and let $T : \mathbb{R}^A \rightarrow \mathbb{R}^A \times \mathbb{R}$ be a continuous linear map. We may identify $\mathbb{R}^A \times \mathbb{R}$ with \mathbb{R}^{A^+} and consider T as a map between \mathbb{R}^A and \mathbb{R}^{A^+} . We say that T is a *bounded surjection* if $T|_{\ell_\infty(A)}$ is a continuous surjection of $\ell_\infty(A)$ onto $\ell_\infty(A^+)$ (with respect to the supremum norm in $\ell_\infty(A)$). Then it follows that T is a surjection onto $\mathbb{R}^A \times \mathbb{R}$ (but we will not use this fact).

For a normed space E , we denote by $B_E(r)$ the closed ball $\{x \in E : \|x\| \leq r\}$.

2.3. LEMMA. *Let X be a separable pseudocompact space and let $T : C_D(X) \rightarrow C_D(X) \times \mathbb{R}$ be a continuous linear surjection, where D is a countable dense subset of X . Then T can be uniquely extended to a continuous bounded linear surjection $T' : \mathbb{R}^D \rightarrow \mathbb{R}^D \times \mathbb{R}$.*

PROOF. Since $C_D(X)$ is a dense linear subspace of \mathbb{R}^D , the map T can be uniquely extended to a continuous linear map $T' : \mathbb{R}^D \rightarrow \mathbb{R}^D \times \mathbb{R}$. We can also consider T as a linear map between $C(X)$ and $C(X) \times \mathbb{R}$. Since the topology in $C_D(X)$ is weaker than the norm topology, it follows that the graph of T is closed in norm. Therefore by the Closed Graph Theorem T is continuous in norm. By the Open Mapping Theorem T is also an open map in the norm topology. It follows that the image $T(B_{C(X)}(1))$ of the unit ball in $C(X)$ is contained in $B_{C(X+)}(R)$ and contains $B_{C(X+)}(r)$, for some $R, r > 0$. For every $\varepsilon > 0$, the intersection $B_{\ell_\infty(D)}(\varepsilon) \cap C_D(X) = \pi_D(B_{C(X)}(\varepsilon))$ is pointwise dense in the pointwise compact ball $B_{\ell_\infty(D)}(\varepsilon)$. We have the same property for X^+ and D^+ . Hence $T'(B_{\ell_\infty(D)}(1))$ is contained in the ball $B_{\ell_\infty(D^+)}(R)$ and contains $B_{\ell_\infty(D^+)}(r)$; this shows that T' is a bounded surjection. ■

We need the following simple observation:

2.4. LEMMA. *Let E be a normed space and $F \subseteq E$ be a proper closed linear subspace of E . Then, for any positive numbers $r < R$, the algebraic sum $B_E(r) + F$ does not contain $B_E(R)$.*

PROOF. It is enough to take a functional x^* on E of norm 1 which is 0 on F , and to consider the images $x^*(B_E(r) + F)$ and $x^*(B_E(R))$. ■

2.5. LEMMA. *Let $T : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega \times \mathbb{R}$ be a bounded continuous linear surjection. Then there exist an $\varepsilon > 0$ and an infinite $A \subseteq \omega$ such that $\lambda(n, \{k \in \omega : k > n\}) > \varepsilon$ for every $n \in A$.*

PROOF. By the Open Mapping Theorem the image $T(B_{\ell_\infty(\omega)}(1))$ of the unit ball in $\ell_\infty(\omega)$ contains $B_{\ell_\infty(\omega^+)}(r)$ for some $r > 0$. Take a positive $\varepsilon < r$. We will prove that $A_\varepsilon = \{n : \lambda(n, \{k \in \omega : k > n\}) > \varepsilon\}$ is infinite.

Suppose the contrary. Let $m = \max(\bigcup\{\text{supp}(n) : n \in A_\varepsilon\} \cup \text{supp}(*)) + 1$. Then, for every $a \in m \cup \{*\}$ (we consider m as $\{i \in \omega : i < m\}$), we have

$$(1) \quad \lambda(a, \{k \in \omega : k \geq m\}) \leq \varepsilon.$$

Let $E = \mathbb{R}^m \times \{0\}^{\omega \setminus m}$ and $F = \{0\}^m \times \mathbb{R}^{\omega \setminus m}$. Obviously, $E + F = \mathbb{R}^\omega$. Hence $T(B_{\ell_\infty(\omega)}(1) \cap E) + T(B_{\ell_\infty(\omega)}(1) \cap F) = T(B_{\ell_\infty(\omega)}(1))$ contains $B_{\ell_\infty(\omega^+)}(r)$. Consider the projection $\pi_{m \cup \{*\}} : \mathbb{R}^{\omega^+} \rightarrow \mathbb{R}^{m \cup \{*\}}$. We have $\dim E = m$ and $\dim \mathbb{R}^{m \cup \{*\}} = m + 1$, hence $\pi_{m \cup \{*\}}(T(B_{\ell_\infty(\omega)}(1) \cap E))$ is contained in a

proper linear subspace of $\mathbb{R}^{m \cup \{*\}}$. On the other hand, from (1) it follows that $\pi_{m \cup \{*\}}(T(B_{\ell_\infty(\omega)}(1) \cap F))$ is contained in the ε -ball in $\mathbb{R}^{m \cup \{*\}}$ (with respect to the supremum norm). Therefore, by Lemma 2.4, the algebraic sum $\pi_{m \cup \{*\}}(T(B_{\ell_\infty(\omega)}(1) \cap E)) + \pi_{m \cup \{*\}}(T(B_{\ell_\infty(\omega)}(1) \cap F))$ cannot contain the r -ball, a contradiction. ■

2.6. LEMMA. *Let D and $E \subseteq D$ be countable dense subsets of a separable pseudocompact space X . Let $T : C_D(X) \rightarrow C_D(X) \times \mathbb{R}$ be a continuous linear surjection. Then there exist an $\varepsilon > 0$, an infinite subset $A \subseteq E$ and a family $\{S_a : a \in A\}$ of finite pairwise disjoint subsets of D with the following properties:*

- (a) $A \cap \bigcup \{S_a : a \in A\} = \emptyset$,
- (b) $\lambda(a, S_a) > \varepsilon$ for every $a \in A$.

Moreover, if D_1, \dots, D_m is a finite partition of the set D , we may additionally require that

- (c) $(\exists i \leq m)(\forall a \in A) [S_a \subseteq D_i]$.

PROOF. By Lemma 2.3 we may consider T as a bounded continuous linear surjection between \mathbb{R}^D and $\mathbb{R}^D \times \mathbb{R}$. Let $D = \{d_n : n \in \omega\}$. Using Lemma 2.5 we may find $\delta > 0$ and an infinite $B \subseteq \omega$ such that $\lambda(d_n, \{d_k : k > n\}) > \delta$ for every $n \in B$. Let $\{n_i : i \in \omega\}$ be an increasing enumeration of B . For every $i \in \omega$, put $P_i = \{d_k \in \text{supp}(d_{n_i}) : k > n_i\}$. So $\lambda(d_{n_i}, P_i) > \delta$. Using suitable refinement of B , we may assume that $n_{i+1} > \max\{k : d_k \in P_i\}$ for all $i \in \omega$. In particular, the sets P_i are disjoint and $\bigcup P_i$ is disjoint from $\{d_n : n \in B\}$.

Now, by induction we will construct distinct points $a_j \in E$ and disjoint finite sets $Q_j \subseteq \text{supp}(a_j)$ for $j \in \omega$. Suppose that a_l and Q_l for $l < j$ have been constructed, or $j = 0$. Let $H = \{a_l : l < j\} \cup \bigcup \{Q_l : l < j\}$. Find $i \in \omega$ such that $d_{n_i} \notin H$ and $P_i \cap H = \emptyset$. Let U_i be the neighborhood of P_i in X such that $\text{Cl}_X(U_i) \cap (H \cup \{d_{n_i}\}) = \emptyset$. By Lemma 2.2 (for $e = d_{n_i}$) and the density of E we can find $a_j \in E \setminus (H \cup \text{Cl}_X(U_i))$ and a finite $Q_j \subseteq \text{supp}(a_j) \cap U_i$ such that $\lambda(a_j, Q_j) > \delta/2$. One can easily verify that the set $\{a_j : j \in \omega\}$ and the sets Q_j satisfy the conditions (a) and (b) (for $\varepsilon = \delta/2$). To obtain (c) we define $B_i = \{j \in \omega : \lambda(a_j, Q_j \cap D_i) > \delta/(2m)\}$ for $i = 1, \dots, m$. One of these sets is infinite, say B_{i_0} . Then the set $A = \{a_j : j \in B_{i_0}\}$ and the sets $S_{a_j} = Q_j$ satisfy (a)–(c) for $\varepsilon = \delta/(2m)$. ■

2.7. LEMMA. *Let $T : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega \times \mathbb{R}$ be a bounded continuous linear surjection. Let A be an infinite subset of ω and let $\{S_n : n \in A\}$ be a family of pairwise disjoint subsets of ω . Then, for every $\delta > 0$, there exists an infinite subset $C \subseteq A$ with the following property:*

$$(\forall n \in C) \left[\lambda\left(n, \bigcup \{S_k : k \in C, k \neq n\}\right) < \delta \right].$$

Proof. First, we will prove that, for every $\varepsilon > 0$ and every infinite $A' \subseteq A$, there exist an $m \in A'$ and an infinite $B \subseteq A'$ such that, for every $n \in B$, we have $\lambda(n, S_m) < \varepsilon$. Suppose not, i.e. there are $\varepsilon > 0$ and $A' \subseteq A$ such that, for every $m \in A'$, the set $B_m = \{n \in A' : \lambda(n, S_m) < \varepsilon\}$ is finite. Let M be the norm of T considered as a map from $\ell_\infty(\omega)$ onto $\ell_\infty(\omega^+)$. Take a natural number $i > M/\varepsilon$. Pick distinct $m_1, \dots, m_i \in A'$ and $n \in A' \setminus \bigcup\{B_{m_j} : j \leq i\}$. Then we have

$$\lambda(n, \omega) \geq \lambda\left(n, \bigcup\{S_{m_j} : j \leq i\}\right) = \sum\{\lambda(n, S_{m_j}) : j \leq i\} \geq i\varepsilon > M.$$

This contradicts the fact that M is the norm of T .

Now, using the above property, we can construct by induction an increasing sequence of points $(n_i) \in A$ and a decreasing sequence of infinite subsets $A_i \subseteq A$, $i \in \omega$, such that:

- (i) $(\forall i \in \omega)(\forall n \in A_i) [\lambda(n, S_{n_i}) < \delta/2^{i+1}]$,
- (ii) $(\forall i \in \omega) [n_{i+1} \in A_i]$,
- (iii) $(\forall i \in \omega) [S_{n_i} \cap \bigcup\{\text{supp}(n_j) : j < i\} = \emptyset]$.

One can easily compute that the set $C = \{n_i : i \in \omega\}$ has the required property. ■

3. Auxiliary properties of $\beta\omega$. We shall formulate some rather standard properties of $\beta\omega$ that we will use in the next sections.

3.1. LEMMA. *Let $\{S_n : n \in \omega\}$ be a family of pairwise disjoint subsets of ω . Then, for every subset C of $\beta\omega \setminus \omega$ of cardinality less than 2^ω , there exists an infinite subset $A \subseteq \omega$ such that $C \cap \text{Cl}_{\beta\omega} \bigcup\{S_n : n \in A\} = C \cap \bigcup\{\text{Cl}_{\beta\omega}(S_n) : n \in A\}$. In particular, if all S_n are finite then $C \cap \text{Cl}_{\beta\omega} \bigcup\{S_n : n \in A\} = \emptyset$.*

Proof. Let \mathcal{A} be an almost disjoint family of cardinality 2^ω of infinite subsets of ω , i.e. $X \cap Y$ is finite for distinct $X, Y \in \mathcal{A}$. For subsets S, T of ω we have $\text{Cl}_{\beta\omega} S \cap \text{Cl}_{\beta\omega} T = \text{Cl}_{\beta\omega}(S \cap T)$. It follows that, for distinct $X, Y \in \mathcal{A}$, we have $\text{Cl}_{\beta\omega} \bigcup\{S_n : n \in X\} \cap \text{Cl}_{\beta\omega} \bigcup\{S_n : n \in Y\} = \text{Cl}_{\beta\omega} \bigcup\{S_n : n \in (X \cap Y)\} = \bigcup\{\text{Cl}_{\beta\omega}(S_n) : n \in (X \cap Y)\}$. Therefore the sets $\text{Cl}_{\beta\omega} \bigcup\{S_n : n \in X\} \setminus \bigcup\{\text{Cl}_{\beta\omega}(S_n) : n \in X\}$, for $X \in \mathcal{A}$, are disjoint. Since \mathcal{A} has cardinality 2^ω we can find $A \in \mathcal{A}$ such that $C \cap (\text{Cl}_{\beta\omega} \bigcup\{S_n : n \in A\} \setminus \bigcup\{\text{Cl}_{\beta\omega}(S_n) : n \in A\}) = \emptyset$, which gives us the required property. ■

For a bounded function $f : \omega \rightarrow \mathbb{R}$, by $\widehat{f} : \beta\omega \rightarrow \mathbb{R}$ we denote the unique continuous extension of f over $\beta\omega$. If X is a subset of ω then $\chi_X : \omega \rightarrow \{0, 1\}$ denotes the characteristic function of X . Lemma 3.1 easily implies the following:

3.2. COROLLARY. *Let $\{(a_\alpha, b_\alpha) : \alpha < \kappa\}$ be a set of pairs of points of $\beta\omega$ of cardinality $\kappa < 2^\omega$. Let $\{S_n : n \in \omega\}$ be a family of pairwise disjoint*

subsets of ω such that, for every $n \in \omega$ and $\alpha < \kappa$, we have $\widehat{\chi}_{S_n}(a_\alpha) = \widehat{\chi}_{S_n}(b_\alpha)$. Then there is an infinite subset $A \subseteq \omega$ such that for every $A' \subseteq A$ and for $S = \bigcup \{S_n : n \in A'\}$, we have $\widehat{\chi}_S(a_\alpha) = \widehat{\chi}_S(b_\alpha)$ for all $\alpha < \kappa$. ■

3.3. LEMMA. Let $\{f_\alpha : \omega \rightarrow \{0, 1\} : \alpha < \kappa\}$ be a set of functions of cardinality $\kappa < 2^\omega$. Let A be an infinite subset of ω . Then there exist two distinct points $a, b \in \text{Cl}_{\beta\omega} A$ such that $\widehat{f}_\alpha(a) = \widehat{f}_\alpha(b)$ for all $\alpha < \kappa$.

PROOF. This follows from the fact that $\text{Cl}_{\beta\omega} A$, being homeomorphic to $\beta\omega$, has the weight 2^ω , and therefore its points cannot be separated by the family of continuous functions of smaller cardinality. ■

4. The first example. Recall that a point $p \in \beta\omega \setminus \omega$ is a weak P -point if p is not in the closure of any countable set $D \subseteq \beta\omega \setminus (\omega \cup \{p\})$; see [vM1]. Kunen [Kun, Theorem 0.1] proved that there exist 2^{2^ω} weak P -points in $\beta\omega \setminus \omega$.

For every infinite subset A of ω we choose a weak P -point p_A in $\beta\omega \setminus \omega$ in such a way that $A \in p_A$ and p_A and $p_{A'}$ are not equivalent (via bijection of ω) for $A \neq A'$. Let $X = \omega \cup \{p_A : A \subseteq \omega, A \text{ infinite}\} \subseteq \beta\omega$. Every dense subset D of X contains ω and every continuous bounded function on D can be uniquely extended to a continuous function on X . Since every sequence in ω has an accumulation point in X , the space X is pseudocompact. Every countable subset $C \subseteq X \setminus \omega$ is closed and discrete in X . The space X has also the following property:

4.1. LEMMA. Let C be a countable subset of X and let \mathcal{A} be an infinite family of pairwise disjoint finite subsets of C . Then \mathcal{A} contains an infinite subfamily which is discrete in C .

PROOF. Since $C \setminus \omega$ is closed and discrete in X , it is enough to apply Lemma 3.1 for the family $\{S \cap \omega : S \in \mathcal{A}\}$ and the set $C \setminus \omega$. ■

4.2. THEOREM. There is no continuous linear surjection of the space $C_p(X)$ onto $C_p(X) \times \mathbb{R}$.

PROOF. Suppose the contrary. Let $\varphi : C_p(X) \rightarrow C_p(X) \times \mathbb{R}$ be a continuous linear surjection. By Lemma 2.1 there exists a countable dense set $D \subseteq X$ such that the map $T = (\pi_D \times \text{id}_{\mathbb{R}})\varphi\pi_D^{-1} : C_D(X) \rightarrow C_D(X) \times \mathbb{R}$ is a continuous linear surjection. The key point of the proof is the following:

CLAIM. There are a $\delta > 0$, a point $p \in X \setminus \omega$, an infinite set $G \subseteq \omega$, and a family $\{V_n : n \in G\}$ of clopen subsets of D such that:

- (i) $\{V_n : n \in G\}$ is discrete in D ,
- (ii) $p \in \text{Cl}_X G$,
- (iii) $\text{supp}(p) \cap \text{Cl}_X(\bigcup \{V_n : n \in G\}) = \emptyset$, where $\text{supp}(p)$ is the support of p with respect to φ ,

- (iv) $(\forall n \in G) [\lambda(n, V_n) > \delta]$,
 (v) $(\forall n \in G) [\lambda(n, \bigcup\{V_k : k \in G \setminus \{n\}\}) < \delta/2]$.

From the above claim it is easy to derive a contradiction: By (i) we can define a continuous function $f : D \rightarrow [-1, 1]$ such that f takes the value 0 outside $\bigcup\{V_n : n \in G\}$, and $f(a) = \lambda_{na}/|\lambda_{na}|$ for all $n \in G$ and $a \in \text{supp}(n) \cap V_n$. From (iv) and (v) we conclude that $Tf(n) > \delta/2$ for all $n \in G$. Let $f' : X \rightarrow [-1, 1]$ be the unique continuous extension of f . From (iii) it follows that f' takes the value 0 on $\text{supp}(p)$. Therefore $\varphi(f')(p) = 0$. Since $\varphi(f')|_G = Tf|_G$, the condition (ii) contradicts the continuity of $\varphi(f')$ at p .

It remains to prove the Claim: Applying Lemma 2.6 for $E = \omega$ we can find ε and $A \subseteq \omega$ and $S_n \subseteq D$, for $n \in A$, satisfying conditions (a) and (b) of 2.6. Using the last condition of 2.6 we may also require that, for all $n \in A$, we have $S_n \subseteq \omega$, or, for all $n \in A$, we have $S_n \cap \omega = \emptyset$. By Lemma 4.1 we may additionally assume that the family $\{S_n : n \in A\}$ is discrete in D . Let $\{n_i : i \in \omega\}$ be an increasing enumeration of A . Put $s_i = \max\{|\lambda_{n_i a}| : a \in S_{n_i}\}$. We will consider three cases:

Case 1: For every $n \in A$ we have $S_n \subseteq \omega$, and $\limsup_{i \rightarrow \infty} s_i = s > 0$. Then we can find an infinite subset $C \subseteq \omega$ such that, for all $i \in C$, there is $k_i \in S_{n_i}$ with $|\lambda_{n_i k_i}| > s/2$. Without loss of generality we may assume that $C = \omega$. Let $B = \{k_i : i \in \omega\}$; by 2.6(a) we have $A \cap B = \emptyset$. Applying Lemma 2.7 (for the sets $\{k_i\}$) we may also assume that

$$(2) \quad (\forall i \in \omega) [\lambda(n_i, B \setminus \{k_i\}) < s/4].$$

We may assume that $\omega \setminus A$ and $\omega \setminus B$ are infinite. Let $\sigma : \omega \rightarrow \omega$ be a bijection such that $\sigma(n_i) = k_i$ for every $i \in \omega$.

Now, consider the support (with respect to φ) of the point p_A . Suppose that there is $q \in \text{supp}(p_A) \setminus \omega$ such that $B \in q$. Since A and B are disjoint, $q \neq p_A$. Therefore there exists $A_q \in p_A$ such that $\sigma(A_q) \notin q$. Let $E = \bigcap\{A_q : q \in \text{supp}(p_A) \setminus \omega, B \in q\} \cap A \in p_A$. Put $F = \sigma(E)$, so that $F \notin q$ for all $q \in \text{supp}(p_A) \setminus \omega$. Let $H = F \setminus (\text{supp}(p_A) \cap \omega)$ and $G = \sigma^{-1}(H)$. Hence $\text{supp}(p_A)$ is disjoint from the closure of H in X and we still have $G \in p_A$. Now, we can define $\delta = s/2$, $p = p_A$ and $V_{n_i} = \{k_i\}$ for $n_i \in G$. It can be easily verified that all conditions of the Claim are satisfied ((i) follows from the discreteness of $\{S_n : n \in A\}$, (v) follows from (2)).

Case 2: For every $n \in A$, $S_n \subseteq \omega$ and $\limsup_{i \rightarrow \infty} s_i = 0$. Put $B = \bigcup\{S_n : n \in A\}$. Applying Lemma 2.7 we may assume that

$$(3) \quad (\forall i \in \omega) [\lambda(n_i, B \setminus S_{n_i}) < \varepsilon/4].$$

Let $r = \sup\{\lambda(n_i, S_{n_i}) : i \in \omega\}$. We have $r < \infty$ since T is bounded.

Again, consider the support of p_A . Put $G = A \setminus \{n \in A : S_n \cap \text{supp}(p_A) \neq \emptyset\}$. Obviously, $G \in p_A$.

Let m' denote the cardinality of $Q = \{q \in \text{supp}(p_A) \setminus \omega, B \in q\}$. Put $m = m' + 1$. By our assumption on (s_i) we may require that, for every $n_i \in G$ and $k \in S_{n_i}$, we have $|\lambda_{n_i k}| < \varepsilon/(4m)$. Take a natural number $p > 4mr/\varepsilon$. For every $n_i \in G$, we can find a partition $S_{n_i} = P_1^i \cup \dots \cup P_p^i$ such that $\lambda(n_i, P_j^i) < \varepsilon/(2m)$ for all $j \leq p$ (some P_j^i may be empty). Put $C_j = \bigcup \{P_j^i : n_i \in G\}$ for all $j \leq p$. So $\{C_j : j \leq p\}$ is a partition of B . For every $q \in Q$ pick $j(q)$ such that $C_{j(q)} \in q$. Let $V_{n_i} = S_{n_i} \setminus \bigcup \{C_{j(q)} : q \in Q\}$ for $n_i \in G$. The choice of G and the sets $C_{j(q)}$ guarantees that the conditions (ii) and (iii) of the Claim are satisfied for $p = p_A$. The inequality (3) implies (v) for $\delta = \varepsilon/2$. The condition (i) again follows from the discreteness of $\{S_n : n \in A\}$. Finally, for all $n_i \in G$ we have

$$(4) \quad \lambda(n_i, V_{n_i}) = \lambda\left(n_i, S_{n_i} \setminus \bigcup \{P_{j(q)}^i : q \in Q\}\right) > \varepsilon - m'(\varepsilon/(2m)) > \varepsilon/2,$$

which shows (iv).

Case 3: For every $n \in A$, $S_n \cap \omega = \emptyset$. We can find disjoint clopen neighborhoods U_n of S_n in D , for every $n \in A$. We may also require that $\{U_n : n \in A\}$ is discrete in D .

Applying Lemma 2.7 (for the sets U_n) we may assume that

$$(5) \quad (\forall n \in A) \left[\lambda\left(n, \bigcup \{U_k : k \in A, k \neq n\}\right) < \varepsilon/2 \right].$$

Put $G = A \setminus \{n \in A : S_n \cap \text{supp}(p_A) \neq \emptyset\}$ and $B = \bigcup \{S_n : n \in G\}$. The set B is closed in X and disjoint from $\text{supp}(p_A)$. Let W be a clopen neighborhood of $\text{supp}(p_A)$ in X disjoint from B . Put $V_n = U_n \setminus W$ for $n \in G$. Then $\text{Cl}_X(\bigcup \{V_n : n \in G\}) \cap \text{supp}(p_A) = \emptyset$. Since $S_n \subseteq V_n$ for $n \in G$, we have

$$(6) \quad (\forall n \in G) [\lambda(n, V_n) > \varepsilon].$$

We put $\delta = \varepsilon$ and again $p = p_A$. Inequalities (5) and (6) give us the last two conditions of the Claim. ■

5. The construction of the compact space K . First, we need to establish some notation. For a subset $Z \subseteq \{\alpha < 2^\omega\}$, $P_Z : \{0, 1\}^{2^\omega} \rightarrow \{0, 1\}^Z$ is the projection; additionally we put $p_\alpha = P_{\{\alpha\}}$ for $\alpha < 2^\omega$. Given a family of functions $\{f_\alpha : \omega \rightarrow \{0, 1\} : \alpha < 2^\omega\}$, we denote by F the diagonal map $\Delta_{\alpha < 2^\omega} f_\alpha : \omega \rightarrow \{0, 1\}^{2^\omega}$. We also define $F_Z = \Delta_{\alpha \in Z} f_\alpha : \omega \rightarrow \{0, 1\}^Z = P_Z \circ F$ for $Z \subseteq \{\alpha < 2^\omega\}$. We put $K = \text{Cl}_{\{0,1\}^{2^\omega}}(F(\omega))$ and $K_Z = \text{Cl}_{\{0,1\}^Z}(F_Z(\omega)) = P_Z(K)$.

Our second example will be the compact space K for a suitably chosen family $\{f_\alpha : \alpha < 2^\omega\}$. We will construct this family by transfinite induction “killing” all potential continuous linear surjections from $C_p(K)$ onto $C_p(K) \times \mathbb{R}$.

For every $n \in \omega$, the function $f_n : \omega \rightarrow \{0, 1\}$ is the characteristic function of the singleton $\{n\}$. Then we have $F_\omega(n) = f_n$ for $n \in \omega$.

Let \mathcal{T} be the family of all pairs (D, T) such that

- (a) D is a countable subset of $\{0, 1\}^{Z_D}$ for some countable $Z_D \subseteq \{\alpha < 2^\omega\}$ such that $\omega \subseteq Z_D$,
- (b) there is a map $\tau_D : \omega \rightarrow D$ such that $\tau_D(n)|_\omega = f_n$ for every $n \in \omega$,
- (c) $T : \mathbb{R}^D \rightarrow \mathbb{R}^D \times \mathbb{R} \approx \mathbb{R}^{D^+}$ is a bounded continuous linear surjection,
- (d) there exist an $\varepsilon > 0$, an infinite subset $A \subseteq \omega$ and a family $\{S_n : n \in A\}$ of finite pairwise disjoint subsets of D with the following properties:
 - (d1) $\tau_D(A) \cap \bigcup\{S_n : n \in A\} = \emptyset$,
 - (d2) $\lambda(\tau_D(n), S_n) > \varepsilon$ for every $n \in A$, where the function $\lambda(\cdot, \cdot)$ is defined by the surjection T ,
 - (d3) $((\forall n \in A)(\forall d \in S_n)[\lambda_{\tau_D(n)d} > 0])$ or $((\forall n \in A)(\forall d \in S_n)[\lambda_{\tau_D(n)d} < 0])$.

The family \mathcal{T} has cardinality 2^ω and we may enumerate it as $\{(D_\alpha, T_\alpha) : \omega \leq \alpha < 2^\omega\}$ in such a way that $\beta < \alpha$ for every $\beta \in Z_{D_\alpha}$. We put $Z_\alpha = Z_{D_\alpha}$ (i.e. D_α is a subset of $\{0, 1\}^{Z_\alpha}$) and $\tau_\alpha = \tau_{D_\alpha}$.

Our construction is based on the following:

5.1. LEMMA. *There exist a family of functions $\{f_\alpha : \omega \rightarrow \{0, 1\} : \alpha \in [\omega, 2^\omega)\}$ and a set of points $\{a_\alpha, b_\alpha \in \beta\omega : \alpha \in [\omega, 2^\omega)\}$ such that the following conditions are satisfied for all $\alpha \in [\omega, 2^\omega)$ (here we use the notation defined above and in Sec. 3):*

- (i) $(\forall \beta, \gamma \leq \alpha)[\widehat{f}_\gamma(a_\beta) = \widehat{f}_\gamma(b_\beta)]$,
- (ii) if $F_{Z_\alpha}(\omega) \subseteq D_\alpha \subseteq K_{Z_\alpha}$ then, for every $E \subseteq \{0, 1\}^{2^\omega}$ such that $P_{Z_\alpha}|_E$ is a bijection of E onto D_α and $P_{Z_\alpha \cup \{\alpha\}}(E) \subseteq K_{Z_\alpha \cup \{\alpha\}}$, for $g = T_\alpha(p_\alpha \circ (P_{Z_\alpha}|_E)^{-1})$ we have $g \circ \widehat{\tau}_\alpha(a_\alpha) \neq g \circ \widehat{\tau}_\alpha(b_\alpha)$.

We will prove this lemma later in this section.

Let $\{f_\alpha : \omega \rightarrow \{0, 1\} : \alpha < 2^\omega\}$ be the family consisting of the functions given by Lemma 5.1 and the previously defined functions f_n , $n \in \omega$. Let K be the compact space generated by this family of functions (see the beginning of this section for the definition of K). Then the space K has the following property:

5.2. THEOREM. *There is no continuous linear surjection of the space $C_p(K)$ onto $C_p(K) \times \mathbb{R}$.*

PROOF. Assume, towards a contradiction, that $\varphi : C_p(K) \rightarrow C_p(K) \times \mathbb{R}$ is a continuous linear surjection. By Lemma 2.1 there exists a countable dense set $E \subseteq K$ such that the map $U = (\pi_E \times \text{id}_\mathbb{R})\varphi\pi_E^{-1} : C_E(K) \rightarrow C_E(K) \times \mathbb{R}$ is a continuous linear surjection. By Lemma 2.3 we may consider U as a

bounded continuous linear surjection between \mathbb{R}^E and $\mathbb{R}^E \times \mathbb{R}$. Further, $F(\omega)$ is a dense subset of K consisting of isolated points, therefore $F(\omega) \subseteq E$. Using Lemma 2.6 (for U , E and $F(\omega)$) we can find an $\varepsilon > 0$, an infinite subset $A \subseteq \omega$ and a family $\{P_n : n \in A\}$ of finite pairwise disjoint subsets of E with the following properties:

- (e1) $F(A) \cap \bigcup\{P_n : n \in A\} = \emptyset$,
- (e2) $\lambda(F(n), P_n) > \varepsilon$ for every $n \in A$,
- (e3) $((\forall n \in A)(\forall e \in P_n) [\lambda_{F(n)e} > 0])$ or $((\forall n \in A)(\forall e \in P_n) [\lambda_{F(n)e} < 0])$.

Take a countable subset $Z \subseteq \{\alpha < 2^\omega\}$ such that $\omega \subseteq Z$ and the projection P_Z is injective on E . Let $D = P_Z(E)$, $\tau_D = F_Z$ and $S_n = P_Z(P_n)$ for $n \in A$. Using the bijection $P_Z|E$ between E and D we can define the bounded continuous surjection $T : \mathbb{R}^D \rightarrow \mathbb{R}^D \times \mathbb{R}$ corresponding to U . From (e1)–(e3) it easily follows that the pair (D, T) satisfies all conditions defining the family \mathcal{T} . Therefore there is an $\alpha < 2^\omega$ such that $(D, T) = (D_\alpha, T_\alpha)$ and $Z = Z_\alpha$. Then the inclusions from the beginning of the condition (ii) are satisfied, and E satisfies the assumptions from (ii). Hence, for $g = T(p_\alpha \circ (P_Z|E)^{-1})$, we have $g \circ \widehat{\tau_D}(a_\alpha) \neq g \circ \widehat{\tau_D}(b_\alpha)$. Observe that $g \circ \tau_D = g \circ F_Z = U(p_\alpha|E) \circ F$. Obviously, $p_\alpha|E \in C_E(K)$, so $q = U(p_\alpha|E) \in C_{E^+}(K^+)$ and $q \circ \widehat{F}(a_\alpha) \neq q \circ \widehat{F}(b_\alpha)$. Let $G : \beta\omega \rightarrow K$ be a continuous extension of the map $F : \omega \rightarrow K$. Then $G = \Delta_{\alpha < 2^\omega} \widehat{f}_\alpha$ and from the condition (i) it follows that $G(a_\alpha) = G(b_\alpha)$. For a continuous function $h : K \rightarrow \mathbb{R}$ we have $h \circ \widehat{F} = h \circ G$. Therefore $q \circ \widehat{F}$ cannot distinguish the points a_α and b_α , a contradiction. ■

It remains to prove Lemma 5.1.

Proof of Lemma 5.1. We will construct the required functions and points by transfinite induction. Suppose that we have constructed f_β and $a_\beta, b_\beta \in \beta\omega$ satisfying the conditions (i) and (ii) for $\beta \in [\omega, \alpha)$ and $\alpha \in [\omega, 2^\omega)$.

First, we consider the trivial case when the inclusions from the beginning of the condition (ii) are not satisfied (observe that, since $Z_\alpha \subseteq [0, \alpha)$, F_{Z_α} and K_{Z_α} are well-defined at this stage of the construction). Then we simply define $f_\alpha \equiv 0$ and take any point $a_\alpha = b_\alpha \in \beta\omega$.

Next, we assume that $F_{Z_\alpha}(\omega) \subseteq D_\alpha \subseteq K_{Z_\alpha}$. From these inclusions and the choice of the functions f_n it follows that $F_{Z_\alpha}(\omega)$ is a dense subset of D_α and K_{Z_α} consisting of isolated points. It also follows that the map τ_α is unique and $\tau_\alpha = F_{Z_\alpha}$. For $n \in \omega$, we denote $\tau_\alpha(n) = F_{Z_\alpha}(n)$ by \tilde{n} . Obviously, K_{Z_α} is a zero-dimensional metrizable compact space. Take ε , A and S_n as in the condition (d) (for (D_α, T_α)). Let $\{n_i : i \in \omega\}$ be an increasing enumeration of A . Refining A if necessary we may assume that

the sequence $(\tilde{n}_i)_i$ converges in K_{Z_α} to the point s . Since the sets S_n are disjoint we can also require that $s \notin S_n$ for all $n \in A$. Therefore we have

$$(7) \quad (\forall n \in A) [S_n \cap \text{Cl}_{K_{Z_\alpha}} \{\tilde{n} : n \in A\} = \emptyset].$$

We will consider two cases:

Case 1: *There exist a point $t \in K_{Z_\alpha}$ and $\delta > 0$ such that, for every neighborhood U of t in K_{Z_α} , the set $\{n \in A : \lambda(\tilde{n}, U \cap S_n) > \delta\}$ is infinite.* Let $\{U_k : k \in \omega\}$ be a decreasing base of neighborhoods of t in K_{Z_α} . Using our assumption we can find an increasing subsequence $(i_k)_k$ such that $\lambda(\tilde{n}_{i_k}, U_k \cap S_{n_{i_k}}) > \delta$. Put $m_k = n_{i_k}$, $B = \{m_k : k \in \omega\} \subseteq A$ and $P_{m_k} = U_k \cap S_{n_{i_k}}$. Then we have

$$(8) \quad (\forall n \in B) [\lambda(\tilde{n}, P_n) > \delta],$$

and $P_{m_k} \subseteq U_k$ for every $k \in \omega$. So the sequence $(P_{m_k})_k$ converges to the point t . Since P_{m_k} are pairwise disjoint we may assume that $t \notin P_{m_k}$ for every $k \in \omega$. Now, we can construct a sequence of pairwise disjoint clopen subsets V_{m_k} of K_{Z_α} such that $P_{m_k} \subseteq V_{m_k} \subseteq U_k \setminus \{t\}$ for $k \in \omega$. Hence $(V_{m_k})_k$ also converges to the point t . Using (7) we may require that

$$(9) \quad (\forall n \in B) [V_n \cap \{\tilde{n} : n \in B\} = \emptyset].$$

We can also assume that

$$(10) \quad (\forall n \in B) [\text{supp}(\tilde{n}) \cap V_n = P_n].$$

Refining our set B again, we may demand that either, for all $n \in B$, $t \notin \text{supp}(\tilde{n})$ or, for all $n \in B$, $t \in \text{supp}(\tilde{n})$, and $\lambda_{\tilde{n}t} \in (\lambda, \lambda + \delta/4)$ for some $\lambda \in \mathbb{R}$. Applying Lemma 2.7 (for the set B and the family $\{V_n \cap D_\alpha : n \in B\}$) we may assume that

$$(11) \quad (\forall n \in B) \left[\lambda \left(\tilde{n}, \bigcup \{V_k \cap D_\alpha : k \in B, k \neq n\} \right) < \delta/4 \right].$$

Let $R_n = \{i \in \omega : \tilde{i} \in V_n\}$ for $n \in B$. Since V_n are clopen in K_{Z_α} , from the part (i) of our inductive assumption it follows that, for all $\beta \in [\omega, \alpha)$ and $n \in B$, we have $\widehat{\chi}_{R_n}(a_\beta) = \widehat{\chi}_{R_n}(b_\beta)$. Applying Corollary 3.2 we may assume (as usual using some refinement of B) that

$$(12) \quad (\forall \beta \in [\omega, \alpha)) (\forall B' \subseteq B) [\widehat{\chi}_{\bigcup \{R_n : n \in B'\}}(a_\beta) = \widehat{\chi}_{\bigcup \{R_n : n \in B'\}}(b_\beta)].$$

Finally, using Lemma 3.3 we can find distinct points $a_\alpha, b_\alpha \in \text{Cl}_{\beta\omega} B$ such that $\widehat{f}_\beta(a_\alpha) = \widehat{f}_\beta(b_\alpha)$ for all $\beta < \alpha$. Take $B' \subseteq B$ such that $a_\alpha \in \text{Cl}_{\beta\omega} B'$ and $b_\alpha \notin \text{Cl}_{\beta\omega} B'$. Then $b_\alpha \in \text{Cl}_{\beta\omega}(B \setminus B')$. We define $f_\alpha = \chi_{\bigcup \{R_n : n \in B'\}}$. We shall verify the conditions (i) and (ii) for f_α and a_α, b_α .

From (9), it follows that $\bigcup \{R_n : n \in B'\} \cap B = \emptyset$, therefore $\widehat{f}_\alpha(a_\alpha) = 0 = \widehat{f}_\alpha(b_\alpha)$. This together with (12) implies (i).

Let E and g be as in the condition (ii). Put $h = p_\alpha \circ (P_{Z_\alpha}|E)^{-1}$; then $g = T_\alpha(h)$. Observe that from the definitions of f_n and K_Z it follows that, for every $n \in \omega$, the point $\tilde{n} = F_{Z_\alpha}(n) \in K_{Z_\alpha}$ has a unique extension $F_{Z_\alpha \cup \{\alpha\}}(n)$ in $K_{Z_\alpha \cup \{\alpha\}}$. Hence, by the inclusion $P_{Z_\alpha \cup \{\alpha\}}(E) \subseteq K_{Z_\alpha \cup \{\alpha\}}$ we have $h(\tilde{n}) = f_\alpha(n)$ for every $n \in \omega$. The convergence of the sequence $(V_{m_k})_k$ implies that every point $d \in D_\alpha$, $d \neq t$, has a neighborhood W (in K_{Z_α}) such that f_α is constant on $\{n : \tilde{n} \in W\}$. Again from $P_{Z_\alpha \cup \{\alpha\}}(E) \subseteq K_{Z_\alpha \cup \{\alpha\}}$ it follows that

$$(13) \quad (\forall d \in D_\alpha \setminus \{t\}) [h(d) = \chi_{\cup\{V_n : n \in B'\}}(d)].$$

For every $n \in B$, put $l_n = \lambda_{\tilde{n}t}h(t)$ if $t \in \text{supp}(\tilde{n})$, and $l_n = 0$ otherwise; see the remark following (10). Since $h(t) \in \{0, 1\}$ (if $t \in D_\alpha$), from that remark it follows that in both cases we have

$$(14) \quad (\exists l \in \mathbb{R})(\forall n \in B) [l_n \in (l, l + \delta/4)].$$

Now, we are ready to estimate the value of $g(\tilde{n})$ for $n \in B$. From (13) and the definition of l_n we obtain

$$(15) \quad g(\tilde{n}) = \sum \left\{ \lambda_{\tilde{n}d} : d \in \text{supp}(\tilde{n}) \cap \bigcup \{V_k : k \in B'\} \right\} + l_n.$$

By the condition (d3) all $\lambda_{\tilde{n}d}$, for $n \in B$ and $d \in P_n$, are of the same sign; assume first that all are positive. Then, for $n \in B'$, we use (8), (10), (11) and (14) to estimate the quantity from (15) as follows:

$$(16) \quad \begin{aligned} g(\tilde{n}) &= \sum \{ \lambda_{\tilde{n}d} : d \in \text{supp}(\tilde{n}) \cap V_n \} \\ &\quad + \sum \left\{ \lambda_{\tilde{n}d} : d \in \text{supp}(\tilde{n}) \cap \bigcup \{V_k : k \in B', k \neq n\} \right\} + l_n \\ &> \sum \{ \lambda_{\tilde{n}d} : d \in P_n \} - \lambda \left(\tilde{n}, \bigcup \{V_k \cap D_\alpha : k \in B, k \neq n\} \right) + l \\ &> \lambda(n, P_n) - \delta/4 + l > \delta - \delta/4 + l = l + 3\delta/4. \end{aligned}$$

Using (11) and (14), for $n \in B \setminus B'$, we obtain

$$(17) \quad \begin{aligned} g(\tilde{n}) &\leq \lambda \left(\tilde{n}, \bigcup \{V_k \cap D_\alpha : k \in B, k \neq n\} \right) + l_n \\ &< \delta/4 + l + \delta/4 = l + \delta/2. \end{aligned}$$

The inequalities (16) and (17) show that $g \widehat{\circ} \tau_\alpha(a_\alpha) \geq l + 3\delta/4 > l + \delta/2 \geq g \widehat{\circ} \tau_\alpha(b_\alpha)$. It is clear that if all $\lambda_{\tilde{n}d}$ are negative for $n \in B$ and $d \in P_n$, then $g \widehat{\circ} \tau_\alpha(a_\alpha) < g \widehat{\circ} \tau_\alpha(b_\alpha)$.

Case 2: For every point $t \in K_{Z_\alpha}$ and $\delta > 0$, there is a neighborhood U of t in K_{Z_α} such that the set $\{n \in A : \lambda(\tilde{n}, U \cap S_n) > \delta\}$ is finite. Using the above assumption, one can easily construct an increasing subsequence (i_k) and corresponding sequences $(U_{n_{i_k}})$ and $(V_{n_{i_k}})$ of pairwise disjoint clopen subsets of K_{Z_α} such that the following conditions will be satisfied for all $k \in \omega$ (one should use (7) to obtain (c4)):

- (c1) $(S_{n_{i_k}} \setminus \bigcup\{U_{n_{i_j}}, V_{n_{i_j}} : j < k\}) \subseteq V_{n_{i_k}}$,
- (c2) $(\text{supp}(\tilde{n}_{i_k}) \setminus (S_{n_{i_k}} \cup \bigcup\{U_{n_{i_j}}, V_{n_{i_j}} : j < k\})) \subseteq U_{n_{i_k}}$,
- (c3) $U_{n_{i_k}} \cap V_{n_{i_k}} = \emptyset$,
- (c4) $V_{n_{i_k}} \cap \{\tilde{n} : n \in A\} = \emptyset$,
- (c5) $A_k = \{n \in A : \lambda(\tilde{n}, (U_{n_{i_k}} \cup V_{n_{i_k}}) \cap S_n) > \varepsilon/2^{k+3}\}$ is finite,
- (c6) $n_{i_k} \in A \setminus \bigcup\{A_j : j < k\}$.

As in the previous case we put $m_k = n_{i_k}$ and $B = \{m_k : k \in \omega\} \subseteq A$. By (c1)–(c3) the sets $P_{m_k} = S_{m_k} \setminus \bigcup\{U_{m_j}, V_{m_j} : j < k\}$ satisfy the condition (10). By (d2), (c5) and (c6) we also have, for every $k \in \omega$,

$$\begin{aligned}
 (18) \quad \lambda(\tilde{m}_k, P_{m_k}) &= \lambda(\tilde{m}_k, S_{m_k}) - \lambda\left(\tilde{m}_k, \left(\bigcup\{U_{m_j}, V_{m_j} : j < k\}\right) \cap S_{m_k}\right) \\
 &> \varepsilon - \sum \{\lambda(\tilde{m}_k, (U_{m_j} \cup V_{m_j}) \cap S_{m_k})\} \\
 &\geq \varepsilon - \sum_{j < k} \varepsilon/2^{j+3} > \varepsilon - \varepsilon/4 = 3\varepsilon/4.
 \end{aligned}$$

As before, applying Lemma 2.7 we may assume that

$$(19) \quad (\forall n \in B) \left[\lambda\left(\tilde{n}, \bigcup\{V_k \cap D_\alpha : k \in B, k \neq n\}\right) < \varepsilon/4 \right].$$

We define the sets R_n (again assuming (12)), the points a_α, b_α , the set B' and the function f_α in the same way as in Case 1. Therefore the condition (i) is satisfied.

Now, let E and g be as in the condition (ii), and let $h = p_\alpha \circ (P_{Z_\alpha}|E)^{-1}$. This time from the construction of the clopen sets U_n and V_n (see (c1) and (c2)) it follows that, for every $n \in B$ and $d \in \text{supp}(\tilde{n})$, the point d has a neighborhood W (in K_{Z_α}) such that f_α is constant on $\{k : \tilde{k} \in W\}$. Hence

$$(20) \quad (\forall n \in B) (\forall d \in \text{supp}(\tilde{n})) [h(d) = \chi_{\bigcup\{V_n : n \in B'\}}(d)],$$

and

$$(21) \quad (\forall n \in B) \left[g(\tilde{n}) = \sum \left\{ \lambda_{\tilde{n}d} : d \in \text{supp}(\tilde{n}) \cap \bigcup\{V_k : k \in B'\} \right\} \right].$$

Repeating similar estimations to those in (16) and (17) (here we need to use (10), (18) and (19)) we obtain $|g \widehat{\circ} \tau_\alpha(a_\alpha) - g \widehat{\circ} \tau_\alpha(b_\alpha)| \geq \varepsilon/4$. ■

6. Remarks. It can be easily observed that, for the space X from Section 4, the Banach space $C(X)$ is isometric to ℓ_∞ . Therefore we have $C(X) \stackrel{1}{\approx} C(X) \times \mathbb{R}$. We do not know if the space $C(K)$, for the compact space K from Section 5, also has this factorization property. By the Stone–Weierstrass theorem the space $C(K)$ may be identified (using the restrictions to $F(\omega)$) with a closed subalgebra of ℓ_∞ generated by the family $\{f_\alpha : \alpha < 2^\omega\}$ and the unit of ℓ_∞ .

We do not know either if, for our examples, $C_p(X) \stackrel{t}{\not\approx} C_p(X) \times \mathbb{R}$; see [Ar3, Problem 2] and [Ar4, Problem 24].

Observe that every function space $C_p(X)$ is linearly homeomorphic to the product $F \times \mathbb{R}$ for some linear subspace F of $C_p(X)$. In particular, $C_p(X) \times \mathbb{R}$ is a continuous image of $C_p(X)$.

Let us point out that from the proof of Theorem 5.2 it follows that the space K from Section 5 has the following property: For every countable dense $D \subseteq K$, we have $C_D(K) \stackrel{1}{\not\approx} C_D(K) \times \mathbb{R}$. For the first example X , the situation is different: $C_\omega(X) \stackrel{1}{\approx} C_\omega(X) \times \mathbb{R}$ (and $C_\omega(X)$ consists of all bounded functions on ω).

In [Ar2, p. 93] Arkhangel'skiĭ suggested to investigate the factorization properties of $C_p(X)$ for the compact space X constructed in [Ber]. Let us note that in this case it can be shown that $C_p(X) \stackrel{1}{\approx} C_p(X) \times \mathbb{R}$.

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