FUNDAMENTA
MATHEMATICAE
153 (1997)

Operators on $C(\omega^\alpha)$ which do not preserve $C(\omega^\alpha)$

by

Dale E. Alspach (Stillwater, Okla.)

Abstract. It is shown that if α,ζ are ordinals such that $1\le\zeta<\alpha<\zeta\omega$, then there is an operator from $C(\omega^{\omega^\alpha})$ onto itself such that if Y is a subspace of $C(\omega^{\omega^\alpha})$ which is isomorphic to $C(\omega^{\omega^\alpha})$, then the operator is not an isomorphism on Y. This contrasts with a result of J. Bourgain that implies that there are uncountably many ordinals α for which for any operator from $C(\omega^{\omega^\alpha})$ onto itself there is a subspace of $C(\omega^{\omega^\alpha})$ which is isomorphic to $C(\omega^{\omega^\alpha})$ on which the operator is an isomorphism.

In an earlier paper [A2] we proved that there is an operator on $C(\omega^{\omega^2})$ which is not an isomorphism on any subspace which is isomorphic to $C(\omega^{\omega^2})$ but the operator is onto $C(\omega^{\omega^2})$. This is in contrast with the situation for $C(\omega)$ and $C(\omega^\omega)$ where there are no surjective operators which do not preserve isomorphically a copy of the space, [P], [A1]. Bourgain [B] proved a very general result which gives an estimate on the size of the ordinal β such that any operator on $C(\omega^{\omega^\alpha})$ which is surjective must be an isomorphism on a subspace isomorphic to $C(\omega^{\omega^\beta})$. Recently Gasparis [G], [G1] has generalized the example in [A2] to the case of operators on $C(\omega^{\omega^{\alpha+1}})$ to show that there are surjective operators on these spaces which do not preserve a copy of $C(\omega^{\omega^{\alpha+1}})$. For most ordinals α this is very far from the estimate given by Bourgain.

Bourgain used the Szlenk index and a combinatorial argument in the proof of his result. Implicit in his proof is the notion of γ-families of sets which was independently developed by Wolfe [W], and in [A2]. The existence of γ-families with associated measures is an indication of the amount of topological disjointness in a subset of $C(K)^*$ whereas the Szlenk index only indicates disjointness. Bourgain essentially shows that a large Szlenk index forces the existence of a γ-family of sets with the size of γ dependent on

1991 *Mathematics Subject Classification*: Primary 46B03; Secondary 06A07, 03E10.
Key words and phrases: ordinal index, Szlenk index, Banach space of continuous functions.

the Szlenk index. The existence of a γ-family is equivalent to a condition on an ordinal index which we have named the Wolfe index. Thus from this view point Bourgain proves that the Szlenk index gives some lower bound on the Wolfe index. In some cases he infers that the two indices are of roughly equivalent size. In this paper we give a very general construction of examples of the type in [A2] and [G] and show that there are many more ordinals for which the Szlenk and Wolfe index are very different.

We will use notation similar to that in [A2]. In particular, if γ is an ordinal, $C(\gamma)$ is the space of continuous functions on the ordinals less than or equal to γ with the order topology, which we denote by $[1,\gamma]$. If K is a topological space, $K^{(\beta)}$ is the β-derived set of K. If $L \subset C(K)^*$, then $L^{(\beta)}$ will be the β-derived set of L with respect to the w*-topology. If K is a countable compact Hausdorff space, then K is homeomorphic to $[1,\omega^\beta n]$, where the cardinality of $K^{(\beta)}$ is n, for some $n \in \mathbb{N}$ (cf. [MS]). It was shown in [BP] that $C(\omega^{\omega^\alpha})$ is isomorphic to $C(\omega^\beta n)$ if and only if $\omega^\alpha \le \beta < \omega^{\alpha+1}$. Thus from the point of view of the isomorphic theory of Banach spaces, the spaces $C(\omega^{\omega^\alpha})$, $\alpha < \omega_1$, are a complete set of representatives of the $C(K)$-spaces for $C(K)$ separable and K countable.

1. A topological construction. In order to define the operator we need to develop a method of constructing special sets of measures on ω^{ω^α} which are homeomorphic to ω^{ω^α} but which have supports which are almost disjoint but are not topologically well separated. In [A2] we used the porcupine topology, [BD], to effect the construction. Here we use a similar construction but with somewhat different notation. The operators that we construct are of the same form as that in our earlier work. Namely, we produce a compact Hausdorff space K and a w*-closed subset L of $C(K)^*$ and we define an operator from $C(K)$ into $C(L)$ by evaluation. In this paper we need to iterate the construction of [A2]. To this end we introduce a general procedure for extending a pair (K,L) by a sequence of spaces K_n, where K and K_n are compact Hausdorff spaces, each K_n has a distinguished point $k_{n,0}$ and L is a set of purely atomic finitely supported probability measures on K.

For each $k \in K$, we let $L(k) = \{l \in L : l(k) \ne 0\} \cup \{\emptyset\}$ and $S(k,L)$ be the one point compactification of $\sum_n \sum_{l\in L(k)} K_n \setminus \{k_{n,0}\}$, where we use $\sum_{i\in I} W_i$ to denote the disjoint sum of topological spaces W_i with the topology generated by sets of the form $\bigcup_{i\in I} G_i$ with G_i open for each $i \in I$. We denote the points of $S(k,L)$ as 4-tuples (k,l,n,j) where $l \in L(k)$ and $j \in K_n$. The point added will be denoted by $(k,\emptyset)$ although it is also $(k,l,n,k_{n,0})$ for any $l \in L(k)$ and n. Note that if $L(k) = \{\emptyset\}$, then $S(k,L(k)) = \{(k,\emptyset)\}$. We want to define a topology on the disjoint union of the sets $S(k,L)$. Intuitively, we want to glue $S(k,L)$ to K at the point k by identifying $(k,\emptyset)$ with k. We also want to extend the measures L by sets of measures L_n on

K_n and make a copy of L_n for each $l \in L(k)$, $n \in \mathbb{N}$. More formally, we make the following definition.

DEFINITION 1.1. Suppose that K and K_n, $n \in \mathbb{N}$, are compact Hausdorff spaces, K_n has a distinguished point $k_{n,0}$ and L, L_n are sets of purely atomic disjointly supported probability measures on K, K_n, respectively, with $\delta_{k_{n,0}} \in L_n$, for each n. Define $(K,L) \otimes \{(K_n, L_n) : n \in \mathbb{N}\}$ to be the pair (K', L') where K' is the compact Hausdorff space and L' is the set of atomic probability measures on K' described below. K' is the set of 4-tuples (k,l,n,j_n), $k \in K$, $l \in L(k)$, $n \in \mathbb{N}$ and $j_n \in K_n$, with the topology generated by sets of the form

$$\bigcup_{k\in K} G_k \cup \bigcup_{k\in G}\{(k,l,n,j_n) : k \in K,\ l \in L(k),\ n \in \mathbb{N},\ j_n \in K_n\} \setminus \bigcup_{(k,l,n)\in F} \{k\} \times \{l\} \times \{n\} \times F_{k,l,n},$$

where G_k is an open subset of

$$\{(k,l,n,j_n) : l \in L(k),\ n \in \mathbb{N},\ j_n \in K_n \setminus \{k_{n,0}\}\} = S(k,L) \setminus \{(k,\emptyset)\}$$

for each k, G is an open set in K, F is a finite set of triples (k,l,n) with $k \in K$, $n \in \mathbb{N}$ and $l \in L(k)$, and $F_{k,l,n}$ is a compact subset of $K_n \setminus \{k_{n,0}\}$. For each $k \in K$ we identify all of the points $(k,l,n,k_{n,0})$ such that $l \in L(k)$, $n \in \mathbb{N}$ with the point $(k,\emptyset)$. (Formally, K' is a set of equivalence classes of 4-tuples, but only the elements with fourth entry $k_{n,0}$ are in non-trivial classes.) Let ϕ be the map from K into K' defined by $\phi(k) = (k,\emptyset)$ and let Φ be the map from $\mathcal{M}(K)$ into $\mathcal{M}(K')$ which is induced by ϕ. Let

$$L' = \Big\{ \sum_{k\in \operatorname{supp} l} l(k) \sum_{j_n \in K_n} l_n(j_n)\delta_{(k,l,n,j_n)} : l \in L,\ n \in \mathbb{N},\ l_n \in L_n\Big\}.$$

In keeping with our identification,

$$\sum_{k\in \operatorname{supp} l} l(k)\delta_{(k,l,n,k_{n,0})} = \sum_{k\in \operatorname{supp} l} l(k)\delta_{(k,\emptyset)} = \Phi(l)$$

for each $l \in L$, $n \in \mathbb{N}$, and $\Phi(l) \in L'$ because

$$\sum_{k\in \operatorname{supp} l} l(k) \sum_{j_n\in K_n} \delta_{k_{n,0}}(j_n)\delta_{(k,l,n,j_n)} = \sum_{k\in \operatorname{supp} l} l(k)\delta_{(k,l,n,k_{n,0})}$$

and we have assumed that $\delta_{k_{n,0}} \in L_n$, for each n.

The next lemma lists some properties of the construction.

LEMMA 1.1. *Suppose that K and K_n, $n \in \mathbb{N}$, are compact Hausdorff spaces, K_n has a distinguished point $k_{n,0}$ and L, L_n, $n \in \mathbb{N}$, are sets of purely atomic finitely supported probability measures on K, K_n, respectively,* *with* $\delta_{k_{n,0}} \in L_n$, *for each* n, *as above. Then if* $(K', L') = (K, L) \otimes \{(K_n, L_n) : n \in \mathbb{N}\}$,

(1) K' *is a compact Hausdorff space and* ϕ *is a homeomorphism of* K *into* K'.

(2) *A net* $(k_d, l_d, n_d, j_d)_{d\in D}$ *in* $K' \setminus \phi(K)$ *converges to* (k, l, n, j) *for some* $j \neq k_{n,0}$ *if and only if there exists* $d_0 \in D$ *such that* $k_d = k$, $l_d = l$, *and* $n = n_d$ *for all* $d \geq d_0$ *and* $(j_d)_{d\in D}$ *converges to* j.

(3) *A net* $(k_d, l_d, n_d, j_d)_{d\in D}$ *in* $K' \setminus \phi(K)$ *converges to* $(k, l, n, k_{n,0}) = (k, \emptyset)$ *if and only if the following hold*:

(a) $(k_d)_{d\in D}$ *converges to* k.

(b) *If* $D_1 = \{d : k_d = k\}$ *is cofinal in* D, *then for each* l *and* n, $D_{1,l,n} = \{d \in D_1 : l_d = l,\ n_d = n\}$ *is not cofinal in* D *or* $(j'_d)_{d\in D}$ *converges to* $k_{n,0}$, *where* $j'_d = j_d$ *if* $d \in D_{1,l,n}$ *and* $j'_d = k_{n,0}$ *otherwise.*

(4) *The map*

$$l \to \Phi(l) = \sum_{k\in \operatorname{supp} l} l(k)\delta_{(k,\emptyset)}$$

for $l \in L$ *is a homeomorphism of* L *into* L' *in the weak** *topology.*

(5) *Each* $l' \in L'$ *is atomic and has finite support.*

(6) *If* L, L_n, $n \in \mathbb{N}$, *are compact in the weak** *topology*, *then* L' *is compact in the weak** *topology.*

(7) *If* (l_d) *is a convergent net in* L_n *with limit* l_0 *and* $l \in L$, *then*

$$\Big(\sum_{k\in \operatorname{supp} l} l(k) \sum_{j_n\in K_n} l_d(j_n)\delta_{(k,l,n,j_n)}\Big)_d$$

converges to

$$\sum_{k\in \operatorname{supp} l} l(k) \sum_{j_n\in K_n} l_0(j_n)\delta_{(k,l,n,j_n)}$$

for each $l \in L$.

Proof. We have given a basis for the topology on K' in the definition above. In order to verify the first property we first observe that $\{(k, \emptyset) : k \in K\}$ is homeomorphic to K. Notice that the basis for the topology of K' given in the definition above defines the topology on $\{(k, \emptyset) : k \in K\}$ to be the topology $\{\phi(G) : G \text{ is open in } K\}$. Thus ϕ is a homeomorphism. If $\mathcal{O}$ is an open cover of K' by basic open sets, then there is a finite subset $\mathcal{O}'$ of $\mathcal{O}$ which covers $\phi(K)$. $K' \setminus \bigcup\{G_i : G_i \in \mathcal{O}'\}$ is contained in a finite union of closed subsets of the form $\{k\} \times \{l\} \times \{n\} \times F_{k,l,n}$, where $F_{k,l,n}$ is a compact subset of $K_n \setminus \{k_{n,0}\}$. The topology on $\{k\} \times \{l\} \times \{n\} \times F_{k,l,n}$ is the topology induced by identifying this with $F_{k,l,n}$ in K_n. Therefore a finite number of additional sets from $\mathcal{O}$ will cover each $\{k\}\times\{l\}\times\{n\}\times F_{k,l,n}$. This proves the first assertion.

For the second, notice that if G is an open set contained in $K_n\setminus\{k_{n,0}\}$ and $j\in G$, then $\{k\}\times\{l\}\times\{n\}\times G$ is an open neighborhood of (k,l,n,j). Thus the net must eventually be in $\{k\}\times\{l\}\times\{n\}\times G$, and (2) follows. Define a map ζ from K' onto $\{(k,\emptyset):k\in K\}$ by $\zeta(k,l,n,j)=k$. Clearly, ζ is continuous and this gives (3)(a), if the net converges. If $D_1=\{d:k_d=k\}$ were cofinal in D, $D_{1,l,n}$ were cofinal in D and $(j'_d)_{d\in D}$ had a convergent subnet with limit $j\neq k_{n,0}$, then there would be an open set G containing j and contained in $K\setminus\{k_{n,0}\}$. However, $\{k\}\times\{l\}\times\{n\}\times G$ would be an open set in K' containing (k,l,n,j) and thus $(k,l,n,j'_d)_{d\in D}$ would converge to (k,l,n,j), which is impossible. Thus (3)(b) holds. Conversely, if we are given a net satisfying (3)(a) and (b) and G' is an open set containing (k,l,n,k_{n_0}), then G' contains a neighborhood of $(k,l,n,k_{n,0})$ of the form

$$H=\bigcup_{k'\in G}\{(k',l',n',j_{n'}):k'\in K,\ l'\in L(k),\ n'\in\mathbb{N},\ j_{n'}\in K_{n'}\}$$
$$\setminus\bigcup_{(k',l',n')\in F}\{k'\}\times\{l'\}\times\{n'\}\times F_{k',l',n'}.$$

Because $(k_d)_{d\in D}$ converges to k, there is a $d_0\in D$ such that $(k_d,l_d,n_d,j_d)\in H\cup\bigcup_{(k',l',n')\in F}\{k'\}\times\{l'\}\times\{n'\}\times F_{k',l',n'}$ for all $d\ge d_0$. Because F is finite, we may assume, by choosing another d_0 and passing to a subset of G if necessary, that $F=\{(k,l',n'):(l',n')\in F'\}$ for some finite set F'. By (b) we know that for each $(l',n')\in F'$ there is a $d_{l',n'}$ such that if $(k_d,l_d,n_d,j_d)=(k,l',n',j_d)$ and $d\ge d_{l',n'}$, then $(k_d,l_d,n_d,j_d)\notin\{k\}\times\{l'\}\times\{n'\}\times F_{k,l',n'}$. If $d\ge d_{l',n'}$, for all $(l',n')\in F'$, and $d\ge d_0$, then $(k_d,l_d,n_d,j_d)\in H$.

Because ϕ is a homeomorphism (4) is immediate. (5) is obvious from the definition and the fact that $(k,l,n,j)=(k',l',n',j')$ if and only if $k=k'$, $l=l'$, $n=n'$ and $j=j'$, or $j=j'=k_{n,0}$ and $k=k'$. To see that L' is compact if $L,\ L_n,\ n\in\mathbb{N}$, are, let $(l'_d)_{d\in D}$ be a net in L', where

$$l'_d=\sum_{k\in\operatorname{supp}l_d}\ \sum_{j\in K_{n(d)}}l_d(k)l''_{n(d)}(j)\delta_{(k,l_d,n(d),j)}$$

for each $d\in D$. Because L and the L_n are compact, we may assume by passing to a subnet that the nets $(l_d)_{d\in D}$ and $(l''_{n(d)})_{d\in D}$ converge to l and l'', respectively. Here we are thinking of $(l''_{n(d)})$ as a net in $\bigcup_{n\in\mathbb{N}}\Phi_n(L_n)$, where Φ_n is the map induced by the natural embedding ϕ_n of K_n into the one point compactification of $\bigcup_{n\in\mathbb{N}}K_n\setminus\{k_{n,0}\}$. Because Φ is w*-continuous, $(\Phi(l_d))_{d\in D}$ converges to $\Phi(l)$. If $\varepsilon>0$ and $k\in\operatorname{supp}l$, let G_k be an open set containing k and such that $l(G_k)<l(k)+\varepsilon$. We may assume that the sets G_k are disjoint. We must consider two cases. First suppose that (l_d)

has a constant subnet. Then $l'_d = \sum_{k\in\operatorname{supp} l}\sum_{j\in K_{n(d)}} l(k)l''_{n(d)}(j)\delta_{(k,l_d,n(d),j)}$ for the elements in the subnet and the limit of the subnet is $\sum_{k\in\operatorname{supp} l} l(k)\sum_{n\in\mathbb{N},j\in K_n} l''(j)\delta_{(k,l,n,j)}$. If there is no constant subnet, then any convergent subnet of points $(k_d,l_d,n_d,j_d)_{d\in D}$ will have a limit of the form $(k,\emptyset)$, where k is the limit of the first coordinates in the subnet, by (2) and (3). We claim that there is a convergent subnet with limit $\sum_{k\in\operatorname{supp} l} l(k)\delta_{(k,\emptyset)}$. Indeed, because (l_d) converges to l, there exists a d_0 such that $l_d(G_k) > l(G_k)-\varepsilon$ for all $k\in\operatorname{supp} l$, $d\ge d_0$. This implies that

$$\begin{aligned} l'_d\Big(\bigcup_{r\in G_k}\{(r,m,n,t) : r\in K,\ m\in L(k),\ n\in\mathbb{N},\ t\in K_n\}\\ \setminus\bigcup_{(r,m,n)\in F}\{r\}\times\{m\}\times\{n\}\times F_{r,m,n}\Big)\\ > l(G_k)-\varepsilon-\sum_{(r,m,n)\in F} l'_d(\{r\}\times\{m\}\times\{n\}\times F_{r,m,n}). \end{aligned}$$

Notice that $l'_d(\{r\}\times\{m\}\times\{n\}\times F_{r,m,n}) = 0$ if $l_d\ne m$ or $r\notin\operatorname{supp} l_d$. Because F is finite, and we have assumed that there is no constant subnet of (l_d), by choosing another $d_1\ge d_0$ we will have $l'_d(\{r\}\times\{m\}\times\{n\}\times F_{r,m,n}) = 0$ for all $(r,m,n)\in F$ and all $d\ge d_1$. Because l'_d is a probability measure and $\varepsilon>0$ is arbitrary, (l'_d) converges to $\Phi(l) = \sum_{k\in\operatorname{supp} l} l(k)\delta_{(k,\emptyset)}$. Thus L' is compact.

(7) is immediate from the definition. ■

Our next lemma will allow us to compute topological information about the spaces K' and L' from the component pieces provided the pieces are properly attached.

LEMMA 1.2. *Let K, L, K_n, L_n, $n\in\mathbb{N}$, K' and L' be as in the previous lemma. In addition assume that K is homeomorphic to $[1,\omega^{\omega^\alpha m}]$, K_n is homeomorphic to $[1,\omega^{\omega^{\beta(n)}m(n)}]$, L (with the w^*-topology) is homeomorphic to $[1,\omega^{\omega^\gamma p}]$, and L_n is homeomorphic to $[1,\omega^{\omega^{\gamma(n)}p(n)}]$. Moreover, assume that*

$$K^{(\omega^\alpha m)} = \{k_0\},\qquad L^{(\omega^\gamma p)} = \{\delta_{k_0}\},$$
$$K_n^{(\omega^{\beta(n)}m(n))} = \{k_{n,0}\},\quad L_n^{(\omega^{\gamma(n)}p(n))} = \{\delta_{k_{n,0}}\}$$

for all n. Let

$$\omega^B M = \sup\{\omega^{\beta(n)}m(n) : n\in\mathbb{N}\}\quad\text{and}\quad \omega^\Gamma P = \sup\{\omega^{\gamma(n)}p(n) : n\in\mathbb{N}\}.$$

Then

(1) *$K'^{(\omega^B M)}\subset\phi(K)$ and if $\bigcup\{\operatorname{supp} l : l\in L\} = K$, then K' is homeomorphic to $[1,\omega^{\omega^B M+\omega^\alpha m}]$ and $K'^{(\omega^B M+\omega^\alpha m)} = \{\phi(k_0)\}$.*

(2) $L'^{(\omega^\Gamma P)} = \Phi(L)$, L' *(with the w^*-topology) is homeomorphic to* $[1, \omega^{\omega^\Gamma P+\omega^\gamma p}]$ *and* $L'^{(\omega^\Gamma P+\omega^\gamma p)} = \{\delta_{\phi(k_0)}\}$.

(3) *If for each $l \in L$, there is a subset H_l of K such that $l(H_l) \geq \varepsilon$, and $(H_l)_{l\in L}$ are disjoint, and for each $n \in \mathbb{N}$, $l'' \in L_n$, there is a subset $H_{n,l''}$ of $K_n \setminus \{k_{n,0}\}$ such that $l''(H_{n,l''}) \geq \varepsilon$ and $(H_{n,l''})_{l''\in L_n}$ are disjoint for each n, then there are disjoint subsets $H'_{l'}$ of K' for each $l' \in L'$ such that $l'(H'_{l'}) \geq \varepsilon$. Moreover, if $l \in L$, then we can define $H'_{\Phi(l)} = \phi(H_l)$ and if*

$$l' = \sum_{k\in\operatorname{supp} l} l(k) \sum_{j_n\in K_n} l_n(j_n)\delta_{(k,l,n,j_n)}$$

for some $l \in L$, $n \in \mathbb{N}$, $l_n \in L_n \setminus \{\delta_{k_{n,0}}\}$, then we can define

$$H'_{l'} = \bigcup_{k\in\operatorname{supp} l} \{k\} \times \{l\} \times \{n\} \times H_{l_n}.$$

Proof. First observe that because K, L, and K_n are countable and the measures in L are finitely supported, K' is countable. If $n \in \mathbb{N}$, $j \in K_n$ and $j \neq k_{n,0}$, then for any $k \in K$, $l \in L$, with $l(k) \neq 0$, $\{(k,l,n,j') : j' \neq k_{n,0}\}$ is an open neighborhood of (k,l,n,j) in K' homeomorphic to $K_n \setminus \{k_{n,0}\}$. Thus (k,l,n,j) is in the same derived sets of K' as of K_n. In particular, $K_n^{(\omega^{\beta(n)}m(n))} = \{k_{n,0}\}$ and thus $K'^{(\omega^B M)} \subset \phi(K)$. If $\bigcup\{\operatorname{supp} l : l \in L\} = K$, then for each $k \in K$, $\{(k,l,n,j) : j \in K_n\} \subset K'$ for some $l \in L$ and therefore $(k,l,n,k_{n,0}) \in K'^{(\omega^\beta M)}$. If k is an isolated point in K, then $\phi(k)$ is the limit only of sequences which are eventually in

$$\{(k,l,n,j) : l \in L,\ l(k) \neq 0,\ n \in \mathbb{N},\ j \in K_n\}.$$

Hence $(k,\emptyset) = (k,l,n,k_{n,0}) \notin K'^{(\omega^B M+1)}$. Because ϕ is a homeomorphism, it follows that $K'^{(\omega^B M)} = \phi(K^{(0)})$. Similarly, $K'^{(\omega^B M+\varrho)} = \phi(K^{(\varrho)})$ for all ϱ. In particular, $K'^{(\omega^B M+\omega^\alpha m)} = \{\phi(k_0)\}$.

Observe that it follows from Lemma 1.1 that L' is countable and compact because K, K_n, L, and L_n are, and thus it is sufficient to consider the derived sets. If $l \in L$, $n \in \mathbb{N}$, then $\{\sum_{k\in\operatorname{supp} l} l(k) \sum_{j\in\operatorname{supp} l_n} l_n(j)\delta_{(k,l,n,j)} : l_n \in L_n\}$ is homeomorphic (by the obvious map) to L_n and

$$\Big\{\sum_{k\in\operatorname{supp} l} l(k) \sum_{j\in\operatorname{supp} l_n} l_n(j)\delta_{(k,l,n,j)} : l_n \in L_n\Big\}^{(\omega^{\gamma(n)}p(n))} = \{\Phi(l)\}.$$

Therefore $\Phi(L) \subset \bigcap_{n\in\mathbb{N}} L'^{(\omega^{\gamma(n)}p(n))} = L'^{(\omega^\Gamma P)}$. If $l_n \in L_n \setminus \{\delta_{k_{n,0}}\}$, then

$$\sum_{k\in\operatorname{supp} l} l(k) \sum_{j\in\operatorname{supp} l_n} l_n(j)\delta_{(k,l,n,j)} - \sum_{k\in\operatorname{supp} l} l(k) l_n(k_{n,0})\delta_{k,\emptyset}$$

is non-zero and is supported in the open set $\bigcup_{k\in\operatorname{supp} l}\{(k,l,n,j) : j \in$

$K_n \setminus \{k_{n,0}\}\}$ and only elements of L' of the form

$$\sum_{k\in\operatorname{supp} l} l(k) \sum_{j\in\operatorname{supp} l'_n} l'_n(j)\delta_{(k,l,n,j)}$$

with $l'_n \in L_n \setminus \{\delta_{k_{n,0}}\}$ are supported in this set. Therefore $\Phi(L) = L'^{(\omega^\Gamma P)}$. Because Φ is a homeomorphism, it follows that $\Phi(L)^{(\varrho)} = L'^{(\omega^B P+\varrho)}$ for all ϱ, proving the second assertion.

For each $l' \notin \Phi(L)$, we have defined $H'_{l'}$ to be a subset of $K' \setminus \phi(K)$. These sets are clearly disjoint. Also if

$$l' = \sum_{k\in\operatorname{supp} l} l(k) \sum_{j\in\operatorname{supp} l_n} l_n(j)\delta_{(k,l,n,j)},$$

then

$$l'(H_{l'}) = \sum_{k\in\operatorname{supp} l} l(k)l_n(H_{l_n}) = l_n(H_{l_n}).$$

Because Φ is induced by the homeomorphism ϕ, it follows that $H'_{\Phi(l)} = \phi(H_l)$, $l \in L$, is a family of disjoint subsets of $\phi(K)$ with $\Phi(l)(H'_{\Phi(l)}) = l(H_l)$. ∎

In order to prove that the evaluation map from $C(K)$ into $C(L)$ is surjective we will need to show that L is equivalent to the usual unit vector basis of l_1. The elements of L are not perturbations of disjointly supported elements and thus the proof uses some special properties of the construction. We introduce a natural ordering on the elements of L which reflects these properties of the construction.

Definition 1.2. Suppose $\mathcal{M}$ is a family of measures on a measurable space $(\Omega,\mathcal{B})$ and for each $\mu \in \mathcal{M}$ there is a set $H_\mu \in \mathcal{B}$ such that $H_\mu \cap H_{\mu'} = \emptyset$ if $\mu \neq \mu'$, and $\mu(H_\mu) \neq 0$. Then $\mu \succ' \mu'$ if and only if there is a scalar $a \in (0, |\mu(H_\mu)/(2\mu'(H_{\mu'}))|]$ such that $\mu_{|\bigcup\{H_{\mu'}:\mu'\neq\mu\}} = a\mu'_{|\bigcup\{H_{\mu'}:\mu'\neq\mu\}}$ and $|\mu'|(H_\mu) = 0$. Define $\mu \succ \nu$ if and only if there is a finite sequence (μ_i) in $\mathcal{M}$ such that $\mu = \mu_0 \succ' \mu_1 \succ' \ldots \succ' \mu_k = \nu$.

Notice that $\mu \succ \mu$ is impossible and the relation $\succ$ is transitive by definition. Thus we can define a partial order on $\mathcal{M}$ by $\mu \succeq \mu'$ if and only if $\mu = \mu'$ or $\mu \succ \mu'$. Although the relation is really on the pairs (μ, H_μ), we will write it as though it were on the measures. This will not present any difficulty because the sets H_μ will be fixed during the construction.

The relation above occurs naturally in the construction of the pairs (K, L). For $(K', L') = (K, L) \otimes \{(K_n, L_n) : n \in \mathbb{N}\}$ as in Lemma 1.1, each $l' \in L'$ which is of the form $\sum_{k\in\operatorname{supp} l} l(k) \sum_{j_n\in K_n} l_n(j_n)\delta_{(k,l,n,j_n)}$ for some $l \in L$, $n \in \mathbb{N}$, $l_n \in L_n$, satisfies $l'_{|K'\setminus(\operatorname{supp} l)\times\{l\}\times\{n\}\times K_n} = l_n(k_{n,0})l$. If we have the sets $(H'_{l'})_{l'\in L'}$, defined as in Lemma 1.2(3), and for $l''\in L_n$, we have $\operatorname{supp} l''\subseteq H_{n,l''}\cup\{k_{n,0}\}$, then $l'\succ' l$.

The next lemma is similar to Proposition IV.13 of [G] or Proposition 4.4 in [G1]. It will be used to show that the sets of measures L that we construct actually are equivalent to the basis of l_1.

LEMMA 1.3. *Suppose that M is a set of mutually singular probability measures on a measurable space $(K,\mathcal{B})$, $\varepsilon>0$, and that (μ_n) is a sequence of (finite) convex combinations of the measures in M and (A_n) is a sequence of disjoint measurable sets. Let $\succ'$ and $\succ$ be defined as above for $\mathcal{M}=\{\mu_n\}$ and $H_{\mu_n}=A_n$. Suppose that $(\mu_n,A_n)_{n=1}^\infty$ satisfy the following*:

(1) *For each $n\in\mathbb{N}$, $\mu_n(A_n)\ge\varepsilon$.*

(2) *For each $n\in\mathbb{N}$, either there is a unique $n'\in\mathbb{N}$ such that $\mu_n\succ'\mu_{n'}$ or for all $n'\ne n$, $\mu_n(A_{n'})=0$.*

(3) *For all $n\ne m$ if it is not the case that $\mu_n\succ\mu_m$, then $\mu_n(A_m)=0$.*

Then $\|\sum c_n\mu_n\|\ge(2\varepsilon/3)\sum|c_n|$ for any sequence of scalars (c_n).

Proof. Because we only use information about the measures on the sets A_n, without loss of generality we may assume that $\mu_n\in\operatorname{co}\{m\in\mathcal{M}: m(A_k)>0, \text{for some } k\}\cup\{0\}$. The relation $\succ$ defines a partial order on $\{\mu_n\}$. Observe that if $\mu\succ'\nu$ and $\mu=\sum_{j\in F}b_j^\mu m_j$ and $\nu=\sum_{j\in G}b_j^\nu m_j$, where $m_j\in\mathcal{M}$ and b_j^μ, b_j^ν are non-zero for all j, then $F\supset G$. Therefore, since each μ_n is a finite convex combination, for any $n(0)\in\mathbb{N}$ there is a unique finite maximal sequence $(\mu_{n(i)})_{i=0}^k$ such that $\mu_{n(0)}\succ'\mu_{n(1)}\succ'\ldots\succ'\mu_{n(k)}$. Let (c_n) be a finite sequence of scalars and let

$$F=\{n:\exists n' \text{ such that } c_{n'}\ne0 \text{ and } \mu_{n'}\succ\mu_n\}.$$

Clearly, F is a finite set. Partition F into sets $(F_j)_{j=0}^J$ such that for each $j<J$ and $n\in F_j$ there is an $n'\in F_{j+1}$ such that $\mu_n\succ'\mu_{n'}$ and for all $n'\ne n$, $n'\in F_j$, $\mu_{n'}$ and μ_n are incomparable. If $\mu_n\succ'\mu_{n'}$, let $a_{n,n'}$ denote the scalar such that $\mu_{n|\bigcup\{A_k:k\ne n\}}=a_{n,n'}\mu_{n'|\bigcup\{A_k:k\ne n\}}$. For notational convenience, let $a_{n,n'}=0$ if it is not the case that $\mu_n\succ'\mu_{n'}$. A simple induction argument using (2) and (3) shows that

$$\begin{aligned}\Big\|\sum c_n\mu_n\Big\|&=\Big\|\sum_{j=0}^J\sum_{n(j)\in F_j}c_{n(j)}\mu_{n(j)}\Big\|\\&=\Big\|\sum_{j=0}^J\sum_{n(j)\in F_j}c_{n(j)}\mu_{n(j)|\bigcup\{A_n:\mu_{n(j)}\succeq\mu_n\}}\Big\|\\&=\Big\|\sum_{j=0}^J\sum_{n(j)\in F_j}\sum_{n:\mu_n\succeq\mu_{n(j)}}c_n\mu_{n|A_{n(j)}}\Big\|.\end{aligned}$$

Another induction argument and the definition of the scalars $a_{n,n'} = 0$ give the following inequality:

$$\Big\|\sum c_n\mu_n\Big\| \ge \sum_{j=0}^{J}\sum_{n(j)\in F_j}\Big|c_{n(j)} + \sum_{n(j-1)\in F_{j-1}} a_{n(j-1),n(j)}(c_{n(j-1)} + \sum_{n(j-2)\in F_{j-2}} a_{n(j-2),n(j-1)}(c_{n(j-2)} + \ldots + \sum_{n(0)\in F_0} a_{n(0),n(1)}c_{n(0)}))\Big|\mu_{n(j)}(A_{n(j)}).$$

Now we split off 1/3 of each term and shift the index on these pieces to combine with the subsequent related term:

$$\begin{aligned}
&\sum_{j=0}^{J}\sum_{n(j)\in F_j}\Big|c_{n(j)} + \sum_{n(j-1)\in F_{j-1}} a_{n(j-1),n(j)}\Big(c_{n(j-1)} \\
&\qquad + \sum_{n(j-2)\in F_{j-2}} a_{n(j-2),n(j-1)}\Big(c_{n(j-2)} + \ldots \\
&\qquad + \sum_{n(0)\in F_0} a_{n(0),n(1)}c_{n(0)}\Big)\Big)\Big|\mu_{n(j)}(A_{n(j)}) \\
&\quad = \sum_{j=0}^{J}\sum_{n(j)\in F_j}\bigg(\frac{2}{3}\Big|c_{n(j)} + \sum_{n(j-1)\in F_{j-1}} a_{n(j-1),n(j)}\Big(c_{n(j-1)} \\
&\qquad + \sum_{n(j-2)\in F_{j-2}} a_{n(j-2),n(j-1)}\Big(c_{n(j-2)} + \ldots \\
&\qquad + \sum_{n(0)\in F_0} a_{n(0),n(1)}c_{n(0)}\Big)\Big)\Big|\mu_{n(j)}(A_{n(j)}) \\
&\qquad + \frac{1}{3}\sum_{\mu_{n(j-1)}\succ'\mu_{n(j)}}\Big|c_{n(j-1)} + \sum_{n(j-2)\in F_{j-2}} a_{n(j-2),n(j-1)}\Big(c_{n(j-2)} \\
&\qquad + \sum_{n(j-3)\in F_{j-3}} a_{n(j-3),n(j-2)}\Big(c_{n(j-3)} + \ldots \\
&\qquad + \sum_{n(0)\in F_0} a_{n(0),n(1)}c_{n(0)}\Big)\Big)\Big|\mu_{n(j-1)}(A_{n(j-1)})\bigg).
\end{aligned}$$

The condition $\mu_{n(j-1)} \succ' \mu_{n(j)}$ is equivalent to $a_{n(j-1),n(j)} \ne 0$ and by the definition of $\succ'$, $2a_{n(j-1),n(j)}\mu_{n(j)}(A_{n(j)}) \le \mu_{n(j-1)}(A_{n(j-1)})$. Therefore by the triangle inequality,

$$\Big\|\sum c_n\mu_n\Big\| \ge \sum_{j=0}^{J}\sum_{n(j)\in F_j}\frac{2}{3}|c_{n(j)}|\mu_{n(j)}(A_{n(j)}) \ge \sum_{j=0}^{J}\sum_{n(j)\in F_j}\frac{2}{3}|c_{n(j)}|\varepsilon. \ \blacksquare$$

2. Construction of the operators. The aim of this section is to produce pairs $(K_\alpha, L_\alpha)_{\alpha<\omega_1}$ by transfinite induction so that K_α is a countable compact Hausdorff space and L_α is a w*-closed subset of the probability measures in $C(K_\alpha)^*$ which is equivalent to the basis of l_1.

Fix an ordinal $\zeta < \omega_1$. Let $\zeta_n \uparrow \omega^\zeta$ and for each $n \in \mathbb{N}$ let $S_n = [1, \omega^{\zeta_n}]$ with the order topology and let $T_n = \{\frac{1}{2}(\delta_\beta + \delta_{\omega^{\zeta_n}}) : \beta \le \omega^{\zeta_n}\}$. Let the distinguished point of S_n be ω^{ζ_n}. Let $S_0 = [1,1]$ and $T_0 = \{\delta_1\}$. Define

$$(K_1, L_1) = (S_0, T_0) \otimes \{(S_n, T_n) : n \in \mathbb{N}\}.$$

It is easy to see that up to a homeomorphism of $[1, \omega^{\omega^\zeta}]$ we could have defined $K_1 = [1, \omega^{\omega^\zeta}]$ and $L_1 = \{\frac{1}{2}(\delta_\beta + \delta_{\omega^{\omega^\zeta}}) : \beta \le \omega^{\omega^\zeta}\}$. We take the distinguished point k_1 of K_1 to be $\phi(1)$ where $1 \in S_0$. Now suppose that we have defined K_γ and L_γ for all $\gamma < \alpha$. Let k_γ denote the distinguished point of K_γ. There are two cases. First assume that $\alpha = \alpha' + 1$ for some α'. Define

$$(K_\alpha, L_\alpha) = (K_1, L_1) \otimes (K_{\alpha'}, L_{\alpha'}).$$

Let the distinguished point be $k_\alpha = \phi(k_1)$. (More formally we should have a sequence of spaces $\{(K_{\alpha_n}, L_{\alpha_n}) : n \in \mathbb{N}\}$ on the right of $\otimes$, but we can take $(K_{\alpha_1}, L_{\alpha_1}) = (K_{\alpha'}, L_{\alpha'})$ and $(K_{\alpha_n}, L_{\alpha_n}) = (S_0, T_0)$ for $n > 1$. These spaces (S_0, T_0) have no effect on (K_α, L_α).) If α is a limit ordinal, let (α_n) be an increasing sequence of ordinals with limit α. Let

$$(K_\alpha, L_\alpha) = (S_0, T_0) \otimes \{(K_{\alpha_n}, L_{\alpha_n}) : n \in \mathbb{N}\}.$$

Let $\phi(1)$, for $1 \in S_0$, be the distinguished point of K_α. The definition for α a limit ordinal depends on the sequence (α_n). However, the properties of the space are not dependent on the sequence and we will assume that whenever we use a sequence approaching α, it is the same one. This completes the definition of the pairs (K_α, L_α). Notice that we actually have such a transfinite family of spaces for each $\zeta < \omega_1$. The choice of ζ will be made in the proof of Theorem 3.5.

Now we must consider the properties of these pairs. First we compute the topological information by using Lemma 1.2. As noted above, K_1 and L_1 are homeomorphic to $[1, \omega^{\omega^\zeta}]$. Notice that we have the following relations. If $K_{\alpha'}$ and $L_{\alpha'}$ are homeomorphic to $[1, \omega^{\omega^\zeta \beta}]$, then $K_{\alpha'+1}$ and $L_{\alpha'+1}$ are homeomorphic to $[1, \omega^{\omega^\zeta(\beta+1)}]$, by Lemma 1.2(1) and (2). If (α_n) is an increasing sequence of ordinals with limit α and K_{α_n} and L_{α_n} are homeomorphic to

$[1,\omega^{\omega^\zeta\beta_n}]$, then K_α and L_α are homeomorphic to $[1,\omega^{\omega^\zeta\beta}]$ where $\beta=\sup\beta_n$. Therefore a straightforward transfinite induction argument shows that K_α and L_α are homeomorphic to $[1,\omega^{\omega^\zeta\alpha}]$ for all $\alpha<\omega_1$.

Next we will show that L_α is equivalent to the standard basis of l_1. We will use Lemmas 1.2(3) and 1.3. For $\delta_1\in T_0$ we take $H_{\delta_1}=S_0$. For each $n\in\mathbb{N}$, $\frac12(\delta_\beta+\delta_{\omega^{\zeta n}})\in T_n$ and $\beta\le\omega^{\zeta n}$, we let $H_{(1/2)(\delta_\beta+\delta_{\omega^{\zeta n}})}=\{\beta\}$. If $l\in L_1$, then $H^1_l=\phi(S_0)$ if $l=\Phi(\delta_1)=\delta_{(1,\emptyset)}$, and $H^1_l=\{\delta_{(1,\delta_1,n,\beta)}\}$ if $l=\frac12(\delta_{(1,\delta_1,n,\beta)}+\delta_{(1,\emptyset)})$, as in Lemma 1.2. Notice that for all $l\ne\delta_{(1,\emptyset)}$, $l\in L_1$, we have $l\succ'\delta_{(1,\emptyset)}$. Thus by Lemma 1.3 with $(\mu_n)=(\delta_{(1,\delta_1,n,\beta)})_{\beta\le\omega^{\zeta n},\,n\in\mathbb{N}}$, L_1 is 3-equivalent to the basis of l_1. Notice that $\Phi(\delta_1)$ is the only element of L_1 which does not have a successor (under $\succ'$) and that if $l\succ' l'$, then $l_{|\bigcup\{H_m:m\ne l\}}=\frac12 l'$.

Assume inductively that for each $\beta<\alpha$, we have defined sets $H^\beta_l\subset K_\beta$ for all $l\in L_\beta$, satisfying the hypothesis of Lemma 1.3 with $\varepsilon=\frac12$, (μ_n) as the point mass measures on K_β and $l\succeq\delta_{k_\beta}$, for all $l\in L_\beta$. Further assume that if $l,l'\in L_\beta$ and $l\succ' l'$, then $l_{|\bigcup\{H_m:m\ne l\}}=\frac12 l'$. Define the sets $H^\alpha_{l'}$, $l'\in L_a$, as in Lemma 1.2(3). Thus Lemma 1.3(1) is satisfied. We need to verify the other hypotheses of Lemma 1.3. In order to handle the successor ordinal case and the limit ordinal case at the same time, let $O_1=K_{\alpha'}$ and $P_1=L_{\alpha'}$, and $O_n=S_0$ and $P_n=T_0$ for all $n>1$, $n\in\mathbb{N}$, in the case $\alpha=\alpha'+1$, and let $O_n=K_{\alpha_n}$ and $P_n=L_{\alpha_n}$ if $\alpha=\lim\alpha_n$. Also let $O_0=K_1$, $P_0=L_1$, or $O_0=S_0$, $P_0=T_0$, respectively. Let o_n be the distinguished point of O_n for $n=0,1,\ldots$ With this notation $(K_\alpha,L_\alpha)=(O_0,P_0)\otimes\{(O_n,P_n):n\in\mathbb{N}\}$. For each $l\in P_n$ let H^n_l be the associated subset of O_n.

Because ϕ is a homeomorphism, it follows that $\Phi(O_0)$ and H^0_l, $l\in\Phi(O_0)$, satisfy the hypothesis of Lemma 1.3. Moreover, $\phi(l_1)\succ'\phi(l_2)$ if and only if $l_1\succ' l_2$. Suppose $l_0,l''_0\in O_0$, $l_n\in P_n$, $l''_{n''}\in P_{n''}$, for some n, n'', and

$$l'=\sum_{k\in\operatorname{supp}l_0}l_0(k)\sum_{j_n\in K_n}l_n(j_n)\delta_{(k,l_0,n,j_n)}$$

and

$$l''=\sum_{k\in\operatorname{supp}l''_0}l''_0(k)\sum_{j_n\in K_{n''}}l''_{n''}(j_n)\delta_{(k,l''_0,n'',j_n)}.$$

Then $l'\succ l_0$ and $l''\succ l''_0$. If $l_0\ne l''_0$ or $n\ne n''$, then $H^\alpha_{l'}\cap H^\alpha_{l''}=\emptyset$ because $H^\alpha_{l'}=\bigcup_{k\in\operatorname{supp}l_0}\{k\}\times\{l_0\}\times\{n\}\times H^n_{l_n}$ and $H^\alpha_{l''}=\bigcup_{k\in\operatorname{supp}l''_0}\{k\}\times\{l''_0\}\times\{n''\}\times H^{n''}_{l''_{n''}}$. Also, $l'(H^\alpha_{l''})=0$ and $l''(H^\alpha_{l'})=0$. If $l_0=l''_0$ and $n=n''$ but $l_n\ne l_{n''}$, then $l'\succ l''$ if and only if $l_n\succ l_{n''}$. Moreover, because P_n and $(H^n_l)_{l\in P_n}$ satisfy the hypotheses of Lemma 1.3,

$$l_0\otimes P_n=\Big\{\sum_{k\in\operatorname{supp}l_0}l_0(k)\sum_{j_n\in K_n}l_n(j_n)\delta_{(k,l_0,n,j_n)}:l_n\in P_n\Big\}$$

and $(H_{l'})_{l'\in l_0\otimes P_n}$ satisfy the same conditions and

$$\sum_{k\in\operatorname{supp} l_0} l_0(k) \sum_{j_n\in K_n} l_n(j_n)\delta_{(k,l_0,n,j_n)} \succ l_0 = \sum_{k\in\operatorname{supp} l_0} l_0(k) \sum_{j_n\in K_n} \delta_{o_n}(j_n)\delta_{(k,l_0,n,j_n)}$$

for all l_n. Observe that Lemma 1.3(2) is therefore satisfied. Indeed, if $l_n \neq \delta_{o_n}$, then there is some $l'_n \in P_n$ such that $l_n \succ' l'_n$ and thus

$$\sum_{k\in\operatorname{supp} l_0} l_0(k) \sum_{j_n\in K_n} l_n(j_n)\delta_{(k,l_0,n,j_n)} \succ' \sum_{k\in\operatorname{supp} l_0} l_0(k) \sum_{j_n\in K_n} l'_n(j_n)\delta_{(k,l_0,n,j_n)}.$$

If $l_n = \delta_{o_n}$, then $l_0 = \sum_{k\in\operatorname{supp} l_0} l_0(k)\sum_{j_n\in K_n}\delta_{o_n}(j_n)\delta_{(k,l_0,n,j_n)}$ and there is some $l'_0 \in O_0$ such that $l_0 \succ' l'_0$ or $l_0 = \delta_{o_0}$. Further, by transitivity of $\succ$ it follows that for all $l \in L_\alpha$, $l \succeq \Phi(\delta_{o_0})$. For Lemma 1.3(3) we need only consider the case of an element of the form $\Phi(l)$ for some $l \in P_0$ and an element of the form $l' = \sum_{k\in\operatorname{supp} l_0} l_0(k)\sum_{j_n\in K_n} l_n(j_n)\delta_{(k,l_0,n,j_n)}$ where $l_0 \in P_0$ and $l_n \in P_n$ for some $n > 1$. In this case there are three possibilities:

(a) $l \succ l_0$,
(b) $l_0 \succeq l$, or
(c) neither (a) nor (b).

Case (b) gives $\sum_{k\in\operatorname{supp} l_0} l_0(k)\sum_{j_n\in K_n} l_n(j_n)\delta_{(k,l_0,n,j_n)} \succeq \Phi(l)$ and so there is nothing to do. In case (c), $\operatorname{supp} l' \subseteq \bigcup_{k\in\operatorname{supp} l_0}\{k\}\times\{l_0\}\times\{n\}\times O_n \cup \operatorname{supp}\Phi(l_0)$ and $H_{\Phi(l)} \subseteq \phi(O_0)\setminus\operatorname{supp}\Phi(l_0)$. Therefore, $\Phi(l)(H_{l'}) = 0$ and $l'(H_{\Phi(l)}) = 0$. In case (a), $\alpha = \alpha'+1$ and thus $l_0 = \delta_{(1,\emptyset)}$ and $l = \frac{1}{2}(\delta_o+\delta_{(1,\emptyset)})$ for some $o \in K_1\setminus\{(1,\emptyset)\}$. Clearly, $\Phi(l)(H_{l'}) = 0$ and $l'(H_{\Phi(l)}) = 0$ in this case also.

We have thus proved the following.

PROPOSITION 2.1. *For each $\zeta < \omega_1$ there is a family of pairs $(K_\alpha, L_\alpha)_{\alpha<\omega_1}$, where for each α, K_α is homeomorphic to $[1,\omega^{\omega^\zeta\alpha}]$ and L_α is a w^*-closed subset of the probability measures in $C(K_\alpha)^*$ which is homeomorphic to $[1,\omega^{\omega^\zeta\alpha}]$ in the w^*-topology. Moreover, L_α is 3-equivalent to the usual basis of l_1. Consequently, the evaluation map $T : C(K_\alpha)\to C(L_\alpha)$ defined by $T(f)(l) = l(f)$, for all $l\in L_\alpha$, is a surjection.*

Remark 2.1. Actually, $[L_\alpha]$ is isometric to l_1. To see this observe that for $\alpha = 1$ the elements $(l_{|H_l})_{l\in L_1}$ are disjointly supported elements of $[L_1]$ with span containing L_1. Thus the normalized sequence is a basis for $[L_1]$ which is 1-equivalent to the basis of l_1. An induction argument shows that for all $\alpha<\omega_1$, $(l_{|H_l})_{l\in L_\alpha}$ are disjointly supported elements of $[L_\alpha]$ with span containing L_α and thus $[L_\alpha]$ is isometric to l_1. Notice that this also means that the argument about the equivalence of L_α to the usual l_1 basis could have been made using $(l_{|H_l})_{l\in L_\alpha}$ in the role of (μ_n) in the application of Lemma 1.3 rather than using the point mass measures.

3. The Wolfe index of operators. In this section we will show that the evaluation operators defined in Proposition 2.1 are actually small in the sense that for most α the ordinals β for which there is a subspace X of $C(K_\alpha)$ which isomorphic to $C(\omega^\beta)$ and for which $T_{|X}$ is an isomorphism are much smaller than $\omega^\zeta\alpha$. The device for computing the possible ordinals β is an ordinal index which was defined in [W] and characterized in [A2].

Definition 3.1. Let K be a compact Hausdorff space, $\varepsilon>0$, and let B be a subset of $C(K)^*$. Let

$$P_0(\varepsilon,B)=\{(\mu,G):\mu\in B,\ G \text{ is open in } K,\ |\mu|(G)\ge\varepsilon\}.$$

If $P_\alpha(\varepsilon,B)$ has been defined, let

$$\begin{aligned}P_{\alpha+1}(\varepsilon,B)=\Big\{(\mu,G)\in P_0(\varepsilon,B):&\text{ there is a sequence}\\ &(\mu_n,G_n)_{n=1}^\infty\subset P_\alpha(\varepsilon,B)\text{ such that }\mu_n\xrightarrow{\mathrm{w}^*}\mu,\\ &G_n\cap G_{n'}=\emptyset,\ \text{for } n\ne n',\ \text{and } \overline{\bigcup G_n}\subset G\Big\}.\end{aligned}$$

For a limit ordinal β let

$$\begin{aligned}P_\beta(\varepsilon,B)=\Big\{(\mu,G)\in P_0(\varepsilon,B):&\text{ there is a sequence of ordinals}\\ &\alpha_n\uparrow\beta\text{ and }(\mu_n,G_n)\subset P_{\alpha_n}(\varepsilon,B)\text{ such that}\\ &\mu_n\xrightarrow{\mathrm{w}^*}\mu,\ G_n\cap G_{n'}=\emptyset,\ \text{for } n\ne n',\ \text{and } \overline{\bigcup G_n}\subset G\Big\}.\end{aligned}$$

The result that we will use here is the following.

Theorem 3.1. *Let T be a bounded operator from $C(K)$ into a separable Banach space X. Then there is a subspace Y of $C(K)$ such that Y is isomorphic to $C(\omega^{\omega^\alpha})$ and $T_{|Y}$ is an isomorphism if and only if there is an $\varepsilon>0$ such that $P_\gamma(\varepsilon,T^*(B_{X^*}))\ne\emptyset$ for all $\gamma<\omega^\alpha$.*

This result is an amalgamation of Theorems 0.2 and 0.3 from [A2]. It follows that we need only bound the Wolfe index. As in the previous section we keep ζ fixed and consider the evaluation operators $T_\alpha: C(K_\alpha)\to C(L_\alpha)$. In this case the expression $T^*(B_{X^*})$ which occurs in Theorem 3.1 is T_α^* of the unit ball of $C(L_\alpha)^*$, which is w*-closed. Also, if $\mu\in B_{C(L_\alpha)^*}$, then $\mu=\sum_{l\in L_\alpha}c_l\delta_l$, where $\sum_{l\in L_\alpha}|c_l|\le 1$. Hence $T^*(\mu)=\sum_{l\in L_\alpha}c_l l$. These observations will allow us to employ the following lemma from [A2] (Lemma 3.2) to reduce to considering only the sets L_α.

Lemma 3.2. *Let L be a w*-closed countable subset of $\{\mu:\mu\in B_{C(K)^*},\ \mu>0\}$ for some countable compact metric space K. Suppose that the eval-*

uation map $T : C(K) \to C(L)$ defined by $(Tf)(l) = l(f)$, for all $l \in L$, is surjective. Then, for $\alpha < \omega_1$, there is an $\varepsilon > 0$ such that $P_\gamma(\varepsilon, \overline{\mathrm{co}}(\pm L)) \neq \emptyset$, for all $\gamma < \omega^\alpha$, if and only if there is an $\varepsilon' > 0$ such that $P_\gamma(\varepsilon', L) \neq \emptyset$, for all $\gamma < \omega^\alpha$.

Before we apply this to the examples let us make a few observations about the sets $P_\gamma(\varepsilon, L_\alpha)$. Because the sequence (μ_n) occurring in the definition is a sequence of distinct elements, for any ordinals γ and η,

$$\{l : (l, G) \in P_{\gamma+\eta}(\varepsilon, L_\alpha)\} \subseteq \{l : (l, G) \in P_\gamma(\varepsilon, L_\alpha)\}^{(\eta)}.$$

Also, because the sets $P_\gamma(\varepsilon, L_\alpha)$ decrease to $\emptyset$,

$$\{l : (l, G) \in P_\gamma(\varepsilon, L_\alpha)\} \setminus \{l : (l, G) \in P_{\gamma+1}(\varepsilon, L_\alpha)\}$$

is dense in $\{l : (l, G) \in P_\gamma(\varepsilon, L_\alpha)\}$ for all γ. Moreover, if $(l, G) \in P_{\gamma+1}(\varepsilon, L_\alpha)$, there exists $((l_n, G_n))_{n=1}^\infty \subseteq P_\gamma(\varepsilon, L_\alpha) \setminus P_{\gamma+1}(\varepsilon, L_\alpha)$ such that $l_n \xrightarrow{\mathrm{w}^*} l$, $l_n(G_n) \geq \varepsilon$, $l(G) \geq \varepsilon$, $G_n \cap G_{n'} = \emptyset$, for $n \neq n'$, and $\overline{\bigcup G_n} \subset G$.

Definition 3.2. For each $\varepsilon > 0$ and $\alpha < \omega_1$ let $\varrho(\varepsilon, \alpha) = \sup\{\gamma : P_\gamma(\varepsilon, L_\alpha) \neq \emptyset\}$.

It is easy to see that $\varrho(\varepsilon, 1) = 0$ for $\alpha = 1$ and $\frac{1}{2} < \varepsilon$, and $\varrho(\varepsilon, 1) = \omega^\zeta$ for $0 < \varepsilon \leq \frac{1}{2}$. Obviously, $\varrho(\varepsilon, \alpha) = 0$ for all α if $\varepsilon > 1$. The lemma below will permit us to estimate $\varrho(\varepsilon, \alpha)$ for all $\alpha < \omega_1$ and $\varepsilon \leq 1$.

Lemma 3.3. *For each $\beta < \omega_1$, $\varrho(\varepsilon, \beta+1) \leq \max(\varrho(2\varepsilon, \beta) + \varrho(\varepsilon, 1), \varrho(\varepsilon, \beta) + 1)$. If $\beta_n \uparrow \beta$, then $\varrho(\varepsilon, \beta) = \lim \varrho(\varepsilon, \beta_n)$.*

Proof. Suppose that $\alpha = \beta + 1$ for some $\beta < \omega_1$. Before we begin estimating ϱ, let us look at the relationship between $P_\gamma(\varepsilon, L_\alpha \setminus \Phi(L_1))$ and pairs (l, G) with $l \in \Phi(L_1)$.

Let $(l_n)_{n\in\mathbb{N}} \subset L_\alpha \setminus \Phi(L_1)^{(1)}$, $l \in \Phi(L_1)$ and let $(G_n)_{n\in\mathbb{N}}$ and G be open subsets of K_α such that $l_n \xrightarrow{\mathrm{w}^*} l$, $l_n(G_n) \geq \varepsilon$, $l(G) \geq \varepsilon$, $G_n \cap G_{n'} = \emptyset$, for $n \neq n'$, and $\overline{\bigcup G_n} \subset G$. Then for some $k \in K$, $l = \frac{1}{2}(\delta_k + \delta_{k_1})$, and by passing to a subsequence we may assume that

$$l_n = \frac{1}{2}\Big(\sum_{j\in K_\beta} l'_n(j)\delta_{(k(n), l''_n, m, j)} + \sum_{j\in K_\beta} l'_n(j)\delta_{(k_1, l''_n, m, j)}\Big),$$

for some $l'_n \in L_\beta$, and $l''_n = \frac{1}{2}(\delta_{k(n)} + \delta_{k_1})$ with $l''_n \xrightarrow{\mathrm{w}^*} l$. (Because of the definition of the pair (K_α, L_α) for α a successor ordinal, only the value 1 of the third index m is of any interest.) For each n let

$$l_n^1 = \sum_{j\in K_\beta} l'_n(j)\delta_{(k(n), l''_n, m, j)} \quad \text{and} \quad l_n^2 = \sum_{j\in K_\beta} l'_n(j)\delta_{(k_1, l''_n, m, j)}.$$

Then $l_n^1(G_n) \geq \varepsilon$ or $l_n^2(G_n) \geq \varepsilon$ for each n and thus for one of (l_n^1) and (l_n^2) there are infinitely many such n.

Suppose that for some $\gamma<\omega_1$, $(l_n,G_n)\in P_\gamma(\varepsilon,L_\alpha)$ for each n. Because L_α is homeomorphic to $[1,\omega^{\omega^\zeta\alpha}]$, there is a closed neighborhood M_n of l_n in L_α and an ordinal γ'', $\omega^\zeta\beta\ge\gamma''\ge\gamma$, such that $M_n^{(\gamma'')}=\{l_n\}$. Moreover, we may assume that there is a closed subset M_n' of L_β such that

$$M_n=\Big\{\frac12\Big(\sum_{j\in K_\beta}l'(j)\delta_{(k(n),l_n'',m,j)}+\sum_{j\in K_\beta}l_n'(j)\delta_{(k_1,l_n'',m,j)}\Big):l'\in M_n'\Big\}.$$

Because $(l_n,G_n)\in P_\gamma(\varepsilon,L_\alpha)$, we have $(l_n,G_n)\in P_\gamma(\varepsilon,M_n)$. Let

$$M_n^1=\Big\{\sum_{j\in K_\beta}l_n'(j)\delta_{(k(n),l_n'',m,j)}:l_n'\in M_n'\Big\}$$
$$M_n^2=\Big\{\sum_{j\in K_\beta}l_n'(j)\delta_{(k_1,l_n'',m,j)}:l_n'\in M_n'\Big\}.$$

Then a transfinite induction argument shows that $(l_n^1,G_n^1)\in P_\gamma(\varepsilon,M_n^1)$ or $(l_n^2,G_n^2)\in P_\gamma(\varepsilon,M_n^2)$ for infinitely many n, where $G_n^1=\{j:(k(n),l_n'',m,j)\in G_n\}$ and $G_n^2=\{j:(k_1,l_n'',m,j)\in G_n\}$. (The argument is essentially the same as the proof of Lemma 3.2.) Because the mapping $\psi:(k,l,m,j)\to j$ is a homeomorphism, $(l_n^i,\psi(G_n^i))\in P_\gamma(\varepsilon,M_n^i)$ for $i=1$ or 2. Thus we must have $\varrho(\varepsilon,L_\beta)\ge\gamma$. In particular, if $l\in\Phi(L_1)^{(0)}$, we have $(l,G)\in P_{\gamma+1}(\varepsilon,L_\alpha)$ only if $P_\gamma(\varepsilon,L_\beta)\ne\emptyset$.

Notice that in the situation above if $\phi(k_0)\notin G$, then $\phi(k_0)\notin\overline{\bigcup G_n}$. Thus for large n, $l_n^2(G_n)=0$. In order for $l_n(G_n)\ge\varepsilon$, we must have $l_n^1(G_n)\ge2\varepsilon$, and $(l_n^1,G_n^1)\in P_\gamma(2\varepsilon,M_n^1)$. Thus $P_\gamma(2\varepsilon,L_\beta)\ne\emptyset$, and $(l_n,G_n)\in P_\gamma(\varepsilon,L_\alpha)$, only if $(l_n',\psi(G_n^1))\in P_\gamma(2\varepsilon,L_\beta)$.

With these observations we can now estimate $\varrho(\varepsilon,\alpha)$.

Suppose that $(l_0,G_0)\in P_\gamma(\varepsilon,L_\alpha)$ for some $\gamma\ge\max(\varrho(2\varepsilon,\beta)+\omega^\zeta+1,\varrho(\varepsilon,\beta)+2)$. Then there exists $(l_{0,n},G_{0,n})\in P_{\gamma_n}(\varepsilon,L_\alpha)$, where $\gamma_n=\gamma-1$ if γ is a successor ordinal, or $\gamma_n\uparrow\gamma$, $\gamma_n\ge\max(\varrho(2\varepsilon,\beta)+\omega^\zeta,\varrho(\varepsilon,\beta)+1)$, if γ is a limit ordinal, such that $l_{0,n}\xrightarrow{\mathrm{w}^*}l_0$, $G_{0,n}\cap G_{0,n'}=\emptyset$, for $n\ne n'$, and $\overline{\bigcup G_{0,n}}\subset G_0$. If $l_{0,n}\in\Phi(L_1)$ for infinitely many n, then for at most one n, $\phi(k_1)\in G_{0,n}$. We can assume, by discarding that one, that there is no such n. Also in this case $\varepsilon\le1/2$, since $\varrho(\varepsilon,1)=0$ for $\varepsilon>1/2$. Because $\gamma\ge\varrho(\varepsilon,\beta)+1\ge\omega^\zeta+1$ and $l_{0,n}\notin\Phi(L_1)^{(\omega^\zeta)}$, there is a w*-open neighborhood O of $l_{0,n}$ such that $P_{\omega^\zeta}(\varepsilon,O\cap\Phi(L_1))=\emptyset$ and $(l_{0,n},G_{0,n})\in P_{\gamma_n}(\varepsilon,O)$. Therefore there exists $((\mu_n,F_n))_{n=1}^\infty\subseteq P_\eta(\varepsilon,O\setminus\Phi(L_1))$ with $\mu_n\xrightarrow{\mathrm{w}^*}l_{0,n}$, $\overline{\bigcup F_i}\subseteq G_{0,n}$, $F_i\cap F_{i'}=\emptyset$ for $i\ne i'$, and $\eta+\xi\ge\gamma_n$ for some $\xi<\omega^\zeta$. Because $\phi(k_1)\notin G_{0,n}$, it follows from the argument above that $P_\eta(2\varepsilon,L_\beta)\ne\emptyset$. Thus $\eta\le\varrho(2\varepsilon,\beta)$. This contradicts the choice of $\gamma\ge\varrho(2\varepsilon,\beta)+\omega^\zeta+1$. Hence $l_{0,n}\in\Phi(L_1)$ for only finitely many n. This implies that $P_{\gamma_n}(\varepsilon,L_\beta)\ne\emptyset$ for all but finitely many n and hence $\gamma_n\le\varrho(\varepsilon,\beta)$, again a contradiction to the choice of γ. Thus we have $\varrho(\varepsilon, \beta+1) \le \max(\varrho(2\varepsilon, \beta) + \omega^\zeta, \varrho(\varepsilon, \beta)+1)$ for $\varepsilon \le 1/2$. If $\varepsilon > 1/2$, then $l_{0,n} \notin \Phi(L_1)$ and thus $\gamma_n \le \varrho(\varepsilon, \beta)$. Hence $\varrho(\varepsilon, \alpha) \le \varrho(\varepsilon, \beta) + 1 \le \max(0+0, \varrho(\varepsilon, \beta)+1) = \max(\varrho(2\varepsilon, \beta) + \varrho(\varepsilon, 1), \varrho(\varepsilon, \beta)+1)$ for $\varepsilon > 1/2$.

In the case when α is a limit ordinal it is easy to see that $\varrho(\varepsilon, \alpha) \le \sup\{\varrho(\varepsilon, \beta) : \beta < \alpha\}$ because we have simply glued the spaces K_{α_n}, $n \in \mathbb{N}$, at their distinguished points to make K_α. ∎

PROPOSITION 3.4. *Suppose $\alpha < \omega_1$ and $\alpha = \beta + \eta$, where $\eta < \omega^{\zeta+1}$ and β is the smallest ordinal for which there exists such an η. Then $\varrho(\varepsilon, \alpha) \le \beta + \omega^\zeta l + \eta$ for $2^{-(l+1)} < \varepsilon \le 2^{-l}$, $l = 0, 1, 2, \ldots$*

Proof. The proof is by induction on α. We have already computed $\varrho(\varepsilon, 1)$ and it clearly satisfies the inequality. Suppose that it is true for all $\alpha' < \alpha$. Fix l and ε.

If α is a limit ordinal and $\alpha_n \uparrow \alpha$, let $\alpha_n = \beta_n + \eta_n$ be the decomposition of α_n with β_n minimal and $\eta_n < \omega^{\zeta+1}$. If the sequence (β_n) is not eventually constant, then $\alpha = \alpha + 0$ is the decomposition of α and $\alpha = \lim \alpha_n = \lim \beta_n = \lim \beta_n + \omega^\zeta l + \eta_n \ge \varrho(\varepsilon, \alpha)$, by Lemma 3.3. If there is some n_0 such that $\beta_n = \beta'$ for all $n \ge n_0$, then $\lim \beta_n + \omega^\zeta l + \eta_n = \beta_0 + \omega^\zeta l + \lim \eta_n$. If $\lim \eta_n < \omega^{\zeta+1}$, then $\alpha = \beta_0 + \lim \eta_n$ is the decomposition of α and the inequality holds; if not, $\eta = 0$ and $\alpha = (\beta_0 + \omega^{\zeta+1}) + 0$ is the decomposition. In this case $\omega^\zeta l + \lim \eta_n = \omega^{\zeta+1}$ and thus $\varrho(\varepsilon, \alpha) \le \beta_0 + \omega^{\zeta+1} = \beta$.

If $\alpha = \alpha' + 1$, and $\alpha' = \beta' + \eta'$ is the decomposition of α', then $\alpha = \beta' + (\eta' + 1)$ is the decomposition of α. By Lemma 3.3, $\varrho(\varepsilon, \alpha) \le \max(\varrho(2\varepsilon, \alpha') + \varrho(\varepsilon, 1), \varrho(\varepsilon, \alpha') + 1)$. If $\varepsilon > 1/2$, $l = 0$, then $\varrho(2\varepsilon, \alpha') + \varrho(\varepsilon, 1) = 0$ and $\varrho(\varepsilon, \alpha') + 1 \le \beta' + \eta' + 1$, as required. If $\varepsilon \le 1/2$, then

$$\begin{aligned}\max(\varrho(2\varepsilon, \alpha') &+ \varrho(\varepsilon, 1), \varrho(\varepsilon, \alpha') + 1) \\ &\le \max(\beta' + \omega^\zeta(l-1) + \eta' + \omega^\zeta, \beta' + \omega^\zeta l + \eta' + 1) \\ &= \beta' + \omega^\zeta l + \eta' + 1 = \beta + \omega^\zeta l + \eta. \ \blacksquare\end{aligned}$$

We now have all of the tools to prove our main result.

THEOREM 3.5. *If $1 \le \zeta < \alpha < \zeta\omega < \omega_1$, then there is an operator T from $C(\omega^{\omega^\alpha})$ onto itself such that if Y is a subspace of $C(\omega^{\omega^\alpha})$ which is isomorphic to $C(\omega^{\omega^\alpha})$ then $T_{|Y}$ is not an isomorphism.*

Proof. Let γ satisfy $\alpha = \zeta + \gamma$. For $(K_{\omega^\gamma}, L_{\omega^\gamma})$ constructed for the ordinal ζ, i.e., for K_1 homeomorphic to $[1, \omega^{\omega^\zeta}]$, and for $2^{-(l+1)} < \varepsilon \le 2^{-l}$, we have $\varrho(\varepsilon, \omega^\gamma) \le \beta + \omega^\zeta l + \eta$, where $\omega^\gamma = \beta + \eta$ and $\eta < \omega^{\zeta+1}$. Therefore $\varrho(\varepsilon, \omega^\gamma) < \beta + \omega^{\zeta+1}$ for every $\varepsilon > 0$. Observe that if $\gamma > 1$, then $\omega^\alpha = \omega^{\zeta+\gamma} > \max(\omega^\gamma, \omega^{\zeta+1})2 \ge \varrho(\varepsilon, \omega^\gamma)$, and if $\gamma = 1$, then $\omega^\zeta l + \omega^\gamma \le \omega^\zeta(l+1) < \omega^{\zeta+1} = \omega^\alpha$. Thus by Theorem 3.1 and Lemma 3.2, there is no subspace Y of $C(\omega^{\omega^\alpha})$ which is isomorphic to $C(\omega^{\omega^\alpha})$ such that $T_{|Y}$ is an isomorphism. ∎

The failure of the condition given in Theorem 3.5 for all ζ is equivalent to $\alpha = \omega^\gamma$ for some $\gamma < \omega_1$. This is still a long way from Bourgain's condition $\omega\omega^\alpha = \omega^{\omega^\alpha}$, which guarantees the existence of subspaces isomorphic to $C(\omega^{\omega^\alpha})$ on which maps of $C(\omega^{\omega^\alpha})$ onto itself would be isomorphisms. On the other hand, the estimate for the inductive step given in Lemma 3.3 is sometimes generous. In specific cases we have $\varrho(\varepsilon, \beta) = \varrho(\varepsilon, \beta + 1)$. Thus it could be that a more careful estimate of $\varrho(\varepsilon, \alpha)$ would yield a stronger result. It also seems likely that there is room for improvement in Bourgain's estimates.

References

[A1] D. E. Alspach, *Quotients of* $C[0,1]$ *with separable dual*, Israel J. Math. 29 (1978), 361–384.
[A2] —, $C(K)$ *norming subsets of* $C[0,1]^*$, Studia Math. 70 (1981), 27–61.
[BP] C. Bessaga and A. Pełczyński, *Spaces of continuous functions IV*, ibid. 19 (1960), 53–62.
[BD] E. Bishop and K. de Leeuw, *The representation of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier (Grenoble) 9 (1959), 305–331.
[B] J. Bourgain, *The Szlenk index and operators on* $C(K)$*-spaces*, Bull. Soc. Math. Belg. Sér. B 31 (1979), 87–117.
[G] I. Gasparis, *Quotients of* $C(K)$ *spaces*, dissertation, The University of Texas, 1995.
[G1] —, *Operators that do not preserve* $C(\alpha)$*-spaces*, preprint.
[MS] S. Mazurkiewicz et W. Sierpiński, *Contributions à la topologie des ensembles dénombrables*, Fund. Math. 1 (1920), 17–27.
[P] A. Pełczyński, *On strictly singular and cosingular operators I. Strictly singular and strictly cosingular operators on* $C(S)$ *spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1965), 31–36.
[W] J. Wolfe, $C(\alpha)$ *preserving operators on* $C(K)$ *spaces*, Trans. Amer. Math. Soc. 273 (1982), 705–719.

Department of Mathematics
Oklahoma State University
Stillwater, Oklahoma 74078
U.S.A.
E-mail: alspach@math.okstate.edu

Received 22 October 1996