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The stability of Markov operators on Polish spaces

by

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Abstract. A sufficient condition for the asymptotic stability of Markov operators acting on measures defined on Polish spaces is presented.

1. Introduction. We study Markov operators defined on a Polish space X. Our goal is to prove sufficient conditions for the asymptotic stability of such operators. The crucial point in proving stability is to show the existence of an invariant measure. When Markov operators are defined on a compact space, the proof of the existence goes as follows. First we construct a positive, invariant functional defined on the space of all continuous bounded functions $f : X \to \mathbb{R}$ and then using the Riesz representation theorem we define an invariant measure. This method was extended by A. Lasota and J. Yorke to the case when X is a locally compact and σ-compact metric space [8]. When X is a Polish space this idea breaks down, since a positive functional may not correspond to a measure. Therefore we base on the concept of tightness. The main idea taken from [8] is nonexpansiveness in the Fortet–Mourier distance. It is known (for details see [1, 6, 7, 8, 9]) that a broad spectrum of Markov processes do not increase the distance between measures transported by the corresponding transition operators. For such operators our results could be applied.

The organization of the paper is as follows. Section 2 contains some notation from the theory of Markov operators. In Section 3 we give some general conditions for asymptotic stability and discuss the condition for nonexpansiveness of P.

2. Preliminaries. Let (X, ϱ) be a *Polish space*, i.e. a separable, complete metric space. Throughout this paper $B(x, r)$ stands for the open ball in X with centre at x and radius r. For every set $C \subset X$ we denote by $\operatorname{diam} C$

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the diameter of C. For every $C \subset X$ and $r > 0$ we denote by $\mathcal{N}^0(C,r)$ the open r-neighbourhood of C, i.e.

$$\mathcal{N}^0(C,r) = \{x \in X : \varrho(C,x) < r\},$$

and by $\mathcal{N}(C,r)$ the closed r-neighbourhood of C, i.e.

$$\mathcal{N}(C,r) = \{x \in X : \varrho(C,x) \le r\},$$

where $\varrho(C,x) = \inf\{\varrho(x,y) : y \in C\}$.

By $\mathcal{M}_{\text{fin}}$ and $\mathcal{M}_1$ we denote the sets of Borel measures (nonnegative, σ-additive) on X such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}_{\text{fin}}$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1$. The elements of $\mathcal{M}_1$ are called *distributions*.

We say that $\mu \in \mathcal{M}_{\text{fin}}$ is *concentrated* on a Borel set $A \subset X$ if $\mu(X \setminus A) = 0$. The set of all distributions concentrated on the Borel set A is denoted by $\mathcal{M}_1^A$.

By $\mathcal{C}_\varepsilon$, $\varepsilon > 0$, we denote the family of all closed sets $C \subset X$ for which there exists a finite set $\{x_1, \ldots, x_n\} \subset X$ such that $C \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$.

As usual, $B(X)$ stands for the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$, and $C(X)$ is the subspace of all bounded continuous functions. In both spaces the norm is

$$\|f\|_0 = \sup_{x \in X} |f(x)|.$$

An operator $P : \mathcal{M}_{\text{fin}} \to \mathcal{M}_{\text{fin}}$ is called a *Markov operator* if it satisfies the following two conditions:

(i) *positive linearity*:

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$$

for $\lambda_1, \lambda_2 \ge 0$ and $\mu_1, \mu_2 \in \mathcal{M}_{\text{fin}}$,

(ii) *preservation of the norm*:

$$P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}_{\text{fin}}.$$

It is easy to prove that every Markov operator can be extended to the space of signed measures

$$\mathcal{M}_{\text{sig}} = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}_{\text{fin}}\}.$$

Namely for every $\nu \in \mathcal{M}_{\text{sig}}$, $\nu = \mu_1 - \mu_2$, we set

$$P\nu = P\mu_1 - P\mu_2.$$

To simplify the notation we write

$$\langle f, \nu \rangle = \int_X f(x)\,\nu(dx) \quad \text{for } f \in B(X),\ \nu \in \mathcal{M}_{\text{sig}}. \tag{2.1}$$

A Markov operator P is called a *Feller operator* if there is a linear operator $U : C(X) \to C(X)$ (dual to P) such that

$$\langle Uf, \mu \rangle = \langle f, P\mu \rangle \quad \text{for } f \in C(X),\ \mu \in \mathcal{M}_{\text{fin}}. \tag{2.2}$$

We may extend U to all bounded measurable (or nonnegative measurable) functions by setting

$$Uf(x) = \langle f, P\delta_x \rangle \quad \text{for } f \in B(X),\ x \in X, \tag{2.3}$$

where $\delta_x \in \mathcal{M}_1$ is the point (Dirac) measure supported at x.

In the space $\mathcal{M}_{\text{sig}}$ we introduce the *Fortet–Mourier norm* (see [5])

$$\|\nu\| = \sup\{|\langle f, \nu \rangle| : f \in \mathcal{F}\},$$

where $\mathcal{F} \subset C(X)$ consists of the functions such that $|f| \le 1$ and $|f(x) - f(y)| \le \varrho(x,y)$. It is known that the convergence

$$\lim_{n\to\infty} \|\mu_n - \mu\| = 0 \quad \text{for } \mu_n, \mu \in \mathcal{M}_1 \tag{2.4}$$

is equivalent to the weak convergence of $(\mu_n)_{n\ge1}$ to μ (see [3, 4]).

A family of distributions $(\mu_n)_{n\ge1}$ is called *tight* if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\mu_n(K) \ge 1 - \varepsilon$ for $n \in \mathbb{N}$.

It is well known (see [2, 3]) that if the family of distributions $(\mu_n)_{n\ge1}$ is tight then there exists a subsequence of integers $(m_n)_{n\ge1}$ and a measure $\mu_* \in \mathcal{M}_1$ such that

$$\lim_{n\to\infty} \|\mu_{m_n} - \mu_*\| = 0.$$

A Markov operator P is called *nonexpansive* if

$$\|P\mu_1 - P\mu_2\| \le \|\mu_1 - \mu_2\| \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1. \tag{2.5}$$

Let P be a Markov operator. A measure $\mu \in \mathcal{M}_{\text{fin}}$ is called *stationary* or *invariant* if $P\mu = \mu$, and P is called *asymptotically stable* if there exists a stationary distribution μ_* such that

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\| = 0 \quad \text{for } \mu \in \mathcal{M}_1. \tag{2.6}$$

Clearly the distribution μ_* satisfying (2.6) is unique.

3. Asymptotic stability. The main aim of this part of the paper is to extend the result proved by A. Lasota and J. Yorke (see [8]) concerning Markov operators defined on locally compact and σ-compact metric spaces to Polish spaces. In order to do it we need the following assumption for a Markov operator $P : \mathcal{M}_{\text{fin}} \to \mathcal{M}_{\text{fin}}$:

(A) For every $\varepsilon > 0$ there is a Borel set $A \subset X$ with $\operatorname{diam} A \le \varepsilon$ and a number $\alpha > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(A) \ge \alpha \quad \text{for } \mu \in \mathcal{M}_1.$$

We start with easy lemmas:

LEMMA 3.1. *If $\|\mu_1 - \mu_2\| \le \varepsilon^2$ for $\mu_1, \mu_2 \in \mathcal{M}_1$ and some $\varepsilon > 0$ then*

$$\mu_1(\mathcal{N}^0(C,\varepsilon)) \ge \mu_2(C) - \varepsilon \quad \text{for every Borel set } C \subset X.$$

Proof. Fix a Borel set $C \subset X$. Define $f(x) = \max(\varepsilon - \varrho(C,x), 0)$. Since $f \in \mathcal{F}$ and $f(x) = 0$ for $x \notin \mathcal{N}^0(C,\varepsilon)$, while $f(x) = \varepsilon$ for $x \in C$, we have

$$\varepsilon\mu_2(C) - \varepsilon\mu_1(\mathcal{N}^0(C,\varepsilon)) \le |\langle f, \mu_1\rangle - \langle f, \mu_2\rangle| \le \|\mu_1 - \mu_2\| \le \varepsilon^2,$$

and the assertion follows. ∎

LEMMA 3.2. *Let P be a nonexpansive Markov operator. Suppose that there exists a measure $\mu \in \mathcal{M}_1$ such that for every $\varepsilon > 0$ there is a set $C_\varepsilon \in \mathcal{C}_\varepsilon$ satisfying $P^n\mu(C_\varepsilon) \ge 1 - \varepsilon$ for $n \in \mathbb{N}$. Then P has an invariant distribution.*

Proof. We first show that $(P^n\mu)_{n\ge1}$ is tight. Fix $\varepsilon > 0$. Let $C_{\varepsilon/2^k} \in \mathcal{C}_{\varepsilon/2^k}$, $k \ge 1$, be such that

$$P^n\mu(C_{\varepsilon/2^k}) \ge 1 - \varepsilon/2^k \quad \text{for } n \in \mathbb{N}.$$

Define $K = \bigcap_{k=1}^\infty C_{\varepsilon/2^k}$. Observe that K is compact and

$$P^n\mu(X \setminus K) = P^n\mu\Big(\bigcup_{k=1}^\infty (X \setminus C_{\varepsilon/2^k})\Big)$$
$$\le \sum_{k=1}^\infty P^n\mu(X \setminus C_{\varepsilon/2^k}) \le \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = \varepsilon \quad \text{for } n \in \mathbb{N}.$$

Set

$$\overline{\mu}_n = \frac{\mu + P\mu + \ldots + P^{n-1}\mu}{n} \quad \text{for } n \in \mathbb{N}. \tag{3.1}$$

By the above, the sequence $(\overline{\mu}_n)_{n\ge1}$ is tight. Hence there exists a subsequence of integers $(m_n)_{n\ge1}$ and a distribution $\overline{\mu}$ such that $\overline{\mu}_{m_n} \to \overline{\mu}$ as $n \to \infty$. Since P is nonexpansive, we get $P\overline{\mu}_{m_n} \to P\overline{\mu}$. From (3.1) it follows that $\|P\overline{\mu}_{m_n} - \overline{\mu}_{m_n}\| \to 0$ as $n \to \infty$ and consequently $P\overline{\mu} = \overline{\mu}$. ∎

We are now in a position to prove the main result of our paper.

THEOREM 3.3. *Let P be a nonexpansive Markov operator. Assume that the condition (A) holds. Then P is asymptotically stable.*

Proof. As in the proof of Theorem 4.1 in [8], (A) gives

$$\lim_{n\to\infty} \|P^n(\mu_1 - \mu_2)\| = 0 \quad \text{for } \mu_1, \mu_2 \in \mathcal{M}_1. \tag{3.2}$$

We proceed to show that for P there exists an invariant measure $\mu_* \in \mathcal{M}_1$.

By Lemma 3.2 it is enough to prove that for every $\varepsilon > 0$ and $\mu \in \mathcal{M}_1$ there exists $C_\varepsilon \in \mathcal{C}_\varepsilon$ such that $P^n\mu(C_\varepsilon) \ge 1 - \varepsilon$ for $n \in \mathbb{N}$. To do this fix $\varepsilon > 0$ and $\mu \in \mathcal{M}_1$. Set $\overline{\varepsilon} = \varepsilon^2/16$. Let $\alpha > 0$ and $A \subset X$ be such that (A) holds for $\overline{\varepsilon}$. If $P^n\mu(A) \ge \alpha/2$, then

$$P^n\mu \ge \frac{\alpha}{2}\nu_n, \tag{3.3}$$

where $\nu_n \in \mathcal{M}_1^A$ is of the form

$$\nu_n(B) = \frac{P^n\mu(B \cap A)}{P^n\mu(A)}. \tag{3.4}$$

Define

$$\delta = \sup\{\gamma \ge 0 : \exists C_{\varepsilon/2} \in \mathcal{C}_{\varepsilon/2} \quad \liminf_{n\to\infty} P^n\mu(C_{\varepsilon/2}) \ge \gamma\}. \tag{3.5}$$

Let $\gamma \ge 0$ and $C_{\varepsilon/2} \in \mathcal{C}_{\varepsilon/2}$ be such that $0 \le \delta - \gamma < \alpha\varepsilon/8$ and

$$\liminf_{n\to\infty} P^n\mu(C_{\varepsilon/2}) \ge \gamma.$$

We are now in a position to show that

$$P^n\nu(\mathcal{N}^0(C_{\varepsilon/2}, \varepsilon/2)) \ge 1 - \varepsilon/2 \quad \text{for } n \in \mathbb{N} \text{ and } \nu \in \mathcal{M}_1^A. \tag{3.6}$$

Suppose that, on the contrary, for some $\nu_0 \in \mathcal{M}_1^A$ and $n_0 \in \mathbb{N}$,

$$P^{n_0}\nu_0(\mathcal{N}^0(C_{\varepsilon/2}, \varepsilon/2)) < 1 - \varepsilon/2. \tag{3.7}$$

By the Ulam theorem, there exists a compact set $K \subset X \setminus \mathcal{N}^0(C_{\varepsilon/2}, \varepsilon/2)$ such that $P^{n_0}\nu_0(K) \ge \varepsilon/2$. Since P is nonexpansive, we have

$$\|P^{n_0}\nu_0 - P^{n_0}\nu\| \le \|\nu_0 - \nu\| \le \operatorname{diam} A \le \varepsilon^2/16$$

for $\nu \in \mathcal{M}_1^A$. Now Lemma 3.1 shows that $P^{n_0}\nu(\mathcal{N}^0(K, \varepsilon/4)) \ge \varepsilon/4$. Putting $B = \mathcal{N}(K, \varepsilon/4)$ we get $B \in \mathcal{C}_{\varepsilon/2}$ and consequently $B \cup C_{\varepsilon/2} \in \mathcal{C}_{\varepsilon/2}$. Applying (3.3) we see that

$$P^{n+n_0}\mu(B) \ge \frac{\alpha}{2}P^{n_0}\nu_n(B) \ge \frac{\alpha\varepsilon}{8}$$

for every sufficiently large n. Since $B \cap C_{\varepsilon/2} = \emptyset$, we deduce that

$$\liminf_{n\to\infty} P^n\mu(B \cup C_{\varepsilon/2}) \ge \liminf_{n\to\infty} P^n\mu(B) + \liminf_{n\to\infty} P^n\mu(C_{\varepsilon/2})$$
$$\ge \gamma + \alpha\varepsilon/8 > \delta,$$

which contradicts the definition of δ. Thus (3.6) holds. Put $C = \mathcal{N}(C_{\varepsilon/2}, \varepsilon/2)$ and note that $C \in \mathcal{C}_\varepsilon$.

We define by induction a sequence of integers $(n_k)_{k\ge0}$ and two sequences of distributions $(\mu_k)_{k\ge0}$, $(\nu_k)_{k\ge0}$. If $k = 0$ we set $n_0 = 0$ and $\mu_0 = \nu_0 = \mu$. If $k \ge 1$ and n_{k-1}, μ_{k-1}, ν_{k-1} are given we choose, according to (A), n_k such that

$$P^{n_k}\mu_{k-1}(A) \ge \alpha/2$$

and we define

$$\nu_k(B) = \frac{P^{n_k}\mu_{k-1}(B\cap A)}{P^{n_k}\mu_{k-1}(A)},$$

$$\mu_k(B) = \frac{1}{1-\alpha/2}\Big(P^{n_k}\mu_{k-1}(B) - \frac{\alpha}{2}\nu_k(B)\Big). \tag{3.8}$$

Observe that $\nu_k \in \mathcal{M}_1^A$. Using (3.8) it is easy to verify by induction that

$$P^{n_1+\ldots+n_k}\mu = \frac{\alpha}{2}P^{n_2+\ldots+n_k}\nu_1 + \frac{\alpha}{2}\Big(1-\frac{\alpha}{2}\Big)P^{n_3+\ldots+n_k}\nu_2 + \ldots + \frac{\alpha}{2}\Big(1-\frac{\alpha}{2}\Big)^{k-1}\nu_k + \Big(1-\frac{\alpha}{2}\Big)^k\mu_k. \tag{3.9}$$

Let $k \in \mathbb{N}$ be such that

$$(1-(1-\alpha/2)^k)(1-\varepsilon/2) \ge 1-\varepsilon.$$

Since $\nu_i \in \mathcal{M}_1^A$, $i=1,\ldots,k$, and (3.6) holds, we have

$$\begin{aligned}P^n\mu(C) &\ge \frac{\alpha}{2}P^{n-n_1}\nu_1(C) + \frac{\alpha}{2}\Big(1-\frac{\alpha}{2}\Big)P^{n-n_1-n_2}\nu_2(C) \\ &\quad + \ldots + \frac{\alpha}{2}\Big(1-\frac{\alpha}{2}\Big)^{k-1}P^{n-n_1-\ldots-n_k}\nu_k(C) \\ &\ge (1-(1-\alpha/2)^k)(1-\varepsilon/2) \ge 1-\varepsilon\end{aligned}$$

for $n \ge n_1+\ldots+n_k$. By the Ulam theorem, we can find a compact set $K \subset X$ such that

$$P^n\mu(K\cup C) \ge 1-\varepsilon \quad \text{for } n \in \mathbb{N}.$$

Since $K\cup C \in \mathcal{C}_\varepsilon$, Lemma 3.2 shows that P has an invariant distribution. ∎

The remaining part of this paper is devoted to nonexpansiveness. We formulate a sufficient condition for the nonexpansiveness of P.

PROPOSITION 3.4. *Let P be a Markov operator. Suppose that P is continuous in the weak topology. Then P is a Feller operator. Moreover, if the operator $U : B(X) \to B(X)$ given by* (2.3) *satisfies $U(\mathcal{F}) \subset \mathcal{F}$, then P is nonexpansive.*

Proof. Let P be as in the statement of the proposition and let $U : B(X) \to B(X)$ be given by (2.3). Obviously U is linear. Further, for every sequence $x_n \to x_0$, $x_n, x_0 \in X$, we have $\delta_{x_n} \to \delta_{x_0}$ in the weak topology. Since P is continuous, we obtain $P\delta_{x_n} \to P\delta_{x_0}$ in the weak topology. Consequently, by the definition of U we have

$$Uf(x_n) \to Uf(x_0) \quad \text{for } f \in C(X).$$

Thus we have verified that $U(C(X)) \subset C(X)$. According to the definition of U we have the equality

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle$$

for $f \in C(X)$ and $\mu = \delta_x$. Since the linear combinations of point measures are dense in $\mathcal{M}_{\text{fin}}$ (in the weak topology) and P is continuous, the equality holds for every $\mu \in \mathcal{M}_{\text{fin}}$. Since $U(\mathcal{F}) \subset \mathcal{F}$, we have

$$\begin{aligned}\|P\mu_1 - P\mu_2\| &= \sup\{\langle f, P\mu_1 - P\mu_2\rangle : f \in \mathcal{F}\} = \sup\{\langle Uf, \mu_1-\mu_2\rangle : f\in\mathcal{F}\} \\ &\le \sup\{\langle f, P\mu_1 - P\mu_2\rangle : f \in \mathcal{F}\} = \|\mu_1-\mu_2\|\end{aligned}$$

for all $\mu_1, \mu_2 \in \mathcal{M}_1$, which completes the proof. ∎

Since the convergence in the Fortet–Mourier distance is equivalent to the weak convergence, the asymptotic stability of a Markov operator acting on measures defined on the Polish space (X,ϱ) may be verified without the precise knowledge of the metric ϱ. An important role is played by the space $C(X)$ of all continuous functions. Following Lasota and Yorke we introduce the definition of essentially nonexpansive Markov operators.

We say that a metric ϱ' is *equivalent* to ϱ if the classes of bounded sets and convergent sequences in both spaces (X,ϱ) and (X,ϱ') are the same. Obviously, if ϱ and ϱ' are equivalent, the space (X,ϱ') is still a Polish space.

We say that a Markov operator P is *essentially nonexpansive* if there exists a metric ϱ' equivalent to ϱ such that P is nonexpansive in (X,ϱ').

Lasota and Yorke discuss precisely such cases (see [8]). Finally we may reformulate our main result.

THEOREM 3.5. *Let P be an essentially nonexpansive Markov operator. Suppose that the condition* (A) *holds. Then P is asymptotically stable.*

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Functional equations in real-analytic functions

by

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Abstract. The equation $\varphi(x) = g(x, \varphi(x))$ in spaces of real-analytic functions is considered. Connections between local and global aspects of its solvability are discussed.

1. Introduction. Given a real-analytic manifold X countable at infinity, $\dim X = m$, we consider an equation

$$\varphi(x) = g(x, \varphi(Fx)) \tag{1}$$

with real-analytic mappings

$$F : X \to X, \quad g : X \times \mathbb{R}^n \to \mathbb{R}^n$$

and an unknown real-analytic vector function

$$\varphi : X \to \mathbb{R}^n.$$

Our aim is to discuss solvability conditions for (1).

The above problem has "local" and "global" aspects. The former means the solvability in a neighborhood of a given point $x_0 \in X$, while the latter deals with the question of *whether* (1) *has a global solution* $\varphi(x)$, $x \in X$, *if it is solvable in a neighborhood of every point* $x_0 \in X$.

It turns out that at least in the case of a linear equation

$$(T\varphi)(x) \equiv \varphi(x) - A(x)\varphi(Fx) = \gamma(x) \tag{2}$$

a collection of local solutions may be used to construct a cocycle "obstructing" global solvability. This situation is similar to the *Stokes phenomena* and *Ecalle–Voronin modules* arising in classification problems of dynamical systems (see [I]).

The construction of the obstructing cocycle and its applications are the main object of the present paper (see Theorem 3.1 and results of Sections 4 and 5).

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