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Banach principle in the space of τ-measurable operators

by

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Abstract. We establish a non-commutative analog of the classical Banach Principle on the almost everywhere convergence of sequences of measurable functions. The result is stated in terms of quasi-uniform (or almost uniform) convergence of sequences of measurable (with respect to a trace) operators affiliated with a semifinite von Neumann algebra. Then we discuss possible applications of this result.

Introduction. The study of measurable operators associated with a von Neumann algebra (vNA) and different types of the almost everywhere convergence for sequences of measurable operators goes back to the celebrated paper of I. Segal [Se]. Since then this branch of the theory of operator algebras has been explored in many different directions. One of them is the so-called non-commutative ergodic theory, which treats the almost everywhere (or norm) convergence of the Cesàro averages along the trajectory (under some kind of contraction in a non-commutative L^p-space) of an operator in L^p. This study was initiated by a number of authors, among whom we mention Lance [La] and Yeadon [Ye]. In the classical ergodic theory, one of the most powerful tools in dealing with the almost everywhere convergence of ergodic averages is the well-known Banach Principle on the convergence of sequences of measurable functions generated by a sequence of linear maps in an L^p-space. This principle is often applied in proofs concerning the almost everywhere convergence of weighted averages, averages along subsequences, moving averages, etc.

In this paper, using the notion of τ-measurable operator, we establish a non-commutative analog of the Banach Principle. Since we do not assume the finiteness of the trace, the result is stated for the quasi-uniform convergence. The proof of the main result of this paper, Theorem 2, can be easily modified for different types of the "almost everywhere" convergences in vNA, in particular, for the bilateral almost uniform (b.a.u.) convergence

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(Theorem 5). In this form, due to the presence of fundamental results on b.a.u. convergence of the ergodic averages of integrable operators [Ye], the non-commutative Banach Principle can be utilized to prove some subsequential and weighted ergodic theorems in the vNA setting. Note that one of the most complete accounts on the almost uniform convergence theorems in vNA's is presented in [Ja].

1. Preliminaries. Let M be a von Neumann algebra acting on a Hilbert space H, and τ a faithful semifinite normal trace on M. Let $B(H)$ be the algebra of all bounded linear operators in H, and let $\|\cdot\|_\infty$ (or $\|\cdot\|$) be the operator norm in $B(H)$. By the *weak topology* in $B(H)$ we mean the topology defined through the system $\{(x\xi,\eta) : \xi,\eta \in H\}$ of seminorms on $B(H)$, where $(\cdot,\cdot)$ is the inner product generated by $\|\cdot\|$. In what follows, we use the letter w to denote this topology.

Algebra of measurable operators. A closed operator x in H with the domain D_x is said to be *affiliated* with M if $y'D_x \subset D_x$ and $y'x \subset xy'$ for every y' from the commutant $M' = \{y' \in B(H) : xy' = y'x \text{ for all } x \in M\}$ of M.

Denote by $P(M)$ the complete lattice of all projections in M. A densely defined closed operator x affiliated with M is called τ-*measurable* if for each $\varepsilon > 0$ there exists $e \in P(M)$ such that $eH \subset D_x$ and $\tau(e^\perp) \le \varepsilon$ ([Ne], [FK]). Note that if x is τ-measurable and $eH \subset D_x$ for some $e \in P(M)$, then $xe \in M$.

If x, y are τ-measurable operators, then $x+y$, xy and x^* are densely defined and preclosed. Moreover, the closures $(x+y)^-$, $(xy)^-$ and x^* are again τ-measurable, and the set $\overline{M}$ of all τ-measurable operators is a *-algebra with respect to these operations.

The *measure topology* on $\overline{M}$ can be defined by the following system of neighborhoods of zero: for $\varepsilon > 0$, $\delta > 0$,

$$V(\varepsilon,\delta) = \{x \in \overline{M} : \|xe\| \le \varepsilon \text{ for some } e \in P(M) \text{ with } \tau(e^\perp) \le \delta\}.$$

One can show (see [FK], Lemmas 3.1 and 3.4) that $V(\varepsilon,\delta)$ is closed for all $\varepsilon > 0$, $\delta > 0$. It is known that $\overline{M}$ is a complete topological *-algebra, and M is dense in it [Ne].

Some general facts

Lemma 1 (see, for example, [BR]). *If $b_\alpha,\ b \in M_+$ and $b_\alpha \to b$ weakly, then*

$$\tau(b) \le \liminf_\alpha \tau(b_\alpha).$$

The next lemma follows immediately from the formula

$$\|x\| = \sup_{\|\xi\|=\|\eta\|=1} |(x\xi,\eta)|, \quad x \in B(H).$$

Lemma 2. *If a net $\{y_\beta\}$ in M w-converges to some $y \in M$, then*

$$\|y\| \le \liminf_\beta \|y_\beta\|.$$

Lemma 3. *Let $b \in M$, $0 \le b \le I$, and f be the spectral projection of b corresponding to the interval* $[1/2,1]$. *Then*

(a) $\tau(f^\perp) \le 2\tau(I-b)$;
(b) *there exists $b^- \in M$ with $\|b^-\| \le 2$ such that $f = bb^-$.*

Proof. (a) By the definition of f, we have

$$(I-b)f^\perp \ge \tfrac{1}{2} f^\perp,$$

which implies the inequality since

$$(I-b) - (I-b)f^\perp = (I-b)f = f(I-b)f \ge 0.$$

Part (b) is obvious. ∎

Corollary 1. *The measure topology in $\overline{M}$ can also be defined by the system*

$$W(\varepsilon,\delta) = \{x \in \overline{M} : \|xb\| \le \delta \text{ for some } b \in M,\ 0 \le b \le I, \text{ with } \tau(I-b) \le \varepsilon\}.$$

Proof. Since, given $\varepsilon > 0$, $\delta > 0$, one has $V(\varepsilon,\delta) \subset W(\varepsilon,\delta)$, it is enough to show that

$$W = W(\varepsilon/2,\delta/2) \subset V(\varepsilon,\delta).$$

Indeed, if $x \in W$, then, for some $b \le I$ in M_+ such that $\tau(I-b) \le \varepsilon/2$, we have $\|xb\| \le \delta/2$. Let f be the spectral projection of b corresponding to the interval $[1/2,1]$. Then, by Lemma 3, $\tau(f^\perp) \le \varepsilon$ and, if $b^- \in M$, $\|b^-\| \le 2$, is such that $f = bb^-$, then

$$\|xf\| = \|xbb^-\| \le 2\|xb\| \le \delta,$$

i.e. $x \in V(\varepsilon,\delta)$. ∎

2. Banach Principle. We begin with presenting a variant of the classical Banach Principle (see, for example, [DS], [Ga], [BJ]).

Theorem 1. *Let (S,Σ,m) be a σ-finite measure space, X be a Banach space, and $\{a_n\}$ a sequence of continuous linear maps from X to the space of all measurable functions on S satisfying the condition*

$$\text{for every } x \in X, \quad \sup_n\{|a_n(x)(s)|\} < \infty \quad \text{almost everywhere (a.e.)}$$

and such that $a_n(x)$ converges a.e. for every x in a dense subset of X. Then this convergence takes place for every $x \in X$.

A sequence $\{a_n\}$ in $\overline{M}$ is said to be convergent to some $\widehat{a} \in \overline{M}$ *almost uniformly* (a.u.) if for every $\varepsilon > 0$ there exists $p \in P(M)$ with $\tau(p^\perp) \le \varepsilon$ such that $\|(a_n-\widehat{a})p\| \to 0$ as $n\to\infty$. A sequence $\{a_n\}$ in $\overline{M}$ converges to $\widehat{a}\in\overline{M}$ *quasi-uniformly* (q.u.) if for every non-zero $p\in P(M)$ there is $q\in P(M)$ such that $0\neq q\le p$ and $\|(a_n-\widehat{a})q\|\to 0$. For properties of different types of convergence in von Neumann algebras each of which, in the commutative case with finite measure, is equivalent to the a.e. convergence, see [Pa]. It is known that the following non-commutative analog of Riesz's theorem holds [Se]. Since, in [Se], it was stated in a slightly different form, we provide a proof.

PROPOSITION 1. *If $y_n\to 0$ in $\overline{M}$ with respect to the measure topology, then $y_{n_k}\to 0$ a.u. for some $\{y_{n_k}\}\subset\{y_n\}$.*

Proof. Since $y_n\to 0$ in the measure topology, given $\varepsilon>0$ and a positive integer k, there exists a sequence $\{e_{nk}\}\subset P(M)$ and a number $N(k)$ such that

$$\tau(e_{nk}^{\perp})\le\varepsilon/2^k\ \ \forall n\quad\text{and}\quad\|y_ne_{nk}\|\le 1/k\ \ \forall n\ge N(k).$$

Therefore, for every k, one can find n_k such that, if we set $e_k=e_{n_kk}$, we have

$$\tau(e_k^{\perp})\le\varepsilon/2^k\quad\text{and}\quad\|y_{n_k}e_k\|<1/k.$$

If we put $e=\wedge_{k=1}^{\infty}e_k$, then $\tau(e^{\perp})\le\varepsilon$ and, moreover, $\|y_{n_k}e\|\le 1/k$ for all k, which means that $y_{n_k}\to 0$ a.u. ∎

The following is a non-commutative generalization of the Banach Principle.

THEOREM 2. *Let M be a von Neumann algebra with a normal faithful semifinite trace τ, and let $\overline{M}$ be the topological *-algebra of all τ-measurable operators. Let X be a Banach space, and $a_n:X\to\overline{M}$ a sequence of continuous linear maps satisfying the condition*

(i) *for all $x\in X$ and non-zero $p\in P(M)$ there is an operator $b\in M_+$, $0\neq b\le p$, such that*

$$\sup_n\{\|a_n(x)b\|\}<\infty.$$

If, for every x from a dense subset $X_0\subset X$,

(ii) $a_m(x)-a_n(x)\to 0$ *q.u.*,

then (ii) *holds on all of X.*

Before we prove the result let us make some remarks.

REMARK 1. If τ is finite, then the q.u. and a.u. convergence are equivalent, while the condition (i) is equivalent to

(i$_a$) for all $x\in X$ and $\varepsilon>0$ there is $b\in M$, $0\le b\le I$, with $\tau(I-b)\le\varepsilon$ and $\sup_n\{\|a_n(x)b\|\}<\infty$.

REMARK 2. If τ is semifinite, then condition (i) is equivalent to

(i$_b$) for all $x\in X$ and $p\in P(M)$ with $\tau(p)<\infty$ and $\sigma>0$ there is $0\le b\le p$ satisfying $\tau(p-b)\le\sigma$ such that $\sup_n\{\|a_n(x)b\|\}<\infty$.

REMARK 3. For an arbitrary $\{y_n\}\subset\overline{M}$ the convergence $y_n\to 0$ q.u. is equivalent to the following one:

(ii$_a$) for $e\in P(M)$ with $\tau(e)<\infty$, $\varepsilon>0$, $\delta>0$, there exists $q\in P(M)$ with $q\le e$ and $\tau(e-q)<\varepsilon$ such that $\|y_nq\|<\delta$ for any $n\ge N(e,\varepsilon,\delta)$.

Proof of Theorem 2. By Remark 3, for $x\in X$ and $p\in P(M)$ with $0<\tau(p)<\infty$, $\varepsilon>0$, $\delta>0$, it is enough to construct a projection $q\le p$ such that $\tau(p-q)<\varepsilon$ and

$$\|(a_m(x)-a_n(x))q\|<\delta$$

for any $m,n\ge N(\varepsilon,\delta,p)$.

For every pair L and k of natural numbers define the set

$$X_{L,k}=\{x\in X:\sup_n\{\|a_n(x)b\|\}\le L\text{ for some }0\le b\le p\text{ with }\tau(p-b)\le\varepsilon_k=\varepsilon/2^{k+3}\}.$$

We shall show that $X_{L,k}$ is closed. Let $x_m\to x$ in X and $x_m\in X_{L,k}$ for every m. There exists a sequence $\{b_m\}\subset M$ with $0\le b_m\le p$ and $\tau(p-b_m)\le\varepsilon_k$ such that

$$\|a_n(x_m)b_m\|\le L$$

for every m, n. Fix n. Since the unit ball in M is w-compact, for some $0\le b\le p$ and a subnet $\{b_\alpha\}\subset\{b_m\}$, $b_\alpha\to b$ weakly. By Lemma 1, we have $\tau(p-b)\le\varepsilon_k$, therefore it remains to show that $\|a_n(x)b\|\le L$. Since $a_n(x_m)\to a_n(x)$ in the measure topology, by Proposition 1, taking into account the continuity of the *-operation with respect to the measure topology, we can assume, without loss of generality, that

$$a_n(x_m)\to a_n(x)$$

in the following sense: for every $\sigma>0$, one can find $h\in P(M)$ with $\tau(h^{\perp})\le\sigma$ such that

$$\|h(a_n(x_m)-a_n(x))\|\to 0$$

as $m\to 0$. In particular, for a given j, there exists $h_j\in P(M)$ with $\tau(h_j^{\perp})\le 1/j$ such that

$$\|h_ja_n(x_m)\|\to\|h_ja_n(x)\|$$

as $m\to\infty$. Therefore, without loss of generality, assume that the double-indexed net

$$h_ja_n(x_m)b_\alpha\to h_ja_n(x)b\quad\text{weakly}$$

for every j, and then, by Lemma 2,

$$\|h_ja_n(x)b\|\le\limsup_{(m,\alpha)}\|h_ja_n(x_m)b_\alpha\|\le L.$$

Taking now $j \to \infty$, we get $h_j \to I$ weakly, hence $h_j a_n(x)b \to a_n(x)b$ weakly, and then, applying Lemma 2 again,

$$\|a_n(x)b\| \le \limsup_j \|h_j a_n(x)b\| \le L.$$

By Remark 2, we have $X = \bigcup_{L=1}^{\infty} X_{L,k}$, therefore, using the Baire category theorem, we find a number L_k, $x_k \in X$ and $\delta_k > 0$ such that for every $x \in X$ satisfying $\|x - x_k\|_X < \delta_k$ there exists an operator $0 \le b_{x,k} \le p$ with $\tau(p - b_{x,k}) \le \varepsilon/2^{k+3}$ for which

$$\sup_n\{\|a_n(x)b_{x,k}\|\} \le L_k.$$

Let $f_{x,k}$ be the spectral projection of $b_{x,k}$ in a von Neumann algebra pMp corresponding to the interval $[1/2, 1]$. Then Lemma 3 implies $\tau(p - f_{x,k}) \le \varepsilon/2^{k+2}$ and

$$\sup_n\{\|a_n(x)f_{x,k}\|\} \le 2L_k.$$

If $\|x - x_k\|_X < \delta_k$ and $g_{x,k} = f_{x,k} \wedge f_{x_k,k}$, then $\tau(p - g_{x,k}) \le \varepsilon/2^{k+1}$ and

$$\sup_n\{\|a_n(x - x_k)g_{x,k}\|\} \le 4L_k.$$

This means that if $\gamma_k = \delta_k/(4L_k)$, then $\|z\|_X \le \gamma_k$ entails the existence of $g_{z,k} \in P(M)$, $0 \le g_{z,k} \le p$, with $\tau(p - g_{z,k}) \le \varepsilon/2^{k+1}$ such that

$$\sup_n\{\|a_n(z)g_{z,k}\|\} \le 1.$$

Since X_0 is dense in X, for every k it is possible to find $z_k \in X$ with $\|z_k\|_X < \gamma_k$ for which

$$x + z_k/k \in X_0.$$

If $y_k = z_k/k$ and $g = \wedge_{k=1}^{\infty} g_{z,k}$, then $g \le p$, $\tau(p - g) \le \varepsilon/2$. Moreover,

$$\|a_n(y_k)g\| \to 0 \quad \text{uniformly in } n.$$

Therefore there exists a number K such that $k \ge K$ would imply

$$\|a_n(y_k)g\| < \delta/3$$

for all n. Fix any $k \ge K$. Taking into account that $x + y_k \in X_0$, by (ii), one may find $q \in P(M)$, $q \le g$, with $\tau(g - q) < \varepsilon/2$ and then a number $N = N(q, \varepsilon, \delta)$ such that

$$\|(a_m(x + y_k) - a_n(x + y_k))q\| < \delta/3$$

for $m, n \ge N$. Finally, we have $q \le p$, $\tau(p - q) < \varepsilon$ and

$$\begin{aligned}&\|(a_m(x) - a_n(x))q\| \\ &\quad \le \|(a_m(x + y_k) - a_n(x + y_k))q\| + \|a_m(y_k)q\| + \|a_n(y_k)q\| < \delta\end{aligned}$$

for all $m, n \ge N$. ∎

REMARK. As we already mentioned, the sets $V(\varepsilon, \delta)$ are closed. On the other hand, one can see that the arguments of [FK] do not prove the closedness of the set

$$X'_{L,k} = \{x \in X : \sup_n\{\|a_n(x)e\|\} \le L \text{ for some } e \in P(M), e \le p, \text{ with } \tau(p - e) \le \varepsilon_k\}.$$

That is the reason we introduce $X_{L,k}$.

3. Applications. As a natural application of the non-commutative Banach Principle we present an alternative ending of proof of the well-known result of Yeadon [Ye] on "almost everywhere" convergence of ergodic averages in the non-commutative L^1-space.

Let M be a vNA with a faithful normal semifinite trace τ. Let $L^1 = L^1(M, \tau)$ be the space of all τ-integrable operators (see, for example, [Se], [Ta]). A positive linear map $\alpha : L^1 \to L^1$ will be called an *absolute contraction* if $\alpha(I) \le I$ and $\tau(\alpha(x)) \le \tau(x)$ for every $x \ge 0$. Note that, in the commutative case, an operator of such a type is called a *Dunford–Schwartz operator*. If α is an absolute contraction in L^1, then, as can be seen in [Ye], $\|\alpha(x)\|_p \le \|x\|_p$ for each $x = x^* \in J$ and all $1 \le p \le \infty$. Moreover, there exist unique continuous extensions $\alpha : L^p \to L^p$ for all $1 \le p < \infty$ and a unique ultra-weakly continuous extension $\alpha : M \to M$. Therefore, for every $x \in L^p$ and any positive integer k, one has

$$\|\alpha^k(x)\|_p \le 2\|x\|_p.$$

Define

$$A_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(x), \qquad x \in \overline{M}.$$

A sequence $\{a_n\} \subset \overline{M}$ is said to be *bilaterally almost uniformly* (b.a.u.) convergent to some $\widehat{a} \in \overline{M}$ if for every $\varepsilon > 0$ there exists $p \in P(M)$ with $\tau(p^\perp) \le \varepsilon$ such that $\|p(a_n - \widehat{a})p\| \to 0$.

In [Ye], the following form of non-commutative individual ergodic theorem was proven.

THEOREM 3. *If α is an absolute contraction in $L^1 = L^1(M, \tau)$, then, for every $x \in L^1$, the averages*

$$A_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} \alpha^k(x)$$

converge b.a.u. in L^1.

The key role in the proof of Theorem 3 is played by the so-called maximal ergodic theorem:

THEOREM 4 [Ye]. *Let α be an absolute contraction in L^1. Then for every $x \in L^1$ and $\varepsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \le 4\varepsilon^{-1}\|x\|_1$ such that $\|eA_n(x)e\| \le 4\varepsilon$ for every n.*

Now, we shall present a non-commutative variant of the Banach Principle adapted to the b.a.u. convergence (the proof is analogous to that of Theorem 2; see [LM]).

THEOREM 5. *Let M be a vNA with the unit I and a faithful normal semifinite trace τ. Let $\overline{M}$ be the topological *-algebra of all τ-measurable operators. For a Banach space X, assume that $a_n : X \to \overline{M}$ is a sequence of continuous linear maps satisfying the condition*

(i) *for every $x \in X$ and $\varepsilon > 0$, it is possible to find an operator $b \in M$, $0 \le b \le I$, with $\tau(I - b) \le \varepsilon$ such that $\sup_n \|ba_n(x)b\| < \infty$.*

If, for every x from a dense subset $X_0 \subset X$,

(ii) *$a_n(x)$ converges b.a.u. in $\overline{M}$,*

then (ii) *holds on all of X.*

We apply this theorem together with Theorem 4 to reprove Theorem 3. First of all, note that $\{A_n(\cdot)\}$ is a sequence of positive linear maps from a Banach space $X = L^1$ to $L^1 \subset \overline{M}$ which are continuous under the norm $\|\cdot\|_1$, hence as maps from L^1 to $\overline{M}$. The condition (i) of Theorem 5 is satisfied via Theorem 4, so it remains to find a dense subset X_0 in L^1 on which the b.a.u. convergence would take place. Such a set can be easily found via a standard argument (see [LM]). Consequently, by Theorem 5, we obtain the b.a.u. convergence of $A_n(x)$ for every $x \in L^1$.

For other applications of Theorems 3–5 we refer the reader to [LM] and [Li].

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