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Weakly mixing but not mixing quasi-Markovian processes

by

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Abstract. Let (f,α) be the process given by an endomorphism f and by a finite partition $\alpha=\{A_i\}_{i=1}^s$ of a Lebesgue space. Let $E(f,\alpha)$ be the class of densities of absolutely continuous invariant measures for skew products with the base (f,α). We say that (f,α) is quasi-Markovian if

$$E(f,\alpha)\subset\Big\{g:\bigvee_{\{B_i\}_{i=1}^s}\operatorname{supp} g=\bigcup_{i=1}^s A_i\times B_i\Big\}.$$

We show that there exists a quasi-Markovian process which is weakly mixing but not mixing. As a by-product we deduce that the set of all coboundaries which are measurable with respect to the "chequer-wise" partition for $\sigma\times S$, where σ is a Bernoulli shift and S is a weakly mixing automorphism, consists of constants.

0. Introduction. The following characterization of processes by using skew products has been introduced in [Ko]. The processes considered are given by pairs (f,α), where f is a positively non-singular measure-preserving map of a Lebesgue probability space $(X,\mathcal{A},q)$ and $\alpha=\{A_i\}_{i=1}^s$ is a finite one-sided generator for f. Let $(Y,\mathcal{B},\varrho)$ be another Lebesgue probability space and let $\{T_i\}_{i=1}^s$ be a family of positively and negatively non-singular maps of Y into Y. The process (f,α) and the family $\{T_i\}_{i=1}^s$ define the skew product map

$$T(x,y)=(f(x),T_{a(x)}(y)),\tag{1}$$

where $a:X\to\{1,\dots,s\}$ is determined by $a(x)=i\Leftrightarrow x\in A_i$. Let $E(f,\alpha)=\{g$: there exists a Lebesgue probability space $(Y,\mathcal{B},\varrho)$ and a family of positively and negatively non-singular maps $\{T_i\}_{i=1}^s$ such that g is the density of an absolutely continuous invariant measure (a.c.i.m.) under the skew product as in (1)$\}$.

We also denote by α the field of unions of elements of α. It has been shown [Ko] that the process (f,α) is

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(i) Bernoulli iff for every $g \in E(f,\alpha)$, g is measurable with respect to $\mathcal{B}$,

(ii) Markovian iff for every $g \in E(f,\alpha)$, g is measurable with respect to $\alpha \times \mathcal{B}$.

Moreover, the class of quasi-Markovian processes has been introduced as a generalization of the Markovian class.

DEFINITION. We say that (f,α) is a *quasi-Markovian process* (q.m.p.) if for every $g \in E(f,\alpha)$ the set $\{g > 0\}$ is measurable with respect to $\alpha \times \mathcal{B}$.

The description of basic properties of q.m.p. may be found in [Ko]. The class of q.m.p. contains the Lasota–Yorke and Misiurewicz type maps (introduced in [LY] and [M] respectively) with Markovian partitions [Ko] as well as mixing Gibbs–Markov maps [AD]. The above examples have the following property: total ergodicity of f implies exactness. In [KK] it has been observed that a totally ergodic q.m.p. is weakly mixing. In [KS] a weakly mixing but not mixing process is considered which satisfies some conditions weaker than the quasi-Markovian property. In this paper we show that some modification of the above process is q.m.p. Therefore there exists a weakly mixing but not mixing q.m.p. The above result is based on Bose's example [Bo2] and the following two facts:

THEOREM 2 (see Section 2). *Let S be a weakly mixing automorphism, with finite entropy, of the space $(X,\mathcal{A},p)$. Let $\mathcal{R}=\{R_0,\ldots,R_{l-1}\}$ be the generating partition for S. Let $(\Omega,\mathcal{G},m,\sigma)$ be a one-sided Bernoulli shift with independent generator $\mathcal{Q}=\{Q_0,\ldots,Q_{l-1}\}$. Take the product dynamical system*

$$(\Omega\times X,\mathcal{G}\times\mathcal{A},m\times p,\sigma\times S)$$

and define a measurable partition $\mathcal{P}=\{P_0,\ldots,P_{l-1}\}$ of $\Omega\times X$ as follows:

$$P_j=\bigcup_{s+t=j\ (\mathrm{mod}\ l)} Q_s\times R_t,\qquad j=0,\ldots,l-1.$$

Then for every $g\in E(\sigma\times S,\mathcal{P})$ the set $\{g>0\}$ is measurable with respect to $\mathcal{B}$.

CONCLUSION 1. *If the partition $\mathcal{P}$ is a one-sided generator for $\sigma\times S$, then $(\sigma\times S,\mathcal{P})$ is q.m.p.*

The next conclusions are related to the ergodic properties of skew product maps.

CONCLUSION 2 (see Theorem 8 of [Ko]). *Let $h\in L^2(m\times p)$ and let $g:\Omega\times X\to S$, where $S=\{z:|z|=1,\ z\in\mathcal{C}\}$ and g is a $\mathcal{P}$-measurable function. Then $h(\sigma\times S)=gh$ a.e. iff $h\equiv$ constant.*

By Conclusion 2 we see that g is a circle-valued $\mathcal{P}$-measurable coboundary of $\sigma\times S$ iff $g\equiv$ constant. Here σ is a one-sided or a two-sided Bernoulli shift.

For the general discussion of coboundaries see [Ba].

CONCLUSION 3 (see Theorems 2 of [Ko] and 2 of [KK]). *If T is a skew product as in (1) for $f=\sigma\times S$ and $\alpha=\mathcal{P}$ and μ is a.c.i.m. for T, then for every T-eigenfunction $H\in L_1(\mu)$ we have*

$$H1_{D_\mu}(\omega,x,y)=\sum_{i=0}^{l-1}1_{P_i}(\omega,x)g_i(y)\qquad m\times p\times\varrho\text{-a.e.}$$

Here $D_\mu=\operatorname{supp}\mu$. If we assume additionally that T_i, $i=0,\ldots,l-1$, are one-one and bimeasurable and $\mu\approx m\times p\times\varrho$, then $H(\omega,x,y)=g(y)$ $m\times p\times\varrho$-a.e.

1. An example of a weakly mixing but not mixing q.m.p. We start with a one-sided version of Bose's $\bar d>\delta$ property (for the definition of the $\bar d>\delta$ property see [Bo2]).

DEFINITION. Let $\delta>0$. We say that the process $(X,\mathcal{A},p,S,R)$ has the *$\bar d^+>\delta$ property* if there is an $X'\subset X$, $p(X')=1$, such that if $x,y\in X'$, $x\neq y$, then

$$\varlimsup_{n\to\infty}\bar d(x_0^n,y_0^n)>\delta.$$

Here

$$\bar d(x_0^n,y_0^n)=\frac1n\operatorname{card}\{i\in[0,n):x_i\neq y_i\}.$$

and x_0^∞,y_0^∞ are the S-R names of x and y, respectively.

Bose [Bo2] has shown the $\bar d>\delta$ property for Chacón's automorphism. We prove the $\bar d^+>\delta$ property for Chacón's automorphism S_0 constructed by using five cuts. We construct S_0 by using the cutting and stacking method, in the following terms:

$$I_0=[0,4/5]=C_0,$$
$$C_{n+1}=S_{5,1}C_n*S_{5,2}C_n*S_{5,3}C_n*S_{5,4}C_n*I_{n+1}*S_{5,5}C_n,$$

where $\{I_n\}_{n=1}^\infty$ is the partition of $[4/5,1]$ into intervals such that $\lambda(I_n)=4/5^{n+1}$ for $n=1,2\ldots$ Let us explain the definition of C_{n+1}. The column C_{n+1} is obtained as follows. We decompose the column C_n into five disjoint subcolumns $S_{5,1}C_n,\ldots,S_{5,5}C_n$ by partitioning the base of C_n into five intervals of the same length. S_0 maps the top of $S_{5,i}C_n$ linearly onto the base of $S_{5,i+1}C_n$ for $i=1,2,3$, and the top of $S_{5,4}C_n$ onto I_{n+1}, next I_{n+1} onto the base of $S_{5,5}C_n$. In other words, we stack the subcolumns

$S_{5,1}C_n, \ldots, S_{5,4}C_n, I_{n+1}, S_{5,5}C_n$ to form the column C_{n+1}. For a description of this method, see [Fr].

According to the above construction we obtain the n-blocks

$$B_0 = 0, \quad B_{n+1} = B_n \otimes B_n \otimes B_n \otimes B_n \otimes 1 \otimes B_n.$$

Here $\otimes$ denotes the concatenation of blocks. The system of n-blocks determines the set $X \subset \{0,1\}^{\mathbb{Z}}$ and the measure μ which is invariant under the shift σ. The dynamical systems (I, λ, S_0) and (X, μ, σ) are isomorphic. This isomorphism is obtained by using the two-sided generator $R = \{I_0, I_0^c\}$ for coding elements of I. Let us denote by C_n' the union of the stack C_n.

REMARK 1. The entropy of S_0 being zero, we conclude that $\{I_0, I_0^c\}$ is a one-sided generator for S_0.

LEMMA 1. *There exists a set $X_0 \subset I$ of full measure such that for every pair $x \neq y \in X_0$ and for every n_0 there exists $n \geq n_0$ such that*

$$\{x, y\} \subset \bigcup_{k=1}^{4} (S_{5,k}C_n)'.$$

Proof. By the definition of C_n there exists n_1 such that for every $n \geq n_1$, $\{x,y\} \subset C_n'$. Consider a pair (x,y) for which the conclusion of the lemma does not hold. Then there exists $n_0 > n_1$ such that for every $n > n_0$, $x \in (S_{5,5}C_n)'$ or $y \in (S_{5,5}C_n)'$. Hence x or y belongs to the set

$$A = \left\{x : \varlimsup_{n\to\infty} \frac{1}{n} \operatorname{card}\{k \leq n : x \in (S_{5,5}C_k)'\} \geq \frac{1}{2}\right\}.$$

FACT 1. *Let J be a union of levels in $S_{5,5}C_n$. Then*

$$\lambda\Big(J \cap \bigcap_{k=1}^{m} (S_{5,5}C_{n_k})'\Big) = 5^{-m}\lambda(J) \quad \textit{for any sequence } n < n_1 < \ldots < n_m.$$

Proof. This is a consequence of the following observation. If J is a union of levels in $(S_{5,5}C_n)'$ and $l > n$, then J is a union of levels in C_l and therefore $\lambda(J \cap (S_{5,5}C_l)') = 5^{-1}\lambda(J)$.

Let r be a real such that $r < 2^{-1}$ and $5^{-1} < r(1-r)^{1/r-1}$. As usual we denote by $[x]$ the integer part of x. For a natural number n we denote by H_n the set of $[rn]$-element subsets of $\{1, \ldots, n\}$. Let $A_s = \bigcap_{j\in s}(S_{5,5}C_j)'$ for $s \in H_n$. By the definition of A we have

$$A \subset \bigcap_k \bigcup_{n\geq k} \bigcup_{s\in H_n} A_s.$$

Now, we will estimate the measure of A. By the Stirling formula we have

$$\operatorname{card} H_n \leq (2\pi[nr](1-r))^{-1/2}((r-1/n)(1-r)^{n/[nr]-1})^{-[nr]}.$$

There exists k_0 such that for $n \geq k_0$,

$$\delta^{-1} = (r-1/n)(1-r)^{n/[nr]-1}5 > 1 \quad \text{and} \quad (2\pi[nr](1-r))^{-1/2} < 1.$$

By Fact 1, $\lambda(\bigcup_{s\in H_n} A_s) \leq \delta^{nr-1}$ for $n \geq k_0$. Consequently,

$$\lambda(A) \leq \sum_{n=k}^{\infty} \delta^{nr-1} = \frac{\delta^{kr-1}}{1-\delta^r}$$

for $k \geq k_0$ and hence $\lambda(A) = 0$. ∎

REMARK 2. We use the 5-fold Chacón automorphism instead of the classical 3-fold construction, since there does not exist any $0 < r < 2^{-1}$ such that $3^{-1} < r(1-r)^{1/r-1}$. The above is necessary in the proof of Lemma 1.

Now, we recall some notions from [Bo2]. We say that B_n *appears at* i_0 in $x \in X$ if $x_{i_0}^{i_0+b_n-1} = B_n$. Given two n-blocks $B_n(x)$ and $B_n(y)$ appearing at i_0 in $x_{-\infty}^{\infty}$ and at j_0 in $y_{-\infty}^{\infty}$ respectively, we say they *overlap* if

$$i_0 - b_n + 1 \leq i_0 \leq j_0 + b_n - 1.$$

Here b_n denotes the length of B_n. We will use two copies of B_n, $\overline{B}_n$ and $\underline{B}_n$, with $\overline{B}_n$ lying above $\underline{B}_n$. We enumerate five copies of B_{n-1} inside B_n starting from the left as $B_{n-1}^{(1)}, \ldots, B_{n-1}^{(5)}$. Finally, for given two copies $\overline{B}_n$ and $\underline{B}_n$, we denote by $\bar{d}(B_n, S_0^j B_n)$ the distance measured over the overlap of $\overline{B}_n$ and $\underline{B}_n$ shifted j indices to the right relative to $\overline{B}_n$.

LEMMA 2. *Let $B = B_n$ $(n \geq 1)$. Then*

(a) $\bar{d}(B, S_0^j B) > 1/22$ *if* $1 \leq j \leq \frac{4}{5}b_n$.

(b) *If* $\bar{d}(B, S_0^j B) \leq 1/22$, *then* $\bar{d}(B, S_0^{j\pm1} B) > 3/10$ *where we allow* $j = 0, 1, \ldots, b_n$.

(c) *If* $j = b_k$ *or* $j = b_k + 1$ *for some* $k < n$, *then* $\bar{d}(B, S_0^j B) > 3/40$.

The proof of Lemma 2 is similar to the proof of Lemma 4.4 of [Bo2] and it is by induction using the definition of B_{n+1}. The set of indices j, i.e. $\{1, \ldots, 2b_n\}$, is divided into 7 parts in the proof of Lemma 4.4 of [Bo2] and every part is considered separately. Since the set of indices j is $\{1, \ldots, 4b_n\}$ in the case of Lemma 2, we divide it into 14 parts. This is the main difference between the proof of Lemma 2 and Lemma 4.4 of [Bo2].

We say that an n-block in x *covers* x_0 if there exist n_0, m, $n_0 \leq 0 \leq m$, such that $x_{n_0}^m = B_n$.

THEOREM 1. *The process (S_0, R) satisfies $\bar{d}^+ > 1/220$.*

Proof. Let $x \neq y \in X_0$ be fixed. We may assume $x_0 \neq y_0$: for this we can replace x, y by $S_0^k x, S_0^k y$ for some $k \geq 0$ (Remark 1). By Lemma 1 there exists a sequence $\{n_k\}$ such that $B_{n_k}(x)$ and $B_{n_k}(y)$ cover x_0 and y_0 in such a manner that $1 \otimes B_{n_k-1}^{(5)}(x)$ and $1 \otimes B_{n_k-1}^{(5)}(y)$ stay on the right with respect to x_0 and y_0, respectively. Now, by using a similar argument to the proof of Lemma 4.5 of [Bo2] we consider two cases.

CASE 1. If $B_{n_k}(x)$ and $B_{n_k}(y)$ are shifted by one index with respect to each other, then

$$\bar{d}(x_0^{N_k}, y_0^{N_k}) > 3/50,$$

by Lemma 2(b). Here $N_k = b_{n_k} - 1$.

CASE 2. If $B_{n_k}(x)$ and $B_{n_k}(y)$ appear shifted by more than one index relative to each other, then

$$\bar{d}(x_0^{N_k}, y_0^{N_k}) > 1/220,$$

by Lemma 2(a). Here $N_k = 2b_{n_k}$. ■

Let σ_0 be a $(659/660, 1/660)$-1-sided Bernoulli shift. For convenience we define σ_0 as a piecewise linear transformation of I as follows:

$$\sigma_0(\omega) = \begin{cases} \frac{660}{659}\omega & \text{for } \omega \in Q_0 = \left[0, \frac{659}{660}\right], \\ 660\omega - 659 & \text{for } \omega \in Q_1 = \left[\frac{659}{660}, 1\right]. \end{cases}$$

Let us form the product dynamical system

$$(I \times I, \mathcal{A} \times \mathcal{A}, \lambda \times \lambda, \sigma_0 \times S_0)$$

and define a measurable partition $\mathcal{P} = \{P_0, P_1\}$ of $I \times I$ as follows:

$$P_0 = Q_0 \times I_0 \cup Q_1 \times I_0^c, \quad P_1 = Q_0 \times I_0^c \cup Q_1 \times I_0.$$

LEMMA 3. *The partition $\mathcal{P}$ is a one-sided generator for $\sigma_0 \times S_0$.*

Proof. By using Remark 1 and Theorem 1 and by a similar argument to the proof of Lemma 3.3 of [Bo2] we get the desired conclusion. ■

COROLLARY 4. *The process $(\sigma_0 \times S_0, \mathcal{P})$ is isomorphic to the process (g, α) where $g : I \to I$ preserves λ and has the following properties: $\alpha = \{[0, a_1], [a_1, 1]\} = \{J_0, J_1\}$ for some $a_1 \in (0, 1)$ and*

(1) *$g_i = g|J_i$ for $i = 0, 1$ are continuous and increasing on J_i,*
(2) *$g(J_i) = [0, 1]$ for $i = 0, 1$,*
(3) *$g_i' \ge \theta > 1$ for almost all $x \in J_i$, $i = 0, 1$,*
(4) *g is weakly mixing but not mixing.*

We get the above by using the argument from [Bo1].

By Conclusions 1, 4 and Lemma 3 we get

CONCLUSION 5. *The process (g, α) is q.m.p.*

2. Proof of Theorem 2. Let S be a weakly mixing automorphism, with finite entropy, of the space $(X, \mathcal{A}, p)$. Let $\mathcal{R} = \{R_0, R_1, \ldots, R_{l-1}\}$ be the generating partition for S. Let $(\Omega, \mathcal{G}, m, \sigma)$ be a one-sided Bernoulli shift with independent generator $Q = \{Q_0, Q_1, \ldots, Q_{l-1}\}$.

Take the product dynamical system

$$(\Omega \times X, \mathcal{G} \times \mathcal{A}, m \times p, \sigma \times S)$$

and define a measurable partition $\mathcal{P} = \{P_0, P_1, \ldots, P_{l-1}\}$ of $\Omega \times X$ as follows:

$$P_j = \bigcup_{s+t=j \ (\mathrm{mod}\ l)} Q_s \times R_t, \quad j = 0, 1, \ldots, l-1.$$

We will assume without loss of generality that $l = 2$, σ is a piecewise linear transformation of I, S is an automorphism of I and the invariant measure is the Lebesgue measure λ. Consider the skew product transformation

$$T(\omega, x, y) = (\sigma(\omega), S(x), T_{a(\omega,x)}(y))$$

related to the process $(\sigma \times S, \mathcal{P})$. Without loss of generality we may also assume that $(Y, \mathcal{B}, \varrho) = (I, \mathcal{A}, \nu)$ where ν is a completed Borel measure on I. Note that the results obtained in Section 3 of [KS] can be generalized, without changes in the proof, to $\sigma \times S$ defined as above. Therefore any a.c.i.m. for T is a product measure of the form $\lambda \times \mu$ and any T-invariant set with respect to $\lambda \times \mu$ has the form $I \times A$ for some $A \in \mathcal{A} \times \mathcal{A}$. Let g be the density of a.c.i.m. for T. Then $g = g(x, y)$. Therefore $\{g > 0\} = I \times D$. As $T(I \times D) = I \times D$ we obtain $S \times T_i(D) \subset D$ for $i = 0, 1$. Let $\lambda \times \mu$ be the invariant measure for T with the given density g. Obviously $\lambda \times \mu|I \times D \approx \lambda \times \lambda \times \nu|I \times D$. Therefore $T|I \times D$ is $\lambda \times \lambda \times \nu$-recurrent. For $\bar{i} = (i_1, \ldots, i_k) \in \{0,1\}^k$ we set $T_{\bar{i}} = T_{i_k} \circ \ldots \circ T_{i_1}$. Now, if $A \subset D$ and $\lambda \times \nu(A) > 0$ then for a.e. $(x, y) \in A$, there exists a sequence (n_k) and $\bar{i}_k = (i_1, \ldots, i_{n_k})$ such that $(S^{n_k}(x), T_{\bar{i}_k}(y)) \in A$ for $k = 1, 2, \ldots$ Moreover

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{k : n_k \le N\} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} 1_{I\times A}(T^k(\omega, x, y))$$
$$= E(1_{I\times A} \mid (\mathcal{A} \times \mathcal{A} \times \mathcal{A})_{\mathrm{inv}})_{\lambda\times\mu}(\omega, x, y) = E(1_A \mid \mathcal{D})_\mu(x, y) \quad \lambda \times \nu\text{-a.e.}$$

Here

$$\mathcal{D} = \{A \in \mathcal{A} \times \mathcal{A} : A \subset D \text{ and } T(I \times A) = I \times A\}.$$

By T-invariance of the set $I \times \{(x, y) \in D : E(1_A \mid \mathcal{D})_\mu(x, y) = 0\}$, we have

$$E(1_A \mid \mathcal{D})_\mu(x, y) > 0 \quad \lambda \times \nu\text{-a.e.}$$

Let $E_y = \{x : (x, y) \in E\}$ for $E \in \mathcal{A} \times \mathcal{A}$ and $y \in Y$.

LEMMA 4. *Let $\lambda \times \nu(E) > 0$ for some $E \in \mathcal{A} \times \mathcal{A}$ and let $\nu(C) > 0$ for some $C \in \mathcal{A}$. Then for every $\varepsilon > 0$ there exists C_ε with $\nu(C_\varepsilon) < \varepsilon$ and $\gamma_\varepsilon > 0$ such that if $y, z \in C - C_\varepsilon$ and $|y - z| < \gamma_\varepsilon$ then $\lambda(E_y \triangle E_z) < 2\varepsilon$.*

Proof. For every $\varepsilon > 0$ there exists a closed set F_ε such that $\lambda \times \nu(F_\varepsilon) > 1 - \varepsilon^2$ and $1_E|F_\varepsilon$ is continuous (by the Lusin theorem). Therefore there exists

$\gamma_\varepsilon > 0$ such that

(2) if $|y-z| < \gamma_\varepsilon$ and $(x,y),(x,z) \in F_\varepsilon$ then $1_E(x,y) = 1_E(x,z)$.

Let $C_\varepsilon = \{y : \lambda(F_{\varepsilon_y}) \le 1-\varepsilon\}$. Here $F_{\varepsilon_y} = \{x : (x,y) \in F_\varepsilon\}$. We have

$$\lambda \times \nu(F_\varepsilon) = \int_Y \lambda(F_{\varepsilon_y})\,d\nu = \Big(\int_{C_\varepsilon} + \int_{C_\varepsilon^c}\Big)\lambda(F_{\varepsilon_y})\,d\nu$$
$$\le (1-\varepsilon)\nu(C_\varepsilon) + 1 - \nu(C_\varepsilon) = 1 - \varepsilon\nu(C_\varepsilon).$$

From $\lambda\times\nu(F_\varepsilon) > 1-\varepsilon^2$, we obtain $\nu(C_\varepsilon) < \varepsilon$. Hence $\lambda(F_{\varepsilon_y}) > 1-\varepsilon$ for $y \in C_\varepsilon^c$. Therefore $\lambda(F_{\varepsilon_y} \cap F_{\varepsilon_z}) > 1-2\varepsilon$ for $y,z \in C_\varepsilon^c$. Let $y,z \in C - C_\varepsilon$ and $|y-z| < \gamma_\varepsilon$. Then

$$\lambda(E_y \triangle E_z) = \Big(\int_{F_{\varepsilon_y}\cap F_{\varepsilon_z}} + \int_{(F_{\varepsilon_y}\cap F_{\varepsilon_z})^c}\Big)|1_E(x,y) - 1_E(x,z)|\,d\lambda(x)$$
$$= \int_{(F_{\varepsilon_y}\cap F_{\varepsilon_z})^c} |1_E(x,y) - 1_E(x,z)|\,d\lambda(x) < 2\varepsilon$$

by (2). ■

Before the proof of Theorem 2 let us present its main idea. We obtain $D = I \times B$ for some $B \in \mathcal{A}$ by showing that, for every $0 < \delta < 1$ and $C = \{y : \delta < \lambda(D_y) < 1-\delta\}$, $\nu(C) = 0$. Then we use the recurrence property of the family $\{S \times T_i\}_{i=0,1}$ on D and the weak mixing property of S to show that $\{D_y\}_{y\in C}$ is a pairwise independent family of sets. By using Lemma 4 we next get $\nu(C) = 0$.

Proof of Theorem 2. We show that there exists $B \in \mathcal{A}$ such that $D = I \times B$. Consider two measurable (by the Fubini theorem) functions $f(y,z) = \lambda(D_y \cap D_z)$ and $h(y) = \lambda(D_y)$. Let the closed set G_ε correspond to the function f by the Lusin theorem. There exists a finite covering $\{I_n \times J_n\}_{n=1}^k$ of G_ε, where I_n, J_n are intervals, such that if $(y,z),(y_1,z_1) \in I_n \times J_n \cap G_\varepsilon$ then $|f(y,z) - f(y_1,z_1)| < \varepsilon$. Similarly, the set H_ε and the covering $\{J_n\}_{n=1}^k$ correspond to h. Here the intervals $\{J_n\}_{n=1}^k$ are chosen uniformly for both purposes. Let

$$A_n = D \cap \Big(I \times \Big(H_\varepsilon \cap \bigcup_{y\in I_n\cap(C-C_\varepsilon)} G_{\varepsilon_y} \cap J_n \cap (C - C_\varepsilon)\Big)\Big),$$

where C_ε is the set from Lemma 4 for $E = D$ and $C = \{y : \delta < \lambda(D_y) < 1-\delta\}$ for some $\delta > 0$. Assume that $\lambda(I_n) \le \gamma_\varepsilon$, $n = 1,\ldots,k$. If $\lambda \times \nu(A_n) > 0$ then for a.e. $(x,z) \in A_n$ there exists a sequence (n_k) such that $(S^{n_k}(x), T_{\bar{i}_k}(z)) \in A_n$ for $k = 1,2,\ldots$ Hence, for a.e. $z \in H_\varepsilon \cap G_{\varepsilon_y} \cap (C - C_\varepsilon) \cap J_n$, $T_{\bar{i}_k}(z) \in H_\varepsilon \cap G_{\varepsilon_{y_k}} \cap (C - C_\varepsilon) \cap J_n$ for some sequence $(\bar{i}_k)$ and $y_k \in I_n \cap (C - C_\varepsilon)$, $k = 1,2,\ldots$ (as $\lambda(A_{n_z}) > 0$). Consequently,

(3) $$|f(y,z) - f(y_k, T_{\bar{i}_k}(z))| < \varepsilon, \quad k = 1,2,\ldots$$

On the other hand

(4) $$|f(y_k, T_{\bar{i}_k}(z)) - f(y, T_{\bar{i}_k}(z))|$$
$$= |\lambda(D_{y_k} \cap D_{T_{\bar{i}_k}(z)}) - \lambda(D_y \cap D_{T_{\bar{i}_k}(z)})| \le \lambda(D_{y_k} \triangle D_y) \le 2\varepsilon,$$

by Lemma 4. Since $S \times T_i D \subset D$ for $i = 0,1$, we have $SD_z \subset D_{T_i(z)}$ for $i = 0,1$ and consequently $S^{n_k}D_z \subset D_{T_{\bar{i}_k}(z)}$. The fact that S preserves λ implies $\lambda(D_z) \le \lambda(D_{T_i(z)})$ for $i = 0,1$. Let $a_z = \lim_{k\to\infty} h(T_{\bar{i}_k}(z))$. As $T_{\bar{i}_k}(z) \in H_\varepsilon \cap J_n$ we have $a_z - h(z) < \varepsilon$. Let us estimate

$$|f(y,z) - \lambda(D_y \cap S^{n_k}D_z)|$$
$$\le |f(y,z) - f(y, T_{\bar{i}_k}(z))| + |\lambda(D_y \cap D_{T_{\bar{i}_k}(z)}) - \lambda(D_y \cap S^{n_k}D_z)|$$
$$\le 3\varepsilon + \lambda(D_{T_{\bar{i}_k}(z)} \triangle S^{n_k}D_z) < 4\varepsilon.$$

The last inequality holds by (3), (4) and by

$$\lambda(D_{T_{\bar{i}_k}(z)} \triangle S^{n_k}D_z) = \lambda(D_{T_{\bar{i}_k}(z)} - S^{n_k}D_z) \le a_z - h(z) \le \varepsilon.$$

Now, as S is weakly mixing and $\lim_{N\to\infty} \frac{1}{N}\operatorname{card}\{k : n_k \le N\} > 0$, without loss of generality we can assume that

$$\lim_{k\to\infty} \lambda(D_y \cap S^{n_k}D_z) = \lambda(D_y)\lambda(D_z)$$

(see [Wa]). Finally, we obtain

$$|\,\lambda(D_y \cap D_z) - \lambda(D_y)\lambda(D_z)\,| \le 4\varepsilon \quad \nu\times\nu\text{-a.e.}$$

on the set $G_\varepsilon \cap (C - C_\varepsilon) \times (C - C_\varepsilon) \cap H_\varepsilon$. Taking $\varepsilon \to 0$ we get

$$\lambda(D_y \cap D_z) = \lambda(D_y)\lambda(D_z) \quad \text{a.e. on } C \times C.$$

By the last equality and the definition of C we have $\lambda(D_y \triangle D_z) \ge \delta^2$ a.e. on $C \times C$, which contradicts the conclusion of Lemma 4 for $\varepsilon < 2^{-1}\delta^2$ and $(y,z) \in (C - C_\varepsilon) \times (C - C_\varepsilon)$ such that $|y - z| < \gamma_\varepsilon$. Hence $\nu(C) = 0$. This implies $D = I \times B$ for some $B \in \mathcal{A}$. ■

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On ideals consisting of topological zero divisors

by

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Abstract. The class $\omega(A)$ of ideals consisting of topological zero divisors of a commutative Banach algebra A is studied. We prove that the maximal ideals of the class $\omega(A)$ are of codimension one.

1. Introduction. Let A be a commutative Banach algebra over the complex field $\mathbb{C}$ with unit e. An element $a \in A$ is a *topological zero divisor* (TZD) if there exists a sequence $b_n \in A$ such that $\|b_n\| = 1$ and $\lim_{n\to\infty} ab_n = 0$. One says that an ideal $I \subset A$ consists of *joint topological zero divisors* (joint TZD) if for every finite collection $a_1, \ldots, a_k \in I$ there exists a sequence (b_n) of normalized elements of A such that

$$\lim_{n\to\infty} \sum_{j=1}^{k} \|a_j b_n\| = 0.$$

The set of all ideals of A consisting of joint TZD is denoted by $\text{ł}(A)$ while $\text{Ł}(A)$ denotes the set of those elements of $\text{ł}(A)$ which are maximal ideals of A. The class $\text{ł}(A)$ was intensively studied in the 70's. The most important results are the following theorems:

THEOREM 1.1 (Żelazko [5]). *The maximal ideals of A which belong to the Shilov boundary $S(A)$ are elements of $\text{Ł}(A)$. If A is a function algebra then $\text{Ł}(A) = S(A)$.*

THEOREM 1.2 (Słodkowski [3]). *If $J \in \text{ł}(A)$ then there exists $I \in \text{Ł}(A)$ such that $J \subset I$.*

V. Müller has proved a result conjectured by W. Żelazko which provides a complete characterization of $\text{ł}(A)$:

THEOREM 1.3 (Müller [2]). *An ideal I of A belongs to $\text{ł}(A)$ if and only if I is not removable.*

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