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The space of real-analytic functions has no basis

by

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Abstract. Let Ω be an open connected subset of $\mathbb{R}^d$. We show that the space $\mathrm{A}(\Omega)$ of real-analytic functions on Ω has no (Schauder) basis. One of the crucial steps is to show that all metrizable complemented subspaces of $\mathrm{A}(\Omega)$ are finite-dimensional.

1. Introduction. Let $\mathrm{A}(\Omega)$ be the space of all complex-valued real-analytic functions on the open set $\Omega \subset \mathbb{R}^d$. This means that $f \in \mathrm{A}(\Omega)$ develops locally into a Taylor series at every point of Ω. The space $\mathrm{A}(\Omega)$ is assumed to be equipped with its natural topology. $\mathrm{A}(\Omega)$ is always a complete ultrabornological reflexive nuclear separable space. Its dual is a nuclear LF-space. The space, being a classical object of study, has attracted some attention in recent time mainly because of its relevance to the theory of partial differential equations but also as an object interesting in itself. As a main result of this paper we will show that for Ω connected the space $\mathrm{A}(\Omega)$ has no basis. We will prove even more, namely, a complemented subspace of $\mathrm{A}(\Omega)$ with basis must be an LB-space (see Theorem 4.1).

A sequence (f_n) of functions which forms a (Schauder) basis in a function space X is not only interesting because of the structure of X. If the space in question is natural in the sense that it appears in problems of classical analysis, then (f_n) is usually very useful from the analytical point of view. For instance, see the special role played by Hermite functions in the space S of rapidly decreasing smooth functions, Chebyshev polynomials in the space of smooth functions on the closed interval $[-1,1]$ or the Franklin system in the Hardy space $H^1(\mathbb{D})$ [39]. Clearly, the space of real-analytic functions is natural so the above remarks apply. Unfortunately, as our paper shows, life is hard and there are no such special real-analytic functions.

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The problem of whether every separable Banach space has a basis appeared in 1931 (probably for the first time) on page 141 of the Polish edition of Banach's book [1] and then it was repeated in the later French version [2, Ch. VII, §3]. It was clear to Banach, Mazur and Schauder (see [30, Ch. III]) that this problem is related to the approximation problem posed in [2, Rem. VI, §1]. The latter question was repeated (in a hidden form) by Mazur in 1936 as Problem 153 of "The Scottish Book" [23] and analyzed in depth by Grothendieck in [10]. Thanks to the paper of Enflo [9] (comp. [20, Sec. 2.d]) we know that there are subspaces of l_p even without the approximation property. Although every nuclear Fréchet space has the approximation property, they also need not have a basis (see Mityagin–Zobin [26], comp. [3], [6], [27]) or the bounded approximation property [8] (comp. [35]). All these counterexamples are artificial in the sense that they were constructed on purpose and they are not "natural" spaces of functions, measures or operators appearing in analysis. The only exception is the highly non-separable space $L(l_2)$ of all bounded operators on l_2 which fails the approximation property [32]. $\mathrm{A}(\Omega)$ is a natural example of a separable complete function space without basis. Although this space is not metrizable it has many nice properties, in particular, the approximation property. For the role of $\mathrm{A}(\Omega)$ in recent research see e.g. [4], [5], [11], [16]–[19], [21], [22].

There are at least two natural ways of defining a topology on $\mathrm{A}(\Omega)$. Due to [22, Prop. 1.9, Th. 1.2] they are equivalent. One is to use the projective limit topology of $\mathrm{proj}_{N\in\mathbb{N}} H(I_N)$, where (I_N) is an increasing exhaustion of Ω by compact sets. Here $H(I_N)$ is the space of germs of holomorphic functions on $I_N \subseteq \mathbb{C}^d$ equipped with the standard topology of a nuclear LB-space. That means $H(I) := \mathrm{ind}_{n\in\mathbb{N}} H^\infty(U_n)$ topologically, where $H^\infty(U_n)$ is the space of bounded holomorphic functions on U_n with the sup-norm $\|\cdot\|_{\infty,U_n}$ and (U_n) forms a fundamental sequence of neighbourhoods of I in $\mathbb{C}^d$. Clearly, the topology of $H(I)$ does not depend on the choice of (U_n) and the topology of $\mathrm{A}(\Omega)$ does not depend on the choice of (I_N). The other canonical way is to use the topology defined by $\mathrm{A}(\Omega) = \mathrm{ind}\, H(U)$, where the inductive limit runs over all neighbourhoods of $\Omega \subseteq \mathbb{R}^d \subseteq \mathbb{C}^d$ in $\mathbb{C}^d$. Here $H(U)$ denotes the space of all holomorphic functions on U with the compact-open topology (see [22, Prop. 1.9, Th. 1.2]). Thus $\mathrm{A}(\Omega)$ is always a complete ultrabornological reflexive nuclear separable webbed space and its dual is a nuclear LF-space.

By a *PLN-space* we mean the projective limit of a sequence of LN-spaces, i.e., LB-spaces with nuclear linking maps. Analogously *PLS-space* means the projective limit of a sequence of LS-spaces, i.e., LB-spaces with compact linking maps. Clearly $\mathrm{A}(\Omega)$ is always a PLN-space. We show that an ultrabornological PLN-space with basis is either an LN-space or it has an infinite-dimensional Fréchet subspace. On the other hand it is either a Fréchet space or it has an infinite-dimensional complemented LN-subspace (see Corollary 2.3). The proof of this result is based on a refinement of the Dynin–Mityagin theorem (see Theorem 2.1).

In order to prove the main result of our paper, we show that for Ω connected, metrizable quotients of $\mathrm{A}(\Omega)$ have property $(\underline{\overline{\overline{\Omega}}})$ (see Theorem 3.4), and metrizable subspaces of $\mathrm{A}(\Omega)$ have $(\underline{\mathrm{DN}})$ (Theorem 3.6). As a consequence we find that metrizable complemented subspaces of $\mathrm{A}(\Omega)$ are finite-dimensional (Theorem 3.7). Of course, this result is also of strong independent interest.

Recall that a locally convex space X is *ultrabornological* if it is an inductive limit of a family of Banach (or Fréchet) spaces or, equivalently, every absolutely convex set absorbing Banach discs is a 0-neighbourhood. For PLS-spaces there is a nice characterization of ultrabornologicity in terms of the representing projective spectra. If $X = \mathrm{proj}_{N\in\mathbb{N}} X_N$ and $i_N^{N+1} : X_{N+1} \to X_N$ are linking maps, then $\sigma : \prod X_N \to \prod X_N$ is defined as $\sigma((x_N)_{N\in\mathbb{N}}) := (x_N - i_N^{N+1} x_{N+1})$ and $\mathrm{Proj}^1 X = \prod X_N / \mathrm{Im}\,\sigma$ (see [28], [29], [37], [36]). If X_N are LS-spaces, then Proj^1 does not depend on the choice of the projective spectrum (X_N) and X is ultrabornological if and only if $\mathrm{Proj}^1 X = 0$ (see [38] and [37, Th. 3.4, Lemma 4.1], [36, Th. 5.7, Th. 6.2]).

By p_B we denote the Minkowski functional of the absolutely convex set B. For other unexplained notions and facts from functional analysis see [24] or [12]. Similarly, [14] is our reference book for complex analysis, [13] for plurisubharmonic functions, and [15] for real-analytic functions.

2. Bases in PLN-spaces. A scalar matrix $A = (a_{j;N,n})$ is called a *Köthe PLB-matrix* if it satisfies:

1. for all j there is N such that for all n: $a_{i;N,n} > 0$;
2. for all j, N, n: $a_{j;N+1,n} \geq a_{j;N,n}$;
3. for all j, N, n: $a_{j;N,n+1} \leq a_{j,N,n}$.

The matrix A is *strictly positive* if there is N_0 such that for every $N \geq N_0$ we have $a_{j;N,n} > 0$ for all $j, n \in \mathbb{N}$. A *Köthe PLB-space* $E_p(A)$ for a Köthe PLB-matrix A is defined to be

$$E_p(A) := \Big\{ x = (x_j) : \ \forall N\ \exists n : \|x\|_{N,n}^{(p)} := \Big(\sum_{j=0}^{\infty} (|x_j| a_{j;N,n})^p \Big)^{1/p} < \infty \Big\}$$

for $1 \leq p < \infty$ with its natural PLB-topology of $E_p(A) = \mathrm{proj}_N E_p^N(A)$, where

$$E_p^N(A) := \operatorname*{ind}_{n\in\mathbb{N}} E_p^{N,n}(A),$$

$$E_p^{N,n}(A) := \{x = (x_j)_{j\in I_N} : \|x\|_{N,n}^{(p)} < \infty\}, \quad I_N := \{j : \ \forall n : a_{j;N,n} > 0\}.$$

Analogously, we define $E_\infty(A)$, where
$$\|x\|_{N,n}^{(\infty)} := \sup_j |x_j| a_{j;N,n}.$$
We omit the superscript in the notation of norms when it produces no misunderstanding.

THEOREM 2.1. *Let X be a PLN-space with an equicontinuous basis $(e_j)_{j\in\mathbb{N}}$. Then $X \cong E_p(A)$, where $1 \le p \le \infty$, for a suitable chosen Köthe PLB-matrix A not depending on p. In fact, we can choose $A = (a_{j;N,n})$ as follows:*
$$a_{j;N,n} := p_{B_{N,n}}(e_j),$$
where $(B_{N,n})$ are the unit balls in LN-spaces X_N, $X = \operatorname{proj} X_N$, satisfying
$$B_{N,n+1} \supseteq B_{N,n}, \quad i_N^{N+1} B_{N+1,n} \subseteq B_{N,n} \quad \text{for } n, N \in \mathbb{N},$$
with $i_N^{N+1} : X_{N+1} \to X_N$ denoting the linking map.

Proof. By the Dynin–Mityagin theorem [12, 21.10.1], $(e_j)_{j\in\mathbb{N}}$ is an absolute basis. Denote by (e_j^*) the biorthogonal functionals.

Observe that $X = \operatorname{proj}_N X_N$, where X_N are LN-spaces. Let P_N be a fundamental family of seminorms on X_N (or corresponding seminorms induced on X) and let $B_N := (b_{j;p})_{j\in\mathbb{N},\, p\in P_N}$ be the Köthe matrix defined by $b_{j;p} = p(e_j)$. As in [12, 14.7.8], we observe that
$$X = \operatorname*{proj}_N \lambda_1(B_N).$$
We will show that for every N there is M such that the (diagonal) linking map $\lambda_1(B_M) \to \lambda_1(B_N)$ factorizes through X_M. Clearly we have
$$p(x) \le \sum_{j=0}^\infty |e_j^*(x)| p(e_j).$$
Thus $\lambda_1(B_M)$ naturally maps into X_M. On the other hand, for every $p \in P_N$ there is a continuous seminorm q on X and a constant C such that
$$\sum_{j=0}^\infty |e_j^*(x)| p(e_j) \le Cq(x).$$
Assume that for any $K > N$ there is $p_K \in P_N$ such that no q can be taken from P_K in the formula above with $p = p_K$. Since P_N is a set of seminorms inducing a topology of an LB-space, there is a seminorm $p \in P_N$ such that
$$p_K(x) \le C_K p(x) \quad \text{for all } K,\ x \in X.$$
Therefore $\sum_{j=0}^\infty |e_j^*(x)| p(e_j)$ cannot be majorized by any continuous seminorm on X; a contradiction.

We have proved that for every N there is M such that for every $p \in P_N$ there is $q \in P_M$ satisfying

$$\sum_{j=0}^\infty |e_j^*(x)| p(e_j) \le Cq(x) \quad \text{for all } x \in X.$$
This implies that X_M maps naturally into $\lambda_1(B_N)$. Set
$$I_N := \{j : \exists p \in P_N : b_{j;p} > 0\} \quad \text{and} \quad \widetilde{B}_N = (b_{j;p})_{j\in I_N,\, p\in P_N}.$$
One easily sees that the spectra (X_N) and $(\lambda_1(\widetilde{B}_N))$ are equivalent and, without loss of generality, we may assume that
$$\ldots \to X_{N+2} \to \lambda_1(\widetilde{B}_{N+1}) \to X_{N+1} \to \lambda_1(\widetilde{B}_N) \to \ldots$$
As a consequence, for every $p \in P_N$ there is $q \in P_{N+1}$ such that the map between the corresponding weighted l_1-spaces $l_1((q(e_j))_{j\in\mathbb{N}})$ and $l_1((p(e_j))_{j\in\mathbb{N}})$ is nuclear. Therefore
$$\sum_{j\in\mathbb{N}} p(e_j)/q(e_j) < \infty.$$
Hence, also $(\lambda_p(\widetilde{B}_N))$ is an equivalent spectrum for any p, $1 \le p \le \infty$.

Let us choose the spectrum with $p = \infty$. The space X_{N+1} has a fundamental sequence of Banach disks and their images are Banach disks in $\lambda_\infty(\widetilde{B}_N)$ contained in the Banach disks of the form
$$C_{N,n} := \{x = (x_j) : \sup_j |x_j| v_{j;N,n} \le 1\}$$
for some sequence $(v_{j;N,n})_{j\in\mathbb{N}}$. Clearly, $(C_{N,n})_{n\in\mathbb{N}}$ gives an LB-topology stronger than the original topology of $\lambda_\infty(\widetilde{B}_N)$ and the map $X_{N+1} \to \lambda_\infty(\widetilde{B}_N)$ factorizes through this LB-space. We have obtained a new equivalent spectrum $(E_\infty^N(V))$ with linking maps factorizing through LN-spaces.

Since for a diagonal map D on l_p, $1 \le p \le \infty$, one can easily calculate the nth diameters of $D(B_{l_p})$, B_{l_p} the unit ball of l_p, as coefficients of D (comp. [31, Sec 9.1]), by nuclearity one can assume that
$$\sum_{j\in\mathbb{N}} v_{j;N,n}/v_{j;N+1,n-1} < \infty.$$
Therefore, also the spectra $(E_p^N(V))$ are equivalent for $1 \le p \le \infty$.

Now, for $p = 1$, we may assume without loss of generality that the linking map $E_1^{N+1}(V) \to E_1^N(V)$ factorizes through X_N. It is easily seen that the map $E_1^{N+1}(V) \to X_N$ factorizes through $E_1^N(A)$ if $A = (a_{j;N,n})$, $a_{j;N,n} = p_{B_{N,n}}(e_j)$.

Using an analogous procedure one finds that the spectra $(E_p^N(A))$ are all equivalent for $1 \le p \le \infty$. ∎

THEOREM 2.2. *Let E be an ultrabornological Köthe PLS-space given by the Köthe PLB-matrix $(a_{j;k,p})$.*

(a) *If E does not admit an infinite-dimensional complemented Fréchet subspace, then it is an LS-space.*

(b) *If E does not admit an infinite-dimensional complemented LS-subspace, then it is a Fréchet space.*

Proof. Let $I \subseteq \mathbb{N}$. We denote by $E(I)$ the subspace of E spanned over the subset I. We define an increasing family of subsets of $\mathbb{N}$ by

$$I_N := \{j : a_{j;N,n} > 0 \text{ for every } n \in \mathbb{N}\}.$$

In general, $E = \prod_{N\in\mathbb{N}} E(I_{N+1} \setminus I_N)$ and each of the spaces in the product can be equipped with a strictly positive matrix. Therefore it suffices to prove the result only for such spaces. Note that complemented subspaces of E are again ultrabornological. Since E is ultrabornological we get by [37, Lemma 4.1] (see also [36, Theorem 6.2])

$$(1) \qquad \forall K\ \exists k, L\ \forall l, M\ \exists m, S\ \forall j \in \mathbb{N}: \quad \min(a_{j;M,m}, a_{j;K,k}) \le S a_{j;L,l}.$$

Since E is a PLS-space, we may assume without loss of generality that

$$\forall L\ \forall l: \quad \lim_{j\to\infty} a_{j;L+1,l-1}/a_{j;L,l} = \infty.$$

Combining this with (1) and taking there $k = K$, $L = K+1$ we may assume without loss of generality that

$$\forall K, l, M\ \exists m: \quad \min(a_{j;M,m}, a_{j;K,K}) \le a_{j;K+1,l}$$

for all except finitely many $j \in \mathbb{N}$. Finally, for all $j \in \mathbb{N}$ we get

$$(2) \qquad \forall K, l, M\ \exists m, S\ \forall j \in \mathbb{N}: \quad \min(S a_{j;M,m}, a_{j;K,K}) \le a_{j;K+1,l}.$$

We now put

$$I_{K,l} := \{j : a_{j;K,K} \le a_{j;K+1,l}\}, \qquad J_{K,l} := \mathbb{N} \setminus I_{K,l}$$

and observe that $J_{K,l} \subseteq J_{K,l+1}$ for all K, l. Therefore (2) shows that $E(J_{K,l})$ is an LB-space. The same holds for $E(J)$ if J is a finite union of the $J_{K,l}$.

To prove (a) we assume that E is not an LB-space. This implies that I is infinite for every finite intersection I of the $I_{K,l}$. This means that we can choose an infinite subset $L \subset \mathbb{N}$ so that for every K and l only finitely many elements of L are not in $I_{K,l}$.

If we fix K then for all $j \in L$ we have, with suitable constants $C_{K,l}$ which take care of the finitely many elements,

$$(3) \qquad a_{j;K,K} \le \inf\{C_{K,l} a_{j;K+1,l} : l \in \mathbb{N}\} =: p_K(j)$$

and $p_K(j)$ defines a seminorm in E. Therefore $E(L)$ is an infinite-dimensional complemented Fréchet subspace of E.

To prove (b) we notice that for all K and l the space $E(J_{K,l})$ is a complemented DF-subspace of E hence, due to the assumption in (b), it is finite-dimensional. Therefore $J_{K,l}$ is finite and, arguing as before, we obtain the estimates (3) for all $j \in \mathbb{N}$. Therefore E is a Fréchet space. ∎

By the weak basis theorem [12, 14.3.4], a basis in an ultrabornological webbed space X is automatically an equicontinuous basis. By Theorems 2.2 and 2.1 we get immediately:

COROLLARY 2.3. *Every ultrabornological PLN-space with basis is either an LN-space or it contains an infinite-dimensional Fréchet subspace. Analogously, it is either a Fréchet space or contains an infinite-dimensional complemented LN-subspace.*

3. Complemented Fréchet subspaces of $A(\Omega)$.

For an open or compact subset $M \subset \mathbb{C}^d$ we denote by $H(M)$ the space of all holomorphic functions on M with its canonical topology. For any function f on a set M we set $\|f\|_M = \sup\{|f(x)| : x \in M\}$ whether it is finite or not. For $z \in \mathbb{C}^d$ we put $|z|_\infty = \max\{|z_j| : j = 1, \ldots, n\}$. The following lemma is, of course, related to the proof of polynomial convexity of compact subsets of $\mathbb{R}^d$ (see [13, Lemma 5.4.1]).

LEMMA 3.1. *Let $I \subset J$ be compact subsets of $\mathbb{R}^d$ and $U \supset I$ be an open bounded subset of $\mathbb{C}^d$. Let $0 < \alpha < 1$. Then there exist finitely many entire functions $f_k : \mathbb{C}^d \to \mathbb{C}$, $k = 1, \ldots, m$, such that the plurisubharmonic function $u := \max(\log|f_1|, \ldots, \log|f_m|)$ is continuous on $\mathbb{C}^d$ and satisfies*

$$I \subset \{z : u(z) < 0\}, \qquad \{z : u(z) \le \alpha\} \subset\subset U,$$
$$J \subset \{z : u(z) < 1\} \subset\subset \mathbb{C}^d.$$

Proof. We choose $0 < \alpha < \alpha' < 1$, put $\varrho = e^{\alpha'}$ and choose $\varepsilon > 0$ so that $\varrho + 2\varepsilon < e$. We find $h \in C(\mathbb{R}^d)$ so that

$$I \subset \{x \in \mathbb{R}^d : |h(x)| < 1 - \varepsilon\}, \qquad \{x \in \mathbb{R}^d : |h(x)| \le \varrho + \varepsilon\} \subset\subset U,$$
$$J \subset \{x \in \mathbb{R}^d : |h(x)| < e - \varepsilon\}.$$

We choose $R > 0$ so that

$$J \cup \overline{U} \subset B = \{z : |z|_\infty \le R - 1\}.$$

Using the Weierstrass approximation theorem we find a polynomial P so that

$$\sup_{|x|_\infty \le R} |h(x) - P(x)| < \varepsilon.$$

Then we have

$$I \subset \{x \in \mathbb{R}^d : |P(x)| < 1\}, \qquad \{x \in \mathbb{R}^d : |x|_\infty \le R,\ |P(x)| \le \varrho\} \subset\subset U,$$
$$J \subset \{x \in \mathbb{R}^d : |P(x)| < e\}.$$

We put $u_1(z) = \max(\log|P(z)|; -1)$. Then u_1 is continuous plurisubharmonic on $\mathbb{C}^d$ and we have

$$I \subset \{z : u_1(z) < 0\}, \qquad \{x \in \mathbb{R}^d : |x|_\infty \le R,\ u_1(x) \le \alpha'\} \subset\subset U,$$
$$J \subset \{z : u_1(z) < 1\}.$$

Set $u_2(z) = \max(\alpha'(|z|_\infty - R + 1); u_1(z))$. Then u_2 is again continuous and plurisubharmonic on $\mathbb{C}^d$ and

$$I \subset \{z : u_2(z) < 0\}, \qquad \{x \in \mathbb{R}^d : u_2(x) \le \alpha'\} \subset\subset U,$$
$$J \subset \{z : u_2(z) < 1\}.$$

Finally we put $u(z) = \max(u_2(z); A|\mathrm{Im}\, z|_\infty - 1)$. Then u is continuous plurisubharmonic on $\mathbb{C}^d$ and obviously

$$I \subset \{z : u(z) < 0\}, \quad J \subset \{z : u(z) < 1\}.$$

We now determine A so that the remaining inclusion of the assertion holds. We may choose $\delta > 0$ so that

(a) $M := \{x \in \mathbb{R}^d : u_2(x) \le \alpha'\} + \{z \in \mathbb{C}^d : |z|_\infty \le \delta\} \subset\subset U$,
(b) for $|z|_\infty \le R$ and $|w|_\infty \le \delta$ we have $|u_2(z+w) - u_2(z)| < \alpha' - \alpha$.

We choose $A > 0$ so that $(1+\alpha')/A < \delta$. If $u(z) \le \alpha$ then, of course, $u_2(z) \le \alpha$, $|z|_\infty \le R$, $|\mathrm{Im}\, z|_\infty < \delta$. Therefore we get

$$u_2(\mathrm{Re}\, z) \le u_2(z) + |u_2(\mathrm{Re}\, z) - u_2(z)| \le \alpha + (\alpha' - \alpha) = \alpha'.$$

This implies

$$z \in \{x \in \mathbb{R}^d : u_2(x) \le \alpha'\} + \{z : |z|_\infty \le \delta\} = M \subset\subset U. \ \blacksquare$$

The following lemma is due to Zaharjuta (see [40], Prop. 2.1.2; cf. [41]). For completeness we give a proof. We first need some notation.

For a function $u = \max\{\log|g_j| : j = 1, \ldots, m\}$, where all g_j are entire functions on $\mathbb{C}^d$, and $0 \le \alpha \le 1$ we put $D_\alpha = \{z : u(z) < \alpha\}$. These are analytic polyhedra. We notice that the u of the previous lemma is of this type, and u will denote a function of this type in what follows. For functions f on D_α we put $|f|_\alpha = \|f\|_{D_\alpha}$.

Lemma 3.2. *For every $0 < \alpha < \alpha' < 1$ there is a constant $C > 0$ so that for every $f \in H(D_{\alpha'})$ with $|f|_{\alpha'} \le 1$ and every $r > 0$ there exist $g \in H(D_{\alpha'})$ and $h \in H(D_1)$ so that $f = g + h$ on $D_{\alpha'}$ and*

$$|g|_0 \le C\frac{1}{r^\alpha}, \quad |h|_1 \le Cr^{1-\alpha}.$$

Proof. We may assume that $D_1 \subset \Delta_0$ where $\Delta_\alpha = \{z : |z|_\infty < e^\alpha\}$ and use the classical method of Oka. Put $\varphi(z) = (z, g_1(z), \ldots, g_m(z)) \in \mathbb{C}^{d+m} = \mathbb{C}^N$ for $z \in D_1$. This defines a biholomorphic map φ from D_1 onto a closed complex submanifold $\varphi(D_1)$ of Δ_1. We have $\varphi(D_\alpha) = \varphi(D_1) \cap \Delta_\alpha$. We choose $\alpha < \alpha'' < \alpha'$. From the Cartan–Oka theory [14, Th. 7.2.7] and the open mapping theorem we conclude: there is a constant $C_0 > 0$ so that for every $\widehat{f} \in H(\varphi(D_{\alpha'}))$ with $\|\widehat{f}\|_{\varphi(D_{\alpha'})} \le 1$ there is $F \in H(\Delta_{\alpha'})$ with $\|F\|_{\Delta_{\alpha''}} \le C_0$.

We expand F into a power series

$$F(z) = \sum_\beta a_\beta z^\beta.$$

Then Cauchy's inequalities yield

$$|a_\beta| \le C_0 e^{-|\beta|\alpha''}, \quad \beta \in \mathbb{N}_0^N.$$

For given $\nu \in \mathbb{N}_0$ we put

$$G(z) = \sum_{|\beta|>\nu} a_\beta z^\beta, \quad H(z) = \sum_{|\beta|\le\nu} a_\beta z^\beta.$$

Then for $z \in \Delta_0$ we obtain

$$|G(z)| \le C_0 \sum_{|\beta|>\nu} e^{-|\beta|\alpha''} \le C_0 e^{-(\nu+1)\alpha} \sum_\beta e^{|\beta|(\alpha-\alpha'')}$$

and, for $z \in \Delta_1$,

$$|H(z)| \le C_0 \sum_{|\beta|\le\nu} e^{(1-\alpha'')|\beta|} \le C_0 e^{\nu(1-\alpha)} \sum_\beta e^{|\beta|(\alpha-\alpha'')}.$$

If $r \ge 1$ we choose $\nu \in \mathbb{N}_0$ so that $e^\nu \le r \le e^{\nu+1}$, and we put $C = C_0 \sum_\beta e^{|\beta|(\alpha-\alpha'')}$.

For f as in the assumption we set $\widehat{f} = f \circ \varphi^{-1}$, obtain F and then G and H. We put $g = G \circ \varphi$ and $h = H \circ \varphi$ and obtain the assertion since for $r \le 1$ it is obvious. $\blacksquare$

Lemma 3.3. *Let $I \subset J$ be connected compact sets in $\mathbb{R}^d$ and $U \supset I$ be an open bounded subset of $\mathbb{C}^d$. Let $0 < \alpha < 1$. Then there exist open sets $V \supset I$, $W \supset J$ in $\mathbb{C}^d$ and $C > 0$ so that for every $f \in H(U)$ with $\|f\|_U \le 1$ and every $r > 0$ there exist $g \in H(V)$, $h \in H(W)$ so that $f = g + h$ on I and*

$$\|g\|_V \le C\frac{1}{r^\alpha}, \quad \|h\|_W \le Cr^{1-\alpha}.$$

Proof. We apply Lemma 3.1 and obtain a plurisubharmonic function u which fulfills the assumptions of Lemma 3.2. We set $V = \{z \in \mathbb{C}^d : u(z) < 0\}$ and $W = \{z \in \mathbb{C}^d : u(z) < 1\}$. Lemma 3.2 then gives the result. $\blacksquare$

A Fréchet space with a fundamental sequence $(\|\cdot\|_n)$ of seminorms defining the topology is said to have *property* $(\overline{\overline{\Omega}})$ if

$$\forall k\ \exists m\ \forall n, \vartheta \in]0,1[\ \exists C\ \forall u \in E' : \quad \|u\|_m^* \le C\|u\|_k^{*\vartheta}\|u\|_n^{*1-\vartheta}.$$

Here $\|\cdot\|^*$ denotes the dual norm for $\|\cdot\|$.

Remark. A Fréchet space E has $(\overline{\overline{\Omega}})$ if and only if

$$\forall V\ \exists U\ \forall W, \gamma > 0\ \exists C\ \forall r > 0 : \quad U \subseteq C\Big(\frac{1}{r^\gamma}V + rW\Big),$$

where U, V, W are 0-neighbourhoods in E.

Theorem 3.4. *Every Fréchet space E which is a quotient of $\mathrm{A}(\Omega)$ has property $(\overline{\overline{\Omega}})$.*

Proof. Let $q : \mathrm{A}(\Omega) \to E$ be a quotient map, and E_k the local Banach spaces of E with unit balls $U_k = \{x \in E : \|x\|_k \le 1\}$.

For given k we may find a compact set $I \subset \Omega$ so that q extends to a continuous linear map $q_k : H(I) \to E_k$. Since $i^k E \subset q_k(H(I))$ Grothendieck's

factorization theorem yields an open neighbourhood U of I so that $i^kE \subset q_k(H(U))$ and for some $m \geq k$ we have
$$q_k(\{f \in H(U) : \|f\|_U \leq C_1\}) \supset i^k U_m.$$
For every $n \geq m$ we find a compact $J \subset \Omega$ so that q extends to a continuous linear map $q_n : H(J) \to E_n$.

For given $0 < \alpha < 1$ we now choose open sets V and W according to Lemma 3.3. For $x \in U_m$ we find $f \in H(U)$ with $\|f\|_U \leq C_1$ so that $q_k f = i^k x$. From Lemma 3.3 we obtain with some $C_2 > 0$ functions $g \in H(V)$ and $h \in H(W)$ so that $f = g + h$ and
$$\|g\|_V \leq C_1C_2\frac{1}{r^\alpha}, \quad \|h\|_W \leq C_1C_2r^{1-\alpha}.$$
Also $D \geq C_1C_2$ may be chosen so large that
$$\|q_k(v)\|_k \leq D\|v\|_V, \quad v \in H(V),$$
$$\|q_n(w)\|_n \leq D\|w\|_W, \quad w \in H(W).$$
Therefore with $\xi = q_k(g)$ and $\eta = q_n(h)$ we have
$$\|\xi\|_k \leq D^2\frac{1}{r^\alpha}, \quad \|\eta\|_n \leq D^2r^{1-\alpha}$$
and $i^kx = \xi + i^k\eta$.

We have proved that
$$U_m \subset D^2\frac{1}{r^\alpha}U_k + D^2r^{1-\alpha}U_n$$
or
$$\|\ \|_m^* \leq D^2\left(\frac{1}{r^\alpha}\|\ \|_k^* + r^{1-\alpha}\|\ \|_n^*\right)$$
for all $r > 0$, which is equivalent to
$$\|\ \|_m^* \leq C\|\ \|_k^{*\vartheta}\|\ \|_n^{*1-\vartheta}$$
with $\vartheta = 1 - \alpha$. ∎

LEMMA 3.5. *Let F be a Fréchet space and Ω connected. Then every continuous linear map $\varphi : F \to \mathrm{A}(\Omega)$ factorizes through a space $H(U)$, where U is an open neighbourhood of Ω.*

Proof. Let (I_n), $I_n \subset \mathring{I}_{n+1}$, be an exhaustion of Ω by compact connected subsets. For every n the induced map $\varphi_n : F \to H(I_n)$ is compact. Hence there is a neighbourhood $U_n := \{z = x+iy : x \in W_n, |y|_\infty < \varepsilon_n\}$, $W_n \subset\subset \Omega$ open and connected, so that φ_n factorizes through $H(U_n)$.

For every $x \in F$ we obtain functions $\varphi_n(x) \in H(U_n)$ which coincide pairwise on $U_n \cap U_m \cap \Omega$. So $\varphi(x)$ may be extended to $H(U)$, where $U = \bigcup_n U_n$.

Therefore φ gives rise to a map $\psi : F \to H(U)$ which is clearly linear and, due to the closed graph theorem, also continuous. It is obvious that $\varphi = \varrho \circ \psi$, where ϱ is the restriction to Ω. ∎

A Fréchet space with a fundamental sequence $(\|\cdot\|_n)$ of seminorms defining the topology is said to have *property* $(\underline{\mathrm{DN}})$ if
$$\exists n\ \forall k\ \exists l, C > 0,\ \tau \in]0,1[: \quad \|x\|_k \leq C\|x\|_n^\tau \|x\|_l^{1-\tau}$$
for every $x \in E$.

THEOREM 3.6. *If Ω is connected then every Fréchet subspace E of $\mathrm{A}(\Omega)$ has property $(\underline{\mathrm{DN}})$.*

Proof. By Lemma 3.5 every Fréchet subspace of $\mathrm{A}(\Omega)$ is isomorphic to a subspace of $H(U)$, where U is an open subset of $\mathbb{C}^d$. Since the latter space has $(\underline{\mathrm{DN}})$ (see [33, Cor. 5.3] or [25, Prop. 2.1]), E also has this property because $(\underline{\mathrm{DN}})$ is inherited by subspaces [33, Lemma 2.2]. ∎

THEOREM 3.7. *If Ω is connected then every complemented Fréchet subspace of $\mathrm{A}(\Omega)$ is finite-dimensional.*

Proof. By Theorems 3.4 and 3.6, E has both $(\overline{\overline{\Omega}})$ and $(\underline{\mathrm{DN}})$. On the other hand, exactly as in [24, 29.21] using [34, Satz 4.4] instead of [24, 29.16] one proves that spaces with both properties above are Banach spaces. This completes the proof by use of the nuclearity of $\mathrm{A}(\Omega)$. ∎

REMARK. There is an alternative proof that complemented Fréchet subspaces E of $\mathrm{A}(\Omega)$ are Banach spaces. Namely, let $T : E \to \mathrm{A}(\Omega)$ be the topological embedding. Since every map $\mathrm{A}(\Omega)' \to E'$ is bounded by [4, Th. 21] and [5, Th. 5] (E has $(\overline{\overline{\Omega}})$!!), T' and also T are bounded.

4. The main result

THEOREM 4.1. *Let Ω be connected. If E is a complemented subspace with basis of $\mathrm{A}(\Omega)$, then E is an LB-space. In particular, $\mathrm{A}(\Omega)$ has no basis.*

Proof. The result follows immediately from Theorem 3.7 and Corollary 2.3. Clearly, $\mathrm{A}(\Omega)$ is not an LB-space because by an easy interpolation argument it has the Fréchet space ω of all sequences as a quotient. ∎

5. Final remarks. In fact Lemma 3.3 can be used to prove for $\mathrm{A}(\Omega)$ the following property which we can define for general PLS-spaces.

A PLS-space $X = \mathrm{proj}_{N\in\mathbb{N}} X_N$, X_N LS-spaces with fundamental sequences of bounded sets $(B_{N,n})_{n\in\mathbb{N}}$, satisfies the *condition* $(\mathrm{P}\overline{\overline{\Omega}})$ if
$$\forall N\ \exists K\ \forall L, k, \gamma > 0\ \exists n, l, C : \quad B_{K,k} \subseteq C\left(\frac{1}{r^\gamma}B_{N,n} + rB_{L,l}\right)$$
for every $r > 0$. Clearly, the condition does not depend on the choice of the representing spectrum $(X_N)_{N\in\mathbb{N}}$ and sequences $(B_{N,n})_{n\in\mathbb{N}}$. For simplicity we have omitted the linking maps in the definition of $(\mathrm{P}\overline{\overline{\Omega}})$: to be precise the

inclusion is valid for images under linking maps in X_M, $M=\min(N,K,L)$. Since any complete quotient of a PLS-space is a PLS-space [7, Th. 1.3], property $(\mathrm{P}\overline{\overline{\Omega}})$ is inherited by such quotients. For Fréchet spaces it coincides with $(\overline{\overline{\Omega}})$.

PROPOSITION 5.1. *If a PLS-space $X=\operatorname{proj} X_N$ satisfies $(\mathrm{P}\overline{\overline{\Omega}})$, then for any sequence (V_N) of 0-neighbourhoods in X_N, we have*

(4) $\forall N\ \exists K,$ *0-neighbourhood* $V\subseteq X_K\ \forall L,\gamma>0\ \exists C$
$$V\subseteq C\left(\frac{1}{r^\gamma}V_N+rV_L\right).$$

In particular, every metrizable quotient of X satisfies $(\overline{\overline{\Omega}})$.

Proof. It is easily seen that for a fixed $\gamma>0$ we have, for some constant C,
$$V_{\gamma,L}:=\bigcap_{r>0}\frac{1}{r^\gamma}V_N+rV_L\supseteq CB_{K,k}.$$
Clearly, since the above inclusion holds for all k, $V_{\gamma,L}$ is a 0-neighbourhood in X_K. Moreover, since X_K is an LB-space, there is a 0-neighbourhood V in X_K which is absorbed by all $V_{m,L}$, $m,L\in\mathbb{N}$. This is the 0-neighbourhood we are looking for.

As is easily seen, condition (4) is inherited by quotients. If the quotient is metrizable then condition (4) becomes $(\overline{\overline{\Omega}})$ if we take as (V_N) a 0-neighbourhood base of the quotient space. ■

The proof of Theorem 3.4 now shows:

THEOREM 5.2. *The space $\mathrm{A}(\Omega)$ satisfies $(\mathrm{P}\overline{\overline{\Omega}})$.*

By Proposition 5.1, we find again that every Fréchet quotient of $\mathrm{A}(\Omega)$ has $(\overline{\overline{\Omega}})$. In fact the only Fréchet quotients of $\mathrm{A}(\Omega)$ the authors know are isomorphic to ω, i.e., the space of all sequences.

Added in proof (September 2000). The paper was written consistently for connected open sets in $\mathbb{R}^d$ but, in fact, $\mathrm{A}(\Omega)$ has no basis for any open set $\Omega\subseteq\mathbb{R}^d$ as the following argument shows.

The same proof as in the paper gives all the results for Ω having finitely many connected components. If $\Omega=\bigcup_{j\in\mathbb{N}}\Omega_j$, Ω_j connected components, and if X is a complemented subspace of $\mathrm{A}(\Omega)=\prod_{j\in\mathbb{N}}\mathrm{A}(\Omega_j)$ with basis, then the proof of Theorem 2.2 implies that $X\simeq\prod_{N\in\mathbb{N}}X_N$, where the X_N have bases and continuous norms. Since the $\mathrm{A}(\Omega_j)$ have continuous norms, it follows easily that each X_N is a complemented subspace of a finite product of $\mathrm{A}(\Omega_j)$ and, via the above mentioned version of Theorem 4.1, X_N is an LB-space. We have proved that for an arbitrary open set Ω, every complemented subspace of $\mathrm{A}(\Omega)$ with basis is a product of LB-spaces, in particular, $\mathrm{A}(\Omega)$ has no basis.

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