STUDIA MATHEMATICA 142 (1) (2000)

A constructive proof of the composition rule for Taylor's functional calculus

by

MATS ANDERSSON and SEBASTIAN SANDBERG (Göteborg)

Abstract. We give a new constructive proof of the composition rule for Taylor's functional calculus for commuting operators on a Banach space.

1. Introduction. Let X be a Banach space, let $\mathcal{L}(X)$ denote the space of bounded operators on X, and suppose that $a_1, \dots, a_n \in \mathcal{L}(X)$ are commuting. If $p(z) = p(z_1, \dots, z_n)$ is a polynomial then $p(a) = p(a_1, \dots, a_n)$ has a well defined meaning. Since the polynomials are dense in $\mathcal{O}(\mathbb{C}^n)$ there is a continuous algebra homomorphism

$$\mathcal{O}(\mathbb{C}^n) \to (a) \subset \mathcal{L}(X), \tag{1.1}$$

where (a) denotes the closed subalgebra of $\mathcal{L}(X)$ that is generated by $a_1, \dots$ $\dots, a_n$. The proper notion of joint spectrum $\sigma(a)$ of the operators $a_1, \dots, a_n$ was introduced by Taylor [9]; it is a compact subset of $\mathbb{C}^n$. Let $(a)''$ denote the subalgebra of $\mathcal{L}(X)$ consisting of all operators that commute with all operators that commute with each a_k. It is easy to see that $(a)''$ is commutative. Taylor proved the following fundamental result in [10].

THEOREM 1.1 (Taylor). *Let $a_1, \dots, a_n$ be commuting operators on a Banach space with joint spectrum $\sigma(a)$. There is a continuous algebra homomorphism $g \mapsto g(a)$ from $\mathcal{O}(\sigma(a))$ into $(a)''$ that extends* (1.1). *Moreover, if $g = (g_1, \dots, g_m)$ is a holomorphic mapping, $g_j \in \mathcal{O}(\sigma(a))$, and $g(a) = (g_1(a), \dots, g_m(a))$, then*

$$\sigma(g(a)) = g(\sigma(a)). \tag{1.2}$$

Taylor's original proof of this theorem was based on representation of holomorphic functions by means of Cauchy–Weyl formulas. Later on, in [11]

2000 *Mathematics Subject Classification*: 47A60, 47A13, 32A25.

Key words and phrases: Taylor spectrum, functional calculus, Cauchy–Fantappiè–Leray formula.

The first author was partially supported by the Swedish Natural Science Researh Council.

and [12], he made the construction with homological methods. Suppose that $g \in \mathcal{O}(\sigma(a))$ is a holomorphic mapping and that $f \in \mathcal{O}(g(\sigma(a)))$. In view of (1.2), both $f \circ g(a)$ and $f(g(a))$ has meaning and it is natural to ask if they coincide. Putinar proved in [7] by homological methods that this question has an affirmative answer.

THEOREM 1.2 (Putinar). *Suppose that $g = (g_1, \ldots, g_m)$ is a mapping, $g_j \in \mathcal{O}(\sigma(a))$, $g(a) = (g_1(a), \ldots, g_m(a))$ and that $f \in \mathcal{O}(g(\sigma(a)))$. Then*

$$f(g(a)) = f \circ g(a). \tag{1.3}$$

A simplified proof appeared in [6].

If a is one single operator and $f \in \mathcal{O}(\sigma(a))$, then $f(a)$ is given by the formula

$$f(a) = \int_{\partial D} f(z)\omega_{z-a}, \tag{1.4}$$

where ω_{z-a} is the resolvent

$$\omega_{z-a} = \frac{1}{2\pi i}(z-a)^{-1}dz.$$

In the case with several commuting operators, the resolvent ω_{z-a} is an $(a)''$-valued cohomology class in $\mathbb{C}^n \setminus \sigma(a)$. In [1] we gave a new constructive proof of Taylor's theorem. From the very definition of the spectrum $\sigma(a)$, we defined, for each $x \in X$, a closed X-valued $(n, n-1)$-form in $\mathbb{C}^n \setminus \sigma(a)$ that represents the class $\omega_{z-a}x$. Then $f(a)$ can be defined by (1.4). The form $\omega_{z-a}x$ is sort of an abstract Cauchy–Fantappiè–Leray kernel. In special situations, for instance outside any Stein compact set that contains $\sigma(a)$, the form $\omega_{z-a}x$ can be realized as a classical Cauchy–Fantappiè–Leray kernel. (Contrary to the convention in [1] we include the factor $(2\pi i)^{-n}$ in the definition of resolvent class here.) This constructive approach is natural if one wants to extend the functional calculus to larger classes of functions. In one variable this was done by Dynkin in [5]; in several variables partial results have been obtained by e.g. Droste; see also the forthcoming papers [3] and [8].

The purpose of this note is to give a proof of Theorem 1.2 along the lines of [1] and [2]. It can be viewed as a continuation of these papers and we keep the same notation.

2. Some auxiliary results. It is sometimes convenient to replace the boundary integral in (1.4) by a smoothed out integral. If $f \in \mathcal{O}(V)$, V a neighborhood of $\sigma(a)$, and if ϕ is a cutoff function that is identically 1 in a neighborhood of $\sigma(a)$ and has support in V, then

$$f(a) = -\int f(z)\overline{\partial}\phi(z) \wedge \omega_{z-a}. \tag{2.1}$$

This immediately follows from (1.4) and Stokes' theorem.

LEMMA 2.1. *Suppose that $T : \mathbb{C}^n \to \mathbb{C}^m$ is linear and $\phi \in \mathcal{O}(\sigma(Ta))$. Then $\phi(Ta) = T^*\phi(a)$.*

The lemma is the special case of Theorem 1.2 when g is linear; for a proof see, e.g., [1], Theorem 3.1.

Let us now consider commuting operators $a_1, \ldots, a_n, b_1, \ldots, b_m$. It follows from Lemma 2.1 that $\sigma(a, b) \subset \sigma(a) \times \sigma(b)$.

We will now recall from [1] and [2] how the resolvent class $\omega_{z-a,w-b}$ in $\mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(a, b)$ for (a, b) can be represented in terms of ω_{z-a} and ω_{w-b}. Let $\widetilde{\omega}_{z-a}x$ be an explicit form in $\mathbb{C}^n \setminus \sigma(a)$ that represents the class $\omega_{z-a}x$. There is a smooth $\overline{\partial}$-closed form $\widetilde{\omega}_{w-b} \wedge \widetilde{\omega}_{z-a}x$ in $\mathbb{C}^n \setminus \sigma(a) \times \mathbb{C}^m \setminus \sigma(b)$ which, for each fixed $z \in \mathbb{C}^n \setminus \sigma(a)$, represents the class $\omega_{w-b} \wedge \widetilde{\omega}_{z-a}x$. Let $\chi(z, w)$ be a function in $\mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(a) \times \sigma(b)$ such that $\{\chi, 1-\chi\}$ is a partition of unity subordinate to the open cover

$$\{\mathbb{C}^n \setminus \sigma(a) \times \mathbb{C}^m, \mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(b)\},$$

of $\mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(a) \times \sigma(b)$. In the set $\mathbb{C}^n \times \mathbb{C}^m \setminus \sigma(a) \times \sigma(b)$, the class $\omega_{z-a,w-b}x$ is then represented by the form

$$\overline{\partial}\chi \wedge \widetilde{\omega}_{w-b} \wedge \widetilde{\omega}_{z-a}x$$

(cf. formula (3.6) in [1]).

Let $\phi(z)$ and $\psi(w)$ be cutoff functions that are identically 1 in neighborhoods of $\sigma(a)$ and $\sigma(b)$ respectively. Moreover, let $G(z, w)$ be holomorphic in a neighborhood of $\sigma(a) \times \sigma(b)$. Then (2.1) applied to the pair (a, b) gives

$$G(a, b)x = -\int_z\int_w G(z, w)\overline{\partial}(\phi \otimes \psi) \wedge \overline{\partial}\chi \wedge \widetilde{\omega}_{w-b} \wedge \widetilde{\omega}_{z-a}x.$$

Integration by parts in this formula yields (cf. formula (3.7) in [1])

$$G(a, b)x = \int_z\int_w G(z, w)\overline{\partial}\psi(w) \wedge \overline{\partial}\phi(z) \wedge \widetilde{\omega}_{w-b} \wedge \widetilde{\omega}_{z-a}x. \tag{2.2}$$

In particular, if $g_1 \in \mathcal{O}(\sigma(a))$, $g_2 \in \mathcal{O}(\sigma(b))$, and $G = g_1 \otimes g_2$, it follows by Fubini's theorem that

$$g_1 \otimes g_2(a, b) = g_1(a)g_2(b). \tag{2.3}$$

3. Proof of Theorem 1.2. Let $a_1, \ldots, a_n$ be a commuting n-tuple of operators, let $g = (g_1, \ldots, g_m)$ be a holomorphic mapping, $g_j \in \mathcal{O}(\sigma(a))$, and let $b = g(a)$.

LEMMA 3.1. *If ϕ is holomorphic at the origin of $\mathbb{C}^m$ and $\Phi(z, w) = \phi(w - g(z))$, then $\Phi \in \mathcal{O}(\sigma(a, b))$ and $\Phi(a, b) = \phi(0)$.*

Proof. It follows from the spectral mapping statement in Theorem 1.1 that $\sigma(a,b) = \{(z,w) : z \in \sigma(a),\ w = g(z)\}$. Therefore $\Phi(z,w)$ is holomorphic in a neighborhood of $\sigma(a,b)$.

There are holomorphic functions $\phi_1, \ldots, \phi_m$ at the origin so that $\phi(\xi) = \phi(0) + \sum \xi_j \phi_j(\xi)$. Therefore,

$$\Phi(z,w) = \phi(0) + \sum_{j=1}^{m} H_j(z,w)\Phi_k(z,w),$$

where $\Phi_j(z,w) = \phi_j(w - g(z))$ and $H_j(z,w) = w_j - g_j(z)$. Now $H_j(a,b) = b_j - g_j(a) = 0$, where the first equality follows from linearity and (2.3), and the second equality follows from our assumption. Since the functional calculus is multiplicative it follows that $\Phi(a,b) = \phi(0)$. ∎

We can now conclude the proof of Theorem 1.2. Assume that $f(w)$ is holomorphic in a neighborhood of $\sigma(b)$. Then $h(z,w,\xi) = f(\xi - (w - g(z)))$ is holomorphic in a neighborhood of $\sigma(a,b) \times \sigma(b) \subset \mathbb{C}^{2m} \times \mathbb{C}^m$, and in view of (2.2) we can therefore write

$$h(a,b,b)x = \int_{\xi}\int_{z,w} f(\xi - (w - g(z)))\overline{\partial}\psi(z,w) \wedge \overline{\partial}\phi(\xi) \wedge \widetilde{\omega}_{z-a,w-b} \wedge \widetilde{\omega}_{\xi-b}x$$

if $\psi(z,w)$ is 1 in a small neighborhood of $\sigma(a,b)$ and $\phi(\xi)$ is 1 in a small neighborhood of $\sigma(b)$. For each fixed ξ we can evaluate the inner integral by Lemma 3.1 to get

$$h(a,b,b)x = -\int_{\xi} f(\xi)\overline{\partial}\phi(\xi) \wedge \omega_{\xi-b}x = f(b)x.$$

Thus $h(a,b,b) = f(b) = f(g(a))$. On the other hand, by the linear mapping $T : (z,\eta) \mapsto (z,w,\xi) = (z,\eta,\eta)$ and Lemma 2.1, we have

$$h(a,b,b) = h(T(a,b)) = T^* h(a,b).$$

Now, $T^* h(z,\eta) = f \circ g(z) \otimes 1$, and hence $T^* h(a,b) = f \circ g(a)$ according to (2.3). Summing up, we get the desired equality $f(g(a)) = f \circ g(a)$. ∎

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Department of Mathematics
Chalmers University of Technology
and
University of Göteborg
S-412 96 Göteborg, Sweden
E-mail: matsa@math.chalmers.se
sebsand@math.chalmers.se

Received November 2, 1999 (4423)