STUDIA MATHEMATICA 141 (3) (2000)

# Characterization of compact subsets of algebraic varieties in terms of Bernstein type inequalities

by

M. BARAN and W. PLEŚNIAK (Kraków)

**Abstract.** We show that in the class of compact sets $K$ in $\mathbb{R}^n$ with an analytic parametrization of order $m$, the sets with Zariski dimension $m$ are exactly those which admit a Bernstein (or a van der Corput–Schaake) type inequality for tangential derivatives of (the traces of) polynomials on $K$.

**0. Introduction.** Markov and Bernstein type inequalities furnish estimates of derivatives of polynomials in terms of their degrees and uniform norms on compact subsets. The multivariate theory of such inequalities is relatively new. It was essentially developed in the eighties (for a detailed survey on this subject see [Pl2]). In particular, in papers [PaPl1], [PaPl2] and [Pl1] Pawłucki and Pleśniak showed that Markov type inequalities are closely related to Hironaka and Łojasiewicz's subanalytic geometry. In [PaPl2] and [Pl1], they also constructed a continuous linear operator extending $\mathcal{C}^\infty$ Whitney jets on Markov compact subsets of $\mathbb{R}^n$ to $\mathcal{C}^\infty$ functions in the whole space $\mathbb{R}^n$. This extended in a relatively simple way earlier results on this topic obtained among others by Mityagin, Seeley, Stein, Tidten and Bierstone. Further important applications of Markov type inequalities to modern multivariate analysis were found by Bos and Milman (see [BoMi1], [BoMi2]) and Baran (see [Ba3], [Ba4], [Ba5]).

In a few recent papers Markov and Bernstein type inequalities have been investigated on algebraic subvarieties of $\mathbb{R}^n$ (see [BLT], [BLMT], [FeNa1], [FeNa2], [FeNa3], [Bru], [BaPl1], [BaPl2], [RoYo]). In particular, in [BaPl1] the authors have characterized semialgebraic curves in $\mathbb{R}^n$ in terms of Bern-

2000 *Mathematics Subject Classification*: 41A17, 41A25, 14P10, 14P05.

*Key words and phrases*: Bernstein and van der Corput–Schaake type inequalities, traces of polynomials on semialgebraic sets, Siciak extremal function, pluricomplex Green function, Zariski dimension.

Research partially supported by grant No 2 P03A 045 14 from KBN (Committee for Scientific Research). The first-named author has also been supported by the European Programme PECO of the French Government.

stein and van der Corput–Schaake type inequalities. The purpose of the present paper is to extend results of [BaPl1] to the case of semialgebraic sets in $\mathbb{R}^n$ of higher dimensions. The main result (Theorem 4.5) shows that in the class of subsets of $\mathbb{R}^n$ with an analytic parametrization of order $m$, compact sets of Zariski dimension $m$ are exactly those which admit Bernstein (or van der Corput–Schaake) type inequalities.

A related characterization of algebraic submanifolds of $\mathbb{R}^n$ in terms of tangential Markov inequalities (with exponent 1) was also proved in [BLMT]. However, our approach essentially differs from that presented in [BLMT] and is mainly based on the concept of an analytic parametrization combined with formulas for Siciak's extremal function of the unit ball in the space $\mathbb{R}^n$ that were established by the first-named author (see [Ba1], [Ba2]). Since the class of sets admitting an analytic parametrization we consider here contains all compact $\mathbb{R}$-analytic manifolds, Theorem 4.5 yields in particular the main result of [BLMT].

**1. Siciak's extremal function for the unit ball in $\mathbb{R}^n$.** In what follows, an important role is played by the Joukowski function

$$g(w)=\tfrac{1}{2}(w+1/w),\qquad w\in\mathbb{C}\setminus\{0\},$$

which establishes a biholomorphism between $\{w\in\mathbb{C}:|w|>1\}$ and $\mathbb{C}\setminus[-1,1]$. Its inverse function $h=g^{-1}:\mathbb{C}\setminus[-1,1]\to\mathbb{C}\setminus\{|w|\le 1\}$ has the form

$$h(z)=z+(z^2-1)^{1/2}$$

if we choose the branch of the square root such that $h(R)>1$ for $R>1$. We shall need the following

LEMMA 1.1. *If* $\alpha\in(-1,1)$, $0<\varepsilon\le 1/2$, *and* $|\beta|\le(1-|\alpha|)/\varepsilon$ *with* $\beta\in\mathbb{R}$, *then*

$$(1-\varepsilon)|\beta|(1-\alpha^2)^{-1/2}\le\frac{1}{\varepsilon}\log|h(\alpha+i\varepsilon\beta)|\le|\beta|(1-\alpha^2)^{-1/2}.$$

Proof. Since $|h(\zeta)|=h\big(\frac12|\zeta-1|+\frac12|\zeta+1|\big)$, applying the de l'Hôpital Rule we check that

$$\lim_{\varepsilon\to 0_+}\frac{1}{\varepsilon}\log|h(\alpha+i\varepsilon\beta)|=|\beta|(1-\alpha^2)^{-1/2}=:\Lambda(\alpha,\beta).$$

On the other hand, since $\log|h|$ is a continuous, subharmonic function in $\mathbb{C}$ that is holomorphic in $\mathbb{C}\setminus[-1,1]$, by the Poisson Formula applied to $\log|h|$ restricted to $\mathbb{R}$, we get, for any $\zeta\in\mathbb{C}$,

$$\log|h(\zeta)|=\frac{|\Im\zeta|}{\pi}\int\limits_{|t|\ge 1}|\zeta-t|^{-2}\log h(|t|)\,dt.$$



Hence we easily obtain

$$\Lambda(\alpha,\beta)=\frac{|\beta|}{\pi}\int\limits_{|t|\ge 1}|t-\alpha|^{-2}\log h(|t|)\,dt.$$

Now, since

$$(1-\varepsilon)|\beta|(t-\alpha)^{-2}\le|\beta|(\varepsilon^2\beta^2+(t-\alpha)^2)^{-1}\le|\beta|(t-\alpha)^{-2}$$

for $|t|\ge 1$, $\alpha\in(-1,1)$, $0<\varepsilon\le 1/2$, and $|\beta|\le(1-|\alpha|)/\varepsilon$, where $\beta\in\mathbb{R}$, the lemma follows.

Let now $E$ be a compact set in the space $\mathbb{C}^n$. For $z\in\mathbb{C}^n$, we set

$$\Phi_E(z)=\sup\{|p(z)|^{1/\deg p}:p\in\mathbb{C}[z],\ \deg p\ge 1,\ \sup|p|(E)\le 1\}.$$

The function $\Phi_E$, called the (*polynomial*) *extremal function* associated with $E$, was introduced by Siciak [Si1]. It has appeared very useful in multivariate complex analysis, in particular in polynomial approximation of analytic functions of several variables. It follows from the definition of $\Phi_E$ that

$$\Phi_E(z)\ge 1\quad\text{in }\mathbb{C}^n,\qquad \Phi_E(z)=1\quad\text{iff}\quad z\in\widehat{E},\tag{1.1}$$

where $\widehat{E}$ is the polynomial hull of $E$. It is a result of Baran [Ba2, Theorem 1.3] that if $E$ is a compact subset of the space $\mathbb{R}^n$ (treated in the whole paper as the *real part* of $\mathbb{C}^n$ so that $\mathbb{R}^n=\mathbb{R}^n+i0\subset\mathbb{C}^n$) then

$$\Phi_E(z)=\sup\{|h(p(z))|^{1/\deg p}:p\in\mathbb{R}[z],\ \deg p\ge 1,\ \sup|p|(E)\le 1\}.\tag{2.2}$$

Let now $X$ be the space $\mathbb{R}^m$ endowed with the Euclidean norm $\|\cdot\|$. Let $\mathbb{B}^m(R):=\{x\in\mathbb{R}^m:\|x\|\le R\}$, $\mathbb{B}^m:=\mathbb{B}^m(1)$, $\mathbb{S}^{m-1}(R):=\partial\mathbb{B}^m(R)$ and $\mathbb{S}^{m-1}:=\mathbb{S}^{m-1}(1)$. Let $X_\vee=(\mathbb{C}^m,\|\cdot\|_\vee)$, where

$$\|x+iy\|_\vee=\sup_{w\in\mathbb{S}^{m-1}}|x\cdot w+iy\cdot w|=\sup_{\theta\in\mathbb{R}}\|(\cos\theta)x-(\sin\theta)y\|,$$

be the injective complexification of $(X,\|\cdot\|)$. By Lundin [Lu], Bedford and Taylor [BeT], and Baran [Ba1], [Ba2], we have

$$\Phi_{\mathbb{B}^m}(z)=\sup_{w\in\mathbb{S}^{m-1}}|h(z\cdot w)|,\qquad z\in\mathbb{C}^m,$$

and by Baran [Ba5],

$$\Phi_{\mathbb{B}^m}(z)=h(\|(z,i)\|_\vee)\le h(\max\{1,\|x\|\}+\|y\|),\tag{1.3}$$

where $\|(z,i)\|_\vee$ is the norm $\|\cdot\|_\vee$ of $(z,i)$ in $\mathbb{C}^{m+1}$ considered as the injective complexification of the Euclidean space $\mathbb{R}^{m+1}$. Hence by (1.2) and Lemma 1.1 we get the following

PROPOSITION 1.2. *If* $\|x\|<1$ *and* $y\in\mathbb{R}^m$, *then*

$$\log\Phi_{\mathbb{B}^m}(x+iy)\le\sup_{w\in\mathbb{S}^{m-1}}|y\cdot w|(1-(x\cdot w)^2)^{-1/2}\le\|y\|(1-\|x\|^2)^{-1/2}.$$

*In particular, if* $\|z\|_\vee\le 1$ *then* $\log\Phi_{\mathbb{B}^m}(z)\le 1$.

Here we have made use of homogeneity of the function

$$\Gamma(\beta) := \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\log|h(\alpha+i\varepsilon\beta)|,$$

which permits us to replace $\varepsilon$ by $t\varepsilon$ with $t>0$ so chosen that the assumption $|\beta|\le(1-|\alpha|)/\varepsilon$ of Lemma 1.1 be satisfied.

In what follows, we shall denote by $\mathbb{D}$ the unit disk $\{z\in\mathbb{C}:|z|<1\}$ and by $\mathbb{T}$ its boundary $\{z\in\mathbb{C}:|z|=1\}$. Let us recall the definition of the generalized Joukowski function:

$$\chi:(\mathbb{C}\setminus\mathbb{D})\times\mathbb{C}^m\ni(\zeta,c)\mapsto\chi(\zeta,c)=\tfrac12(\zeta c+\zeta^{-1}\overline{c})\in\mathbb{C}^m$$

that was introduced in [Ba1]. For a fixed $c\in\mathbb{C}^m$, let

$$\chi_c(\zeta)=\chi(\zeta,c).$$

Note that if $a,b\in\mathbb{R}$ then $\chi(\zeta,a+ib)=g(\zeta)a+i\widehat{g}(\zeta)b$, where, as previously, $g(\zeta)=\frac12(\zeta+\zeta^{-1})$, and $\widehat{g}(\zeta)=\frac12(\zeta-\zeta^{-1})$. We have the following formula for Siciak's extremal function of the unit ball in $\mathbb{R}^m$ (see [Ba1]):

$$\Phi_{\mathbb{B}^m}(\chi_c(\zeta))=|\zeta|\quad\text{ if } |\zeta|\ge1 \text{ and } c\in\mathbb{S}_{\vee}^{m-1}, \tag{1.4}$$

where $\mathbb{S}_{\vee}^{m-1}$ is the unit sphere in $X_{\vee}$. Set

$$\mathcal{E}_R=\{z\in\mathbb{C}^m:\Phi_{\mathbb{B}^m}(z)\le R\},\qquad \mathcal{E}^{(\delta)}=\{z\in\mathbb{C}^m:\operatorname{dist}(z,\mathbb{B}^m)\le\delta\}.$$

Then by (1.3) we have

$$\mathcal{E}_R\supset\mathcal{E}^{((g(R)-1)/2)}. \tag{1.5}$$

Lemma 1.3. *$\chi_c(\mathbb{T})\subset\mathbb{B}^m$ if and only if $c\in\mathbb{B}_{\vee}^m$, the unit ball in $X_{\vee}$.*

Proof. We have

$$\|x+iy\|_{\vee}=\sup_{\theta}\|(\cos\theta)x-(\sin\theta)y\|.$$

Hence, if we put $c=x+iy$ then

$$\sup_{\theta}\|\chi_c(e^{i\theta})\|=\sup_{\theta}\|(\cos\theta)x-(\sin\theta)y\|\le1\quad\text{iff}\quad c\in\mathbb{B}_{\vee}^m.$$

Corollary 1.4. *If $x\in\mathbb{B}^m(\sqrt{1-\lambda^2})$, $v\in\mathbb{S}^{m-1}(\lambda)$ and $c=x+iv$, then $\chi_c(\mathbb{T})\subset\mathbb{B}^m$. Moreover, if $x\in\mathbb{S}^{m-1}$ and $v\in x^{\perp}\cap\mathbb{B}^m$, then we also have $\chi_c(\mathbb{T})\subset\mathbb{B}^m$.*

**2. Pluricomplex Green function on analytic sets.** Let now $E$ be a compact set in the space $\mathbb{C}^n$. We set

$$V_E(z)=\sup\{u(z):u\in\mathcal{L}(\mathbb{C}^n),\ u|_E\le0\},\qquad z\in\mathbb{C}^n,$$

where $\mathcal{L}(\mathbb{C}^n)=\{u\in\mathrm{PSH}(\mathbb{C}^n):\sup_{z\in\mathbb{C}^n}[u(z)-\log(1+\|z\|)]<\infty\}$ is the *Lelong class* of plurisubharmonic functions of minimal growth. The function $V_E$ is called the (*plurisubharmonic*) *extremal function* associated with $E$. If



the set $E$ is non-pluripolar, the upper semicontinuous regularization $V_E^*(z)=\limsup_{w\to z}V_E(w)$ of $V_E$ is known to be a multidimensional counterpart of the classical *Green function* for $\mathbb{C}\setminus\widehat{E}$, where $\widehat{E}$ is the polynomially convex hull of $E$, since it is a solution of the homogeneous complex *Monge–Ampère equation* $(dd^cV_E^*)^n=0$ in $\mathbb{C}^n\setminus\widehat{E}$, which reduces in the one-dimensional case to the *Laplace equation* (for details see [K]). We also note (see [Si2]) that

$$\begin{aligned}V_E(z)&\\ &=\sup\Big\{\frac{1}{\deg p}\log|p(z)|:p\in\mathbb{C}[z_1,\dots,z_n],\ \|p\|_E\le1,\ \deg p\ge1\Big\}\\ &=\log\Phi_E(z),\qquad z\in\mathbb{C}^N,\end{aligned}\tag{2.1}$$

where $\Phi_E$ is Siciak's extremal function and $\|p\|_E:=\sup|p|(E)$.

We recall that a subset $A$ of $\mathbb{C}^n$ is said to be *pluripolar* if there exists a plurisubharmonic function $u$ on $\mathbb{C}^n$ such that $A\subset\{u=-\infty\}$. By Josefson's theorem [Jos], $A$ is pluripolar if and only if it is *locally pluripolar*, i.e. if for each point $a\in A$ there exist an open neighbourhood $U$ of $a$ and a plurisubharmonic function $u$ on $U$ such that $A\cap U\subset\{u=-\infty\}$.

Let now $\mathbb{M}$ be a (locally) analytic set in $\mathbb{C}^n$ such that the set $\mathbb{M}_{\mathrm{reg}}$ of regular points of $\mathbb{M}$ is a complex submanifold of $\mathbb{C}^n$ of pure dimension $k$, where $k\le n$. A function $u$ defined on $\mathbb{M}$ is said to be plurisubharmonic on $\mathbb{M}$ if it is plurisubharmonic on $\mathbb{M}_{\mathrm{reg}}$ and locally bounded above on $\mathbb{M}$. Let $N$ be a subset of $\mathbb{M}$. Then $N$ is said to be *pluripolar* in $\mathbb{M}$ if there exists a plurisubharmonic function $u$ on $\mathbb{M}$ such that $N\cap\mathbb{M}_{\mathrm{reg}}\subset\{u=-\infty\}$.

We have the following (see [BaPl2, Lemma 0.1])

Lemma 2.1. *Let $E$ be a non-pluripolar compact subset of $\mathbb{C}^k$ and let $f$ be an analytic map defined in an open neighbourhood of $E$, with values in a locally analytic subset $\mathbb{M}$ of $\mathbb{C}^n$ of dimension $\min(k,n)$, where we set $\mathbb{M}=\mathbb{C}^n$ if $k>n$. If $\operatorname{rank}_V f:=\sup_{z\in V}\operatorname{rank}_z f=\min(k,n)$ for a connected component $V$ of $U$ such that $V\cap E$ is non-pluripolar, then $f(E)$ is a non-pluripolar subset of $\mathbb{M}$.*

In Section 3, we shall deal with real algebraic subsets of $\mathbb{R}^n$. (Let us recall that in the whole paper $\mathbb{R}^n$ is treated as a subset of $\mathbb{C}^n$ such that $\mathbb{C}^n=\mathbb{R}^n+i\mathbb{R}^n$.) Therefore we shall also need a "real" version of Lemma 2.1, which can be easily obtained by the standard complexification method (cf. [BaPl2, Corollary 0.2]).

Corollary 2.2. *Let $E$ be a non-pluripolar compact subset of $\mathbb{R}^k$ and let $f$ be a real-analytic map defined in an open neighbourhood of $E$, with values in a real algebraic subset $\mathbb{M}$ of $\mathbb{R}^n$ of dimension $\min(k,n)$, where we set $\mathbb{M}=\mathbb{R}^n$ if $k\ge n$. If $\operatorname{rank}_E f=k$ then $f(E)$ is a non-pluripolar subset of (the complexification $\widetilde{\mathbb{M}}$ of) $\mathbb{M}$.*

We shall also need the following beautiful characterization of algebraic sets in $\mathbb{C}^N$ due to Sadullaev [Sa].

SADULLAEV'S CRITERION 2.3. *An analytic subset $A$ of $\mathbb{C}^n$ is algebraic if and only if the function $\Phi_E$ (or else the function $V_E$) is locally bounded in $A$ for some (and hence for each) non-pluripolar compact subset $E$ of $A$.*

**3. Van der Corput–Schaake type inequalities.** Let $\mathbb{M} \subset \mathbb{R}^n$ be an algebraic set with $\dim \mathbb{M} = m$, $1 \le m \le n$, such that $\mathbb{M} = \{x \in \mathbb{R}^n : P(x) = 0\}$ for a polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$. (For background of the theory of real algebraic sets, see e.g. [BeRi].) Let $\widetilde{\mathbb{M}}$ be the complexification of $\mathbb{M}$. Let $\phi$ be an analytic map defined in an open connected neighbourhood $U$ of $\mathbb{B}^m$ in $\mathbb{R}^m$, with values in $\mathbb{M}$, such that $\operatorname{rank}_U \phi = m$. Then by Corollary 2.2 the set $N := \phi(\mathbb{B}^m)$ is a non-pluripolar compact subset of $\widetilde{M}$. By the Bernstein–Walsh–Siciak theorem (see [Si1]) there exists $R_0 > 1$ such that $\phi$ extends to an analytic map $\widetilde{\phi}$ in an open neighbourhood of the set $\mathcal{E}_{R_0} = \{z \in \mathbb{C}^m : V_{\mathbb{B}^m} \le \log R_0\}$, with values in $\mathbb{C}^n$. Then, since $\widetilde{\phi}(\mathcal{E}_{R_0}) \subset \widetilde{\mathbb{M}}$, by Sadullaev's Criterion 2.3 we get

$$C_0 := \sup V_N(\widetilde{\phi}(\mathcal{E}_{R_0})) < \infty. \tag{3.1}$$

Define

$$R_0^* := \sup\{R > 1 : \chi(\{1 \le |\zeta| \le R\} \times \mathbb{B}_\vee^m) \subset \mathcal{E}_{R_0}\},$$

where $\chi$ is the generalized Joukowski function, and set

$$C_1 := C_0/\log R_0^*$$

with the constant $C_0$ defined by (3.1). We have

LEMMA 3.1. *If $1 \le |\zeta| \le R_0^*$ and $c \in \mathbb{B}_\vee^m$ then*

$$V_N(\widetilde{\phi} \circ \chi_c(\zeta)) \le C_1 \log |\zeta|.$$

Proof. By the definition of $R_0^*$, the function $\widetilde{\phi} \circ \chi_c$ is analytic in an open neighbourhood of the ring $\{1 \le |\zeta| \le R_0^*\}$. Let now $u \in \mathcal{L}(\mathbb{C}^n)$, $u \le 0$ on $N$. Then the function

$$v(\zeta) := u \circ \widetilde{\phi} \circ \chi_c(\zeta) - M \log |\zeta|$$

is subharmonic in an open neighbourhood of $\{1 \le |\zeta| \le R_0^*\}$. By Lemma 1.3, $v(\zeta) \le 0$ for $|\zeta| = 1$. Moreover, by the definition of the function $V_E$, $v(\zeta) \le V_N(\widetilde{\phi} \circ \chi_c(\zeta)) \le C_1 \log R_0^*$ if $|\zeta| = R_0^*$. Hence by the maximum principle we get $v(\zeta) \le 0$ for $1 \le |\zeta| \le R_0^*$, whence we derive the assertion of the lemma.

Lemma 3.1 together with (1.4) and (2.1) yields the following important

COROLLARY 3.2. $V_N(\widetilde{\phi}(z)) \le C_1 V_{\mathbb{B}^m}(z)$ *for* $z \in \mathcal{E}_{R_0^*}$.



We are now ready to prove a version of the classical van der Corput–Schaake inequality [CS1], [CS2] that is a precise form of the Bernstein inequality [Bern], on "good" pieces of algebraic sets. For $v \in \mathbb{R}^m$ and $t \in \mathbb{B}^m$, we set

$$\mathcal{T}(t, v) := D_v \phi(t).$$

If $\mathbb{M}$ is smooth at $\phi(t)$ then $\mathcal{T}(t, v)$ is a vector of the tangent space $T_{\phi(t)}\mathbb{M}$. For $t \in \mathbb{B}^m$, we define

$$\mathbb{V}_t = \{y \in \mathbb{B}^m : t + iy \in \mathbb{B}_\vee^m\}.$$

PROPOSITION 3.3. *If $t \in \mathbb{B}^m$ and $v \in \mathbb{V}_t$ then for any polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ we have*

$$|D_{\mathcal{T}(t,v)} P(x)| \le C_1 (\deg P)(\|P\|_N^2 - P^2(x))^{1/2},$$

*where $x = \phi(t)$ and $\|P\|_N := \sup |P|(N)$.*

Proof. Let $\varepsilon > 0$ be so small that $R(\varepsilon) := \varepsilon + \sqrt{1+\varepsilon^2} \le R_0^*$. Set $c = t + iv$. By Lemma 3.1 we have

$$V_N(\phi \circ \chi_c(R(\varepsilon))) = V_N(\phi(\sqrt{1+\varepsilon^2}\, t + i\varepsilon v)) \le C_1 \log R(\varepsilon).$$

Hence if $P \in \mathbb{R}[x_1, \ldots, x_n]$, $\deg P \ge 1$, $\|P\|_N \le 1$ and $0 < \varepsilon < 1$ then by (1.2) and (2.1) we get

$$\frac{1}{\deg P} \log |h(\alpha P \circ \phi \circ \chi_c(R(\varepsilon)))| \le C_1 \log(R(\varepsilon)). \tag{3.3}$$

Now, if $\varepsilon$ is sufficiently small, we can write

$$\phi(\sqrt{1+\varepsilon^2}\, t + i\varepsilon v) = \phi(\sqrt{1+\varepsilon^2}\, t) + i\varepsilon D_v \phi(\sqrt{1+\varepsilon^2}\, t) + O(\varepsilon^2)$$

and

$$\begin{aligned} P \circ \phi \circ \chi_c(R(\varepsilon)) &= P(\phi(\sqrt{1+\varepsilon^2}\, t)) \\ &\quad + i\varepsilon D_{\mathcal{T}_\varepsilon(x,v)} P(\phi(\sqrt{1+\varepsilon^2}\, xt)) + O(\varepsilon^2), \end{aligned} \tag{3.4}$$

where $\mathcal{T}_\varepsilon(x, v) = D_v \phi(\sqrt{1+\varepsilon^2}\, x) \to \mathcal{T}(x, v)$ as $\varepsilon \to 0_+$. By (3.3) and (3.4), and by Lemma 1.1, we get

$$\alpha |D_{\mathcal{T}(t,v)} P(x)| \le C_1 (\deg P)(1 - \alpha^2 P^2(x))^{1/2}$$

with $0 < \alpha < 1$, whence by letting $\alpha \to 1_-$ we obtain the required inequality in case $\|P\|_N \le 1$. The general case easily reduces to the above one.

By Proposition 3.3 and by Corollary 1.4 we get

COROLLARY 3.4. *If $t \in \mathbb{B}^m(\alpha)$, $0 < \alpha < 1$ and $x = \phi(t)$, then for any $v \in \mathbb{S}^{m-1}$,*

$$|D_{\mathcal{T}(t,v)} P(x)| \le \frac{C_1}{\sqrt{1-\alpha^2}} (\deg P)(\|P\|_N^2 - P^2(x))^{1/2}.$$

*If* $t \in \mathbb{B}^m$, $v \in \mathbb{S}^{m-1}$, $v \cdot t = 0$ *and* $\phi(t) = x$, *then*

$$|D_{T(t,v)}P(x)| \le C_1(\deg P)(\|P\|_N^2 - P^2(x))^{1/2}.$$

**4. Main result.** Let $E$ be a subset of $\mathbb{C}^n$ and let $X(E)$ be the Zariski closure of $E$, i.e. $X(E)$ is the smallest algebraic subset of $\mathbb{C}^n$ that contains $E$. By the *Zariski dimension* of $E$ we shall mean the dimension of $X(E)$. Consider the space $\mathcal{P}_k(E) = \{p|_E : p \in \mathbb{C}[z_1, \ldots, z_n],\ \deg p \le k\}$. It is clear that $\mathcal{P}_k(E) = \mathcal{P}_k(X(E))$. Define $\delta_k = \delta_k(E) := \dim \mathcal{P}_k(E)$. One can prove (cf. [BaPl1, (1.7)])

PROPOSITION 4.1. *An irreducible closed analytic subset $E$ of $\mathbb{C}^n$ of pure dimension $m$ is algebraic if and only if $\delta_k(E) = O(k^m)$ for $k$ large enough.*

We recall that in the whole paper $\mathbb{R}^n = \{(\Re z_1, \ldots, \Re z_n) : (z_1, \ldots, z_n) \in \mathbb{C}^n\}$. In what follows, we will be assuming that $K$ is a compact subset of $\mathbb{R}^n$.

DEFINITION 4.2. $K$ is said to have an *analytic parametrization of dimension* $m$, $1 \le m \le n$, if there exist $\varrho > 1$, $r \in \mathbb{N}$ and real-analytic maps $\phi_j = (\phi_{j1}, \ldots, \phi_{jn}) : \mathbb{B}^m(\varrho) \to K$, $j = 1, \ldots, r$, such that for each $j$ we have $\operatorname{rank} \phi_j = m$ and

$$K = \bigcup_{j=1}^{r} \phi_j(\mathbb{B}^m).$$

REMARK. If $\phi : \mathbb{B}^m \to K = \phi(\mathbb{B}^m)$ is any analytic parametrization of $K$, i.e. $\phi$ is an analytic map defined in an open neighbourhood $U$ of $\mathbb{B}^m$, with values in $\mathbb{R}^n$, then in general $\phi$ does not fit the requirements of Definition 4.2. The point is that it may happen that $K \subsetneq \phi(U)$. However, in such a case we can compose $\phi$ with an appropriately chosen spherical coordinate system $h_m : \mathbb{R}^m \to \mathbb{B}^m$ (cf. e.g. [K, p. 22]) so that $\phi \circ h_m$ does satisfy all the requirements of Definition 4.2. If e.g. $m = 2$, we can define

$$h_2(s,t) = \cos\frac{\pi}{\sqrt{2}}s\left(\cos\frac{\pi}{\sqrt{2}}t, \sin\frac{\pi}{\sqrt{2}}t\right)$$

in order to get $h_2 : \mathbb{R}^2 \to \mathbb{B}^2$ with

$$h_2(\mathbb{B}^2) = h_2\left(\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \times \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]\right) = \mathbb{B}^2,$$

and $\operatorname{rank} h_2 := \sup_{(s,t)} \operatorname{rank}_{(s,t)} h_2 = 2$.

REMARK. In Definition 4.2, instead of considering an analytic parametrization defined in a neighbourhood of the unit ball $\mathbb{B}^m$, we could be working with an analytic parametrization defined in an open neighbourhood of the $m$-dimensional cube $\mathbb{I}^m = [-1,1]^m$. For, taking the map



$$l_m : \mathbb{R}^m \ni (t_1, \ldots, t_m) \mapsto \left(\sin\frac{\pi\sqrt{m}}{2}t_1, \ldots, \sin\frac{\pi\sqrt{m}}{2}t_m\right)$$

we have $\operatorname{rank} l_m = m$ and

$$l_m(\mathbb{R}^m) = l_m(\mathbb{B}^m) = l_m\left(\left[-\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}\right]^m\right) = \mathbb{I}^m.$$

Hence if $\phi : U \to \mathbb{R}^m$, where $U$ is an open connected neighbourhood of $\mathbb{I}^m$, is an analytic map with $\operatorname{rank} \phi = m$ such that $\phi(U) = \phi(\mathbb{I}^m) \subset K$, then $\phi \circ l_m : \mathbb{R}^m \to K$, $\operatorname{rank} \phi \circ l_m = m$ and

$$\phi \circ l_m(\mathbb{R}^m) = \phi \circ l_m(\mathbb{B}^m) = \phi(\mathbb{I}^m) \subset K.$$

EXAMPLE 4.3. Let $\mathbb{M}$ be an $m$-dimensional real-analytic manifold. We recall that a subset $K$ of $\mathbb{M}$ is said to be *semianalytic* if for each point $x \in \mathbb{M}$ there exist an open neighbourhood $U$ of $x$ in $\mathbb{M}$ and a finite number of real-analytic functions $f_{ij}$, $g_{ij}$ in $U$ such that

$$K \cap U = \bigcup_i \bigcap_j \{f_{ij} = 0,\ g_{ij} > 0\}.$$

A subset $K \subset \mathbb{M}$ is said to be *subanalytic* if for each point $x \in \mathbb{M}$ there exists an open neighbourhood $U$ of $x$ in $\mathbb{M}$ such that $K \cap U$ is the projection of a relatively compact semianalytic subset of $\mathbb{M} \times \mathbb{N}$, where $\mathbb{N}$ is a real-analytic manifold (see [DŁS]). For another (equivalent) definition of a subanalytic set, see [Hir]. By the famous Hironaka Rectilinearization Theorem [Hir, Theorem 7.1] one can prove (see [PaPl1, Corollary 6.2]) the following

COROLLARY. *If $K$ is a compact, subanalytic subset of $\mathbb{M}$ of pure dimension $m$, then there exist a finite number of real-analytic maps $\phi_k : \mathbb{R}^m \to \mathbb{M}$ such that $\bigcup_k \phi_k(\mathbb{I}^m) = K$.*

Hence, if we take $\mathbb{M}$ to be an $m$-dimensional real-analytic submanifold of $\mathbb{R}^n$ then, in view of the above two remarks, every compact subanalytic subset of $\mathbb{M}$, of pure dimension $m$, admits an analytic parametrization in the sense of Definition 4.2.

Other examples of sets with an analytic parametrization are furnished by the following

EXAMPLE 4.4. Let $K_j$ be a compact subset of $\mathbb{R}^{n_j}$ that admits a (global) analytic parametrization $\phi^j = (\phi_1^j, \ldots, \phi_{n_j}^j)$ of dimension $m_j$, defined in an open neighbourhood of the cube $\mathbb{I}^{m_j}$, for $j = 1, 2$, respectively. Then $\phi = (\phi_1^1, \ldots, \phi_{n_1}^1, \phi_1^2, \ldots, \phi_{n_2}^2)$ is an analytic parametrization of dimension $m_1 + m_2$ of $K_1 \times K_2 \subset \mathbb{R}^{n_1+n_2}$, defined in an open neighbourhood of $\mathbb{I}^{m_1+m_2}$.

It is clear that every map $\phi_j$ of an analytic parametrization of $K$ extends analytically to an open neighbourhood of $\mathbb{B}^m(\varrho)$ in $\mathbb{C}^m$. In what follows, we will be keeping the same symbol $\phi_j$ to denote this extension. The main result of this paper reads as follows.

THEOREM 4.5. *Let $K$ be a compact subset of $\mathbb{R}^n$ with an analytic parametrization $\{\phi_j\}_{j=1}^r$ of dimension $m$, $1 \le m \le n$, with parameters $r \in \mathbb{N}$ and $\varrho > 1$. Then the following conditions are equivalent*:

(i) *The Zariski dimension of $K$ is $m$.*

(ii) *There exist positive constants $C_2$ and $\delta_2$ such that*

$$V_K(\phi_j(z)) \le C_2\delta \quad \text{for } \operatorname{dist}(z, \mathbb{B}^m) \le \delta \le \delta_2,\ z \in \mathbb{C}^m,\ j = 1, \ldots, r.$$

(iii) *There exist positive constants $C_3$ and $\delta_3$ such that for every polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ of degree at most $d$,*

$$|P(\phi_j(z))| \le C_3\|P\|_K \quad \text{for } \operatorname{dist}(z, \mathbb{B}^m) \le \delta_3/d,\ z \in \mathbb{C}^m,\ j = 1, \ldots, r.$$

(iv) (Bernstein Inequality) *There exists a constant $C_4 > 0$ such that for each polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$,*

$$|D_{\mathcal{T}_j(t,v)}P(x)| \le C_4(\deg P)\|P\|_K,$$

*for $x \in K_j := \phi_j(\mathbb{B}^m)$, $t \in \phi^{-1}(x) \cap \mathbb{B}^m$ and $v \in \mathbb{S}^{m-1}$, $j = 1, \ldots, r$. Here $\mathcal{T}_j(t, v) = D_v\phi_j(t)$.*

(iv′) (van der Corput–Schaake Inequality) *There exists a constant $C_4' > 0$ such that for each polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$,*

$$|D_{\mathcal{T}_j(t,v)}P(x)| \le C_4'(\deg P)(\|P\|_K^2 - P^2(x))^{1/2},$$

*for $x \in K_j = \phi_j(\mathbb{B}^m)$, where $t \in \phi_j^{-1}(x) \cap \mathbb{B}^m$, and $v \in \mathbb{S}^{m-1}$, $j = 1, \ldots, r$.*

Proof. (i)⇒(ii). Let $X(K)$ be the Zariski closure of $K$ in $\mathbb{C}^n$. Assume that $\dim X(K) = m$. Let $\{\phi_j\}_{j=1}^r$ be an analytic parametrization of $K$ with parameter $\varrho > 1$. Then every $\phi_j$ extends to a holomorphic map, which will be still denoted by $\phi_j$, defined in an open, connected neighbourhood $U$ of $\mathcal{E}_{R_0} = \{z \in \mathbb{C}^m : V_{\mathbb{B}^m} \le \log R_0\}$ with $R_0 = h(\varrho)$. Moreover, since the ball $\mathbb{B}^m$ is a non-pluripolar subset of $\mathbb{C}^m$, by Corollary 2.2, for each $j$, $K_j = \phi_j(\mathbb{B}^m)$ is non-pluripolar in $X(K) \supset \phi_j(U)$. Hence by Sadullaev's Criterion 2.3 we have

$$C_0 := \max_{1\le j\le r} \|V_K\|_{\phi_j(\mathcal{E}_{R_0})} < \infty.$$

Observe that

$$\sqrt{\varrho}\,\mathcal{E}_{h(\sqrt{\varrho})} \subset \mathcal{E}_{R_0}.$$

Set

$$C_1 = \frac{C_0}{\log R_0} \quad \text{and} \quad R_1 = \sup\{R > 1 : \mathcal{E}_R \subset \sqrt{\varrho}\,\mathcal{E}_{h(\sqrt{\varrho})}\}.$$

Further, define

$$\delta_0 := \min\left\{\frac{\sqrt{\varrho} - 1}{2}, \frac{1}{2}(g(R_1) - 1)\right\}$$



and

$$C_2 := C_1(\varrho - (1 + \delta_0^2)^{-1/2}).$$

Suppose now that $\operatorname{dist}(z, \mathbb{B}^m) \le \delta \le \delta_0$. Then by Corollary 3.2 and by the choice of $R_1$,

$$V_K(\phi_j(z)) \le C_1 V_{\mathbb{B}^m}(z/\sqrt{\varrho})$$

for $z \in \mathcal{E}_{R_1}$, whence by (1.5), the same inequality holds for $z \in \mathcal{E}^{((g(R_1)-1)/2)}$. Applying Proposition 1.2 we derive the inequality

$$V_{\mathbb{B}^m}((x + z)/\sqrt{\varrho}) \le \|z\|(\varrho - (1 + \delta_0)^2)^{-1/2}$$

for $\|z\| \le \delta_0$ and $x \in \mathbb{B}^m$, which gives the required estimate in (ii).

We shall now prove (i)⇒(iv′). Indeed, if $K$ is a subset of $\mathbb{R}^n$ of Zariski's dimension $m$, with an analytic parametrization $\{\phi_j\}_{j=1}^r$, where $\phi_j$ are analytic in a neighbourhood of $\mathbb{B}^m(\varrho)$ for some $\varrho > 1$, with values in $\mathbb{R}^n$, then by Corollary 3.4, for $\alpha = 2/(1 + \varrho)$ and for any polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$, we get

$$|D_{\mathcal{T}_j(t,v)}P(x)| \le \frac{C_1}{\sqrt{1 - \alpha^2}}(\deg P)(\|P\|_K^2 - P^2(x))^{1/2},$$

where $x = \phi_j(t)$, $j \in \{1, \ldots, r\}$ and $v \in \mathbb{S}^{m-1}$.

The implication (ii)⇒(iii) easily follows from (2.1). To prove (iii)⇒(iv) apply Cauchy's Integral Formula to the function $g(s) := P \circ \phi_j(t + sv)$ defined in an open neighbourhood of $0 \in \mathbb{C}$. The implication (iv′)⇒(iv) is evident.

Hence, in order to complete the proof of Theorem 4.3 it suffices to show that (iv)⇒(i). To this end, let $K$ be a compact subset of $\mathbb{R}^n$ such that $K = \phi(\mathbb{B}^m)$, where $\phi = (\phi_1, \ldots, \phi_n)$ is an analytic map defined in an open neighbourhood $U$ of the unit ball $\mathbb{B}^m$ in $\mathbb{R}^m$ with $\operatorname{rank}_U \phi = m$. Let $\delta_k = \dim \mathcal{P}_k(K)$. Let $\{\widehat{e}_1, \ldots, \widehat{e}_{\delta_k}\}$ be a basis of $\mathcal{P}_k(X)$ and let $\{\zeta_1^{(k)}, \ldots, \zeta_{\delta_k}^{(k)}\} \subset K$ be a system of *extremal points* of $K$ with respect to the basis $\{\widehat{e}_j\}$, of order $\delta_k$. This means that

$$V_k(K) := |V(\zeta_1^{(k)}, \ldots, \zeta_{\delta_k}^{(k)})| = \sup\{|V(x_1, \ldots, x_{\delta_k})| : \{x_1, \ldots, x_{\delta_k}\} \subset K\},$$

where $V(x_1, \ldots, x_{\delta_k}) := \det[\widehat{e}_i(x_j)]$ is the *generalized Vandermonde determinant.* Then by an argument of Siciak we show (see [BaPl1, (1.5)]) that $V_k(K) > 0$. Choose $t_j \in \phi^{-1}(\zeta_j^{(k)}) \cap \mathbb{B}^m$, $j = 1, \ldots, \delta_k$, and define

$$(4.1)\qquad r = \sup\{R > 0 : \mathbb{B}^m(t_i, R) \cap \mathbb{B}^m(t_j, R) = \emptyset \text{ for } i \ne j,\ i, j \in \{1, \ldots, \delta_k\}\},$$

where $\mathbb{B}^m(a, R)$ denotes the closed ball in $\mathbb{R}^m$ of radius $R$, centred at $a$. Then we have

(1) $r \le 1$,
(2) $\bigcup_{j=1}^{\delta_k} \mathbb{B}^m(t_j, r) \subset \mathbb{B}^m(0, 1 + r)$.

Hence

$$\operatorname{vol}(\mathbb{B}^m(0,1+r)) = M_1(1+r)^m \geq M_2\delta_k r^m \tag{4.2}$$

with some positive constants $M_1$ and $M_2$ independent of $k$. Observe that one can find two indices $i$ and $j$, $i \neq j$, such that $|t_i - t_j| = 2r$, since otherwise $r$ would not satisfy (4.1). Then by (4.2), $|t_i - t_j| \leq M_3/\delta_k^{1/m}$ with a constant $M_3 > 0$ which does not depend on $k$. We may assume that $i = 1$ and $j = 2$. Since $V_k(K) > 0$, the formula

$$Q_k(x) := \frac{V(x, \zeta_2^{(k)}, \ldots, \zeta_{\delta_k}^{(k)})}{V(\zeta_1^{(k)}, \ldots, \zeta_{\delta_k}^{(k)})},$$

where

$$V(x, \zeta_2^{(k)}, \ldots, \zeta_{\delta_k}^{(k)}) := \det\begin{pmatrix} \widehat{e}_1(x) & \widehat{e}_1(\zeta_2^{(k)}) & \ldots & \widehat{e}_1(\zeta_{\delta_k}^{(k)}) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{e}_{\delta_k}(x) & \widehat{e}_{\delta_k}(\zeta_2^{(k)}) & \ldots & \widehat{e}_{\delta_k}(\zeta_{\delta_k}^{(k)}) \end{pmatrix},$$

defines a polynomial $Q_k \in \mathcal{P}_k(K)$ such that $|Q_k(\zeta_1^{(k)})| = 1$ and $Q_k(\zeta_2^{(k)}) = 0$, and by the Bernstein Inequality (iv) we get

$$\begin{aligned} 1 &= |Q_k(\zeta_1^{(k)}) - Q_k(\zeta_2^{(k)})| = |Q_k(\phi(t_1)) - Q_k(\phi(t_2))| \\ &\leq Mk|t_1 - t_2| \leq MM_3k/\delta_k^{1/m}. \end{aligned} \tag{4.3}$$

Suppose now that the Zariski closure of $K$ is of dimension $> m$. Then by Proposition 4.1 we would have

$$\limsup_{k\to\infty} \frac{\delta_k}{k^m} = \infty,$$

which contradicts (4.3) for $k$ large enough. The proof of the theorem is complete.

REMARK. Since every compact real-analytic manifold admits an analytic parametrization in the sense of Definition 4.2, the equivalence (i)⇔(iv) of the above theorem yields the main result of [BLMT].

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: baran@im.uj.edu.pl
plesniak@im.uj.edu.pl

*Received November 23, 1998*
*Revised version April 3, 2000* (4210)

STUDIA MATHEMATICA 141 (3) (2000)

# On Bárány's theorems of Carathéodory and Helly type

by

EHRHARD BEHRENDS (Berlin)

**Abstract.** The paper begins with a self-contained and short development of Bárány's theorems of Carathéodory and Helly type in finite-dimensional spaces together with some new variants. In the second half the possible generalizations of these results to arbitrary Banach spaces are investigated. The Carathéodory–Bárány theorem has a counterpart in arbitrary dimensions under suitable uniform compactness or uniform boundedness conditions. The proper generalization of the Helly–Bárány theorem reads as follows: if $\mathcal{C}_n$, $n = 1, 2, \ldots$, are families of closed convex sets in a bounded subset of a separable Banach space $X$ such that there exists a positive $\varepsilon_0$ with $\bigcap_{C\in\mathcal{C}_n}(C)_\varepsilon = \emptyset$ for $\varepsilon < \varepsilon_0$, then there are $C_n \in \mathcal{C}_n$ with $\bigcap_n (C_n)_\varepsilon = \emptyset$ for all $\varepsilon < \varepsilon_0$; here $(C)_\varepsilon$ denotes the collection of all $x$ with distance at most $\varepsilon$ to $C$.

**1. Introduction.** The simplest version of Bárány's Carathéodory theorem is often illustrated as follows: imagine in the plane three triangles, the first with red, the second with blue and the third with green vertices; if all contain a point $z$, then it is possible to choose a red, a blue and a green vertex such that $z$ is in the convex hull of these three points. The surprising feature is that even in this innocent two-dimensional setting there seems to be no really simple proof of this combinatorial fact.

The $d$-dimensional *Carathéodory–Bárány theorem* reads as follows: if $\Delta_i$, $i = 0, \ldots, d$, are subsets of $\mathbb{R}^d$ for which the convex hull $\mathrm{co}(\Delta_i)$ of $\Delta_i$ contains a common point $z$, then one may choose $x_i \in \Delta_i$ for $i = 0, \ldots, d$ such that $z$ is in $\mathrm{co}(\{x_0, \ldots, x_d\})$. By a duality argument one can deduce the following *Helly–Bárány theorem*: if $\mathcal{C}_i$, $i = 0, \ldots, d$, are finite families of compact convex subsets of $\mathbb{R}^d$ such that $\bigcap_{C\in\mathcal{C}_i} C = \emptyset$ for every $i$, then it is possible to find $C_i \in \mathcal{C}_i$ with $\bigcap_i C_i = \emptyset$.

These theorems—which obviously contain the classical Carathéodory and Helly theorems as special cases—were published in 1982 in [3]. Since then a number of refinements and applications have been studied (see the

2000 *Mathematics Subject Classification*: 46B20, 46B22, 52A20, 52A35.
*Key words and phrases*: Bárány, Carathéodory, Helly, Helly-type theorem, Krein–Milman theorem, RNP.